2.3(3): Newtonian Theory of Precessing Orbits

The usual Newtonian theory produces the static ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(1)$$

but the observed orbit is the precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(2)$$

The difference between eq. (1) and eq. (2) is:

$$\boxed{\theta \rightarrow x\theta} \quad -(3)$$

Therefore a Newtonian theory of the precessing ellipse follows from eq. (3).

The lagrangian that gives eq. (1) is:

$$\mathcal{L} = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) + \frac{mMG}{r} \quad -(4)$$

so

$$L = mr^2\frac{d\theta}{dt} \quad -(5)$$

from the Euler Lagrange equation. Therefore the lagrangian that gives eq. (2) is obtained using eq. (3):

$$\mathcal{L}_1 = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + x^2r^2\left(\frac{d\theta}{dt}\right)^2\right) + \frac{mMG}{r} \quad -(5)$$

giving

$$L_1 = mx^2r^2\frac{d\theta}{dt} \quad -(6)$$

The total energy that gives eq. (1) is

$$E = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) - \frac{mMG}{r} \quad -(7)$$

So the total energy that gives eq. (2) is:

$$E_1 = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + x^2 r^2 \left(\frac{d\theta}{dt} \right)^2 \right) - \frac{mMG}{r} \quad -(8)$$

From eq. (8):

$$\frac{dr}{dt} = \left(\frac{2}{m} \left(E_1 - \frac{L_1^2}{2 m x^2 r^2} - U \right) \right)^{1/2}, \quad -(9)$$

$$U = -\frac{mMG}{r} \quad -(10)$$

Now use:

$$\frac{d\theta}{dr} = \frac{d\theta}{dt} \frac{dt}{dr} \quad -(11)$$

$$= \frac{L_1}{m x^2 r^2} \left(\frac{2}{m} \left(E_1 - \frac{L_1^2}{2 m x^2 r^2} - U \right) \right)^{-1/2}$$

So

$$\boxed{\frac{dr}{d\theta} = \frac{m x^2 r^2}{L_1} \left(\frac{2}{m} \left(E_1 - \frac{L_1^2}{2 m x^2 r^2} - U \right) \right)^{1/2} \quad -(12)}$$

From eq. (2):

$$\frac{dr}{d\theta} = \frac{x \epsilon}{\alpha} r^2 \sin(x\theta) \quad -(13)$$

$$= \frac{x \epsilon}{\alpha} r^2 \left(1 - \frac{1}{\epsilon^2} \left(\frac{\alpha}{r} - 1 \right)^2 \right)^{1/2}$$

Comparing eqs. (12) and (13):

$$\left(\frac{x\epsilon}{\alpha}\right)^2\left(r^2-\frac{1}{\epsilon^2}(\alpha-r)^2\right) \quad -(14)$$

$$= \frac{x^4 r^2}{L_1^2}\left(2m\left(E_1+\frac{mMG}{r}-\frac{L_1^2}{2mx^2r^2}\right)\right)$$

so:

$$\boxed{\alpha = \frac{L_1^2}{x^2m^2MG}} \quad -(15) \checkmark$$

$$\boxed{\epsilon = \left(1+\frac{2\cancel{x^2}E_1L_1^2}{x^2m^3M^2G^2}\right)^{1/2}} \quad -(16)$$

This is a classical or Newtonian description of the precessing ellipse.

Usually it is asserted that such a description is not possible, but it is easily possible

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