203(9): Applications of the General Equation of Orbits

The equation is first checked for self consistency. It is:

$$\boxed{\frac{1}{r^2}\left(1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right)-\left(\frac{p}{L}\right)^2 = \text{constant of motion.}} \quad -(1)$$

Here:

$$p = A^{1/2}\gamma m \frac{dr}{dt} \quad -(2)$$

$$L = \gamma m r^2 \frac{dr}{dt}\frac{d\theta}{dr} \quad -(3)$$

where

$$A = 1 + r^2\left(\frac{d\theta}{dr}\right)^2 \quad -(4)$$

So:

$$\left(\frac{p}{L}\right)^2 = \frac{A}{r^4\left(\frac{d\theta}{dr}\right)^2} = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2} \quad -(5)$$

QED. The great advantage of eq. (1) is that it is fully relativistic and can be applied to any orbit, i.e. any $dr/d\theta$. It expresses all orbits in terms of the ratio of p and L squared, a constant of motion of any orbit.

Examples

1) For the precessing ellipse:

$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{x\epsilon}{\alpha}\right)^2 r^4 \sin^2(x\theta) \quad -(6)$$

$$= \left(\frac{x\epsilon}{\alpha}\right)^2 r^4 \left(r^2 - \frac{1}{\epsilon^2}(\alpha - r)^2\right)$$

So:

$$\boxed{\frac{1}{r^2}\left(1 + r^2\left(\frac{x\epsilon}{\alpha}\right)^2 \sin^2(x\theta)\right) = \text{constant of motion} = \left(\frac{p}{L}\right)^2}$$

$$-(7)$$

where

$$\left(\frac{p}{L}\right)^2 = \frac{E^2 - m^2c^4}{c^2L^2}. \quad -(8)$$

2) Logarithmic Spiral

$$\left(\frac{dr}{d\theta}\right)^2 = \alpha^2 r^2 \quad -(9)$$

and

$$\boxed{\left(\frac{p}{L}\right)^2 = \frac{E^2 - m^2c^4}{c^2L^2} = \frac{1+\alpha^2}{r^2}} \quad -(10)$$

3) Hyperbolic Spiral

$$\boxed{\left(\frac{p}{L}\right)^2 = \frac{E^2 - m^2c^4}{c^2L^2} = \frac{1}{r^2}\left(1 + \frac{r^2}{r_0^2}\right)} \quad -(11)$$

) Newtonian

$$\frac{2mT}{L^2} = \left(\frac{p}{L}\right)^2 \quad -(12)$$

The logarithmic spiral is:

$$r = r_0 e^{\alpha\theta} \quad - (13)$$

and the hyperbolic spiral is:

$$r = \frac{r_0}{\theta} \quad - (14)$$

For the log spiral, in order for eq. (10) to be true:

$$\left(1 + \alpha^2\right) = y r^2 \quad - (15)$$

where y is a constant. For the hyperbolic spiral:

$$\frac{1}{r^2} + \frac{1}{r_0^2} = \left(\frac{p}{L}\right)^2 \quad - (16)$$

so

$$\frac{1}{r_0^2} = \left(\frac{p}{L}\right)^2 - \frac{1}{r^2}. \quad - (17)$$

From eq. (15), the log spiral is:

$$\boxed{r = r_0 \exp\left(\left(y r^2 - 1\right)^{1/2} \theta\right)} \quad - (18)$$

From eq. (17), the hyperbolic spiral is:

$$\boxed{\frac{1}{r} = \left(\left(\frac{p}{L}\right)^2 - \frac{1}{r^2}\right)\theta \quad - (19)}$$

In eq. (18):

$$y = \left(p / L\right)^2 \quad - (20)$$

So in this theory the log spiral orbit must be:

$$\frac{r}{r_0} = \exp\left(\left(\left(\frac{E^2 - m^2c^4}{c^2L^2}\right)r^2 - 1\right)^{1/2}\theta\right) \quad -(21)$$

and the hyperbolic spiral orbit must be:

$$\frac{1}{r} = \left(\frac{E^2 - m^2c^4}{c^2L^2} - \frac{1}{r^2}\right)\theta \quad -(22)$$

According to this analysis, the spiral type orbits observed in astronomy are not simple spirals, but are of type (21) and (22).

