03(7): All orbits can be described by Special Relativity.

To show this start with eq. (16) of note 203(5):

$$\frac{E^2}{mc^2} = mA\left(\frac{dr}{d\tau}\right)^2 + mc^2 \quad -(1)$$

and define the relativistic momentum by:

$$\boxed{p = mA^{1/2}\frac{dr}{d\tau}} \quad -(2)$$

Then eq. (1) is:

$$\boxed{E^2 = p^2c^2 + m^2c^4} \quad -(3)$$

or

$$\boxed{E = \gamma mc^2} \quad -(4)$$

the relativistic momentum is:

$$\boxed{\underline{p} = \gamma m\underline{v}} \quad -(5)$$

where

$$\boxed{\underline{v} = A^{1/2}\frac{dr}{dt}} \quad -(6)$$

The Lorentz factor is:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(7)$$

where v is given by eq. (6).

With these definitions:

2)
$$p^{\mu} p_{\mu} = m^2 c^2 \quad -(8)$$

where
$$p^{\mu} = \left(\frac{E}{c} ; \underline{p} \right), \quad p_{\mu} = \left(\frac{E}{c}, -\underline{p} \right) \quad -(9)$$

~~The theory of orbits is therefore special relativity.~~

The special relativity of free particles is regained when:
$$A = 1. \quad -(10)$$

The <u>general orbit</u> is described by:
$$r = f(\theta) \quad -(11)$$
where $f(\theta)$ is any function of θ. Therefore:
$$\frac{dr}{d\theta} = f'(\theta) \quad -(12)$$
and
$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 \left(1 + \frac{r^2}{(f'(\theta))^2} \right) \quad -(13)$$

Therefore in general:
$$\boxed{A = 1 + \frac{r^2}{(f'(\theta))^2}} \quad -(14)$$

From this equation:

$$\left(f'(\theta)\right)^2 = \frac{r^2}{A-1} \quad -(15)$$

However, from eq (15) of note 203(6):

$$\left(f'(\theta)\right)^2 = \frac{r^4}{A}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \quad -(16)$$

So:

$$\frac{1}{b^2} - \frac{1}{a^2} = \left(\frac{A}{A-1}\right) \cdot \frac{1}{r^2} \quad -(17)$$

By definition:

$$\frac{1}{b^2} = \left(\frac{E}{cL}\right)^2, \quad \frac{1}{a^2} = \left(\frac{mc}{L}\right)^2 \quad -(18)$$

So

$$\frac{1}{b^2} - \frac{1}{a^2} = \left(\frac{E}{cL}\right)^2 - \left(\frac{mc}{L}\right)^2$$

$$= \frac{1}{L^2}\left(\frac{E^2}{c^2} - m^2c^2\right)$$

$$= \frac{1}{L^2}\left(\frac{E^2 - m^2c^4}{c^2}\right)$$

$$= \left(\frac{p}{L}\right)^2 \quad -(19)$$

from eq. (3). This is a constant of motion. So:

$$\boxed{\frac{A}{A-1} = \left(\frac{pr}{L}\right)^2} \quad -(20)$$

for all orbits

4) Some Spiral Orbits

1) Logarithmic

$$r = r_0 e^{\alpha\theta}, \quad \frac{dr}{d\theta} = \alpha r. \quad -(21)$$

2) Hyperbolic

$$r = \frac{r_0}{\theta}, \quad \frac{dr}{d\theta} = \frac{r^2}{r_0}. \quad -(22)$$

3) Archimedes

$$r = a + b\theta, \quad \frac{dr}{d\theta} = b. \quad -(23)$$

4) Fermat

$$r = r_0 \theta^{1/2}, \quad \frac{dr}{d\theta} = \frac{1}{2}\frac{r_0^2}{r} \quad -(24)$$

5) Lituus

$$r^2 = \frac{r_0^2}{\theta}, \quad \frac{dr}{d\theta} = \frac{r^3}{2r_0^2} \quad -(25)$$

In each case:

$$A = 1 + r^2\left(\frac{dr}{d\theta}\right)^{-2} \quad -(26)$$

so:

$$A(\text{logarithmic}) = 1 + \frac{1}{\alpha^2} \quad -(27)$$

$$A(\text{hyperbolic}) = 1 + \left(\frac{r_0}{r}\right)^2 \quad -(28)$$

$$A(\text{Archimedean}) = 1 + \left(\frac{r}{b}\right)^2 \quad -(29)$$

$$A(\text{Fermat}) = 1 + \left(\frac{2r^2}{r_0^2}\right)^2 \quad -(30)$$

$$A(\text{Lituus}) = 1 + \left(2\frac{r_0^2}{r^2}\right)^2 \quad -(31)$$

[illegible] eq. ( ) :

L = [illegible] = constant $-(32)$

5)

Cross Check

Use:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 = c^2 d\tau^2 \quad -(32)$$

So:

$$\mathcal{L} = \frac{1}{2}mc^2 = \frac{1}{2}mc^2\left(\frac{dt}{d\tau}\right)^2 - \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{m}{2}\left(\frac{d\theta}{d\tau}\right)^2 r^2 \quad -(33)$$

$$E = mc^2\left(\frac{dt}{d\tau}\right), \quad L = mr^2\frac{d\theta}{d\tau} \quad -(34)$$

So

$$mc^2 = \frac{E^2}{mc^2} - m\left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{mr^2} \quad -(35)$$

and

$$m\left(\frac{dr}{d\tau}\right)^2 = m\left(\frac{dr}{d\theta}\right)^2\left(\frac{d\theta}{d\tau}\right)^2 = \frac{mL^2}{m^2r^4}\left(\frac{dr}{d\theta}\right)^2 \quad -(36)$$

$$= \frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2}.$$

So

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{mr^4}{L^2}\left(\frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2}\right) \quad -(37)$$

$$= r^4\left(\frac{1}{b^2} - \frac{1}{a^2} - \frac{1}{r^2}\right) \quad -(38)$$

where $b = \frac{cL}{E}$, $a = \frac{L}{mc}$.

Comparing eqs. (16) and (37):

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{1}{b^2} - \frac{1}{a^2} - \frac{1}{r^2}\right) = \frac{r^4}{A}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \quad -(39)$$

So:

$$\frac{1}{b^2} - \frac{1}{a^2} = \left(\frac{A}{A-1}\right)\frac{1}{r^2} \quad \checkmark\checkmark \quad -(40)$$

which is eq. (17), <u>QED</u>.

In eq. (19) use:

$$p = A^{1/2}\gamma m (dr/dt) \quad -(41)$$

$$L = mr^2 \frac{d\theta}{d\tau} = mr^2 \frac{d\theta}{dr}\frac{dr}{dt}\frac{dt}{d\tau}$$

$$= \gamma m r^2 \frac{dr}{dt}\frac{d\theta}{dr} \quad -(42)$$

So

$$\boxed{\left(\frac{pr}{L}\right)^2 = \frac{A}{r^2}\left(\frac{dr}{d\theta}\right)^2 = \frac{A}{A-1}} \quad -(43)$$

and:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^2}{A-1} \quad -(44) \quad \checkmark\checkmark$$

which is eq. (15), <u>Q.E.D.</u>