203(10): Simultaneous Solution of ~~the~~ two Orbital Equations.

The two orbital equations are:

$$\frac{1}{r^2}\left(1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right) = \left(\frac{p}{L}\right)^2 \quad -(1)$$

and

$$\left(\frac{dr}{dt}\right)^2\left(1+r^2\left(\frac{d\theta}{dr}\right)^2\right) = v^2 \quad -(2)$$

From eq. (1):

$$\left(\frac{dr}{d\theta}\right)^2 = r^2\left(r^2\left(\frac{p}{L}\right)^2 - 1\right) \quad -(3)$$

in which:

$$\left(\frac{p}{L}\right)^2 = \frac{E^2-m^2c^4}{c^2L^2} = \frac{(\gamma^2-1)m^2c^4}{c^2L^2} \quad -(4)$$

so

$$\boxed{p^2 = (\gamma^2-1)m^2c^2}. \quad -(5)$$

Here:

$$\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2} \quad -(6)$$

where

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \quad -(7)$$

Using eq. (3) in eq. (2):

$$\boxed{v^2 = \left(\frac{dr}{dt}\right)^2\left(1+\frac{1}{\left(\frac{rp}{L}\right)^2-1}\right) = \text{constant}} \quad -(8)$$

2)

The easiest equation to use is eq. (2):

$$\boxed{v^2 = \left(\frac{dr}{dt}\right)^2 \left(1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right) = \text{constant}} \quad - (9)$$

This shows that as:

$$\frac{d\theta}{dr} \rightarrow 0 \quad - (10)$$

then

$$v \rightarrow \frac{dr}{dt} = \text{constant} \quad - (11)$$

which is the condition of special relativity, one frame moves at constant v with respect to another.

The constant kinetic lagrangian is:

$$\boxed{\mathcal{L} = T := \frac{p^2}{2m} = \frac{1}{2}\left(\gamma^2 - 1\right) mc^2} \quad - (12)$$

In the limit:

$$v \ll c \quad - (13)$$

$$\mathcal{L} = T \rightarrow \frac{1}{2}\left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right) mc^2$$

$$\sim \left(1 + \frac{1}{2} mv^2 - 1\right)$$

$$= \frac{1}{2} mv^2, \quad - (14)$$

= constant classical kinetic energy, QED.

The method is based on the metric:

3)

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(15)$$

which is the Minkowski metric in the plane:

$$dZ^2 = 0. \quad -(16)$$

The constant kinetic lagrangian is:

$$\mathcal{L}_1 = T = \frac{1}{2} mc^2 = \frac{1}{2} mc^2 \left(\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2 \right)$$

$$-(17)$$

Comparing eqs. (12) and (17):

$$\mathcal{L} = (\gamma^2 - 1) \mathcal{L}_1 \quad -(18)$$

The metric (15) uses a constant v as is well known. To describe orbits it is constrained by any function:

$$f = \frac{d\theta}{dr} \quad -(19)$$

i.e. by any orbit.

Therefore:

$$\boxed{ds^2 = c^2 dt^2 - dr^2 (1 + r^2 f^2)} \quad -(20)$$

where $f(\theta, r)$ is observed in astronomy, or for any planar rotation in the laboratory.

Notes

The velocity v is defined by:

$$v^2 = \frac{d\underline{r} \cdot d\underline{r}}{dt^2} = \left(\frac{dr}{dt}\right)^2 \left(1 + r f^2\right) \quad -(21)$$

$$= \text{constant of motion}$$

Comparing eqs. (8) and (21) gives eq. (3), QED.
The constant of motion is the product in eq. (21).

$$\boxed{\text{So every orbit is characterized by } (p/L)^2 \text{ from eq. (1).}}$$

This means that every orbit is characterized by:

$$\left(\frac{p}{L}\right)^2 = \frac{E^2 - m^2c^4}{c^2L^2} \quad -(22)$$

$$:= x$$

Thus:

$$x := \frac{1}{r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right), \quad -(23)$$

a constant of motion.

From eq. (8) the function $\frac{dr}{dt}$ is defined

5) in terms of the two constants v^2 and $(p/L)^2$:

$$\left(\frac{dr}{dt}\right)^2 = \frac{v^2}{1 + \dfrac{1}{r^2\left(\frac{p}{L}\right)^2 - 1}}$$

$$= \frac{v^2\left(r^2\left(\frac{p}{L}\right)^2 - 1\right)}{r^2\left(\frac{p}{L}\right)^2}$$

$$= \left(\frac{L}{p}\right)^2 \frac{v^2}{r^2}\left(r^2\left(\frac{p}{L}\right)^2 - 1\right)$$

$$\boxed{\left(\frac{dr}{dt}\right)^2 = v^2\left(1 - \left(\frac{L}{rp}\right)^2\right)} \quad -(24)$$