Limit of large r

In this limit, for the logarithmic spiral:

$$\alpha^2 \xrightarrow[r \to \infty]{} \left(\frac{p}{L}\right)^2 r^2 - 1 \to \infty \quad -(23)$$

resulting in a very large pitch spiral. For the hyperbolic spiral:

$$\left(\frac{p}{L}\right)^2 \xrightarrow[r \to \infty]{} \frac{1}{r_0^2} \quad -(24)$$

The relativistic linear momentum is defined by:

$$p = A^{1/2}\gamma m \frac{dr}{dt} = \left(\frac{E^2 - m^2c^4}{c^2}\right)^{1/2} \quad -(25)$$

$$= \text{constant of motion}$$

For each orbit it is possible to define formally a constant kinetic energy:

5)

$$T := \frac{p^2}{2m} = \frac{mA\gamma^2}{2}\left(\frac{dr}{dt}\right)^2 \quad (26)$$

$$= \frac{1}{2}\left(\gamma^2 - 1\right)mc^2$$

This is a self consistent equation because of eq. (4), so:

$$m\left(1+r^2\left(\frac{d\theta}{dr}\right)^2\right)\gamma^2\left(\frac{dr}{dt}\right)^2 = \left(\gamma^2-1\right)mc^2 \quad (27)$$

where $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$; $v^2 = \frac{d\underline{r}\cdot d\underline{r}}{dt^2}$ — (28)

and $d\underline{r}\cdot d\underline{r} = dr^2 + r^2 d\theta^2$ — (29)

so $m\left(1+r^2\left(\frac{d\theta}{dr}\right)^2\right)\left(\frac{dr}{dt}\right)^2 = \left(\frac{\gamma^2-1}{\gamma^2}\right)mc^2$ — (29)

$$= \left(\frac{v}{c}\right)^2 mc^2 = mv^2$$

$$\text{so} \quad v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dr}\right)^2\left(\frac{dr}{dt}\right)^2 \quad (30)$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \text{constant} \quad (31)$$

QED

$$\boxed{\text{Eq. (30) is: } v^2 = \left(\frac{dr}{dt}\right)^2\left(1 + r^2\left(\frac{d\theta}{dr}\right)^2\right) = \text{constant}}$$

For a free particle in the limit:

$$d\theta / dr \to 0 \quad (32)$$

) then:

$$v \rightarrow \frac{dr}{dt} \quad - (33)$$

Eq. (31) is equivalent to:

$$\underline{v} = \frac{d\underline{r}}{dt} \quad - (34)$$

in cylindrical polar coordinates, where:

$$\left.\begin{aligned} \underline{v} &= \frac{d\underline{r}}{dt} = \frac{d}{dt}\left(r\,\underline{e}_r\right) \\ &= \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r \\ &= \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \end{aligned}\right\} \quad - (35)$$

so:

$$v^2 = \underline{v} \cdot \underline{v} = \dot{r}^2 + r^2\dot{\theta}^2 \quad - (36)$$

Eq. (31) therefore means that the orbit $d\theta/dr$ is equivalent to a movement of the frame of reference, a movement that generate a constant velocity squared v^2 and a constant kinetic energy.

The two equations (1) and (31) are equivalent.

The orbit is a constraint that transforms special relativity into general relativity, i.e. it transforms the Minkowski metric into a

7) constrained Minkowski metric. The constraint is the orbit itself. This is a completely general result for any orbit.

The law of orbits is that v^2 as defined by eq. (31) is constant.

For a log spiral for example:

$$v^2 = \left(\frac{dr}{dt}\right)^2 \left(1 + \frac{1}{\alpha^2}\right) = \text{constant} \quad -(37)$$

so if α is constant, then:

$$v = \frac{dr}{dt} = \text{constant} \quad -(38)$$

For a hyperbolic spiral:

$$\frac{d\theta}{dr} = \frac{r_0}{r^2} \quad -(39)$$

so

$$v^2 = \left(\frac{dr}{dt}\right)^2 \left(1 + \frac{r_0}{r^2}\right) \quad -(40)$$

and

$$v^2 \xrightarrow[r \to \infty]{} \text{constant} \quad -(41)$$

So these are simple explanations of the velocity curve: