203(4): Full Details of the Newtonian Calculation of the Perihelion Precession.

The basis of the calculation is:

$$\theta \rightarrow x\theta . \qquad -(1)$$

The lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d(x\theta)}{dt} \right)^2 \right) + \frac{mMG}{r}$$

$$= \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + x^2 r^2 \left(\frac{d\theta}{dt} \right)^2 \right) + \frac{mMG}{r} \qquad -(2)$$

where

$$U = -\frac{mMG}{r} \qquad -(3)$$

The Euler Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \qquad -(4)$$

and the angular momentum is:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = x^2 m r^2 \frac{d\theta}{dt} . \qquad -(5)$$

The total energy is:

$$E = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + x^2 r^2 \left(\frac{d\theta}{dt} \right)^2 \right) + U$$

$$= \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{L^2}{2 m x^2 r^2} + U \qquad -(6)$$

So:

$$\frac{dr}{dt} = \left(\frac{2}{m} \left(E - U - \frac{L^2}{2 m x^2 r^2} \right) \right)^{1/2} \qquad -(7)$$

Now use:

$$\frac{d\theta}{dr} = \frac{d\theta}{dt} \frac{dt}{dr} \qquad -(8)$$

where

$$\frac{d\theta}{dt} = \frac{L}{m x^2 r^2} \qquad -(9)$$

2)

So:

$$\frac{dr}{d\theta} = \frac{m x^2 r^2}{L}\left(\frac{2}{m}\left(E - U - \frac{L^2}{2m x^2 r^2}\right)\right)^{1/2} \quad -(10)$$

and

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{2 m x^4 r^4}{L^2}\left(E - U - \frac{L^2}{2 m x^2 r^2}\right) \quad -(11)$$

The astronomical observations give the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon\cos(x\theta)} \quad -(12)$$

so:

$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha} r^2 \sin(x\theta). \quad -(13)$$

From eq. (12):

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(14)$$

Use:

$$\sin^2(x\theta) + \cos^2(x\theta) = 1 \quad -(15)$$

so

$$\sin^2(x\theta) = 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 \quad -(16)$$

and

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{x^2\epsilon^2}{\alpha^2} r^4\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right) \quad -(17)$$

Comparing eqs. (11) and (17):

$$\frac{2 m x^4}{L^2}\left(E - U - \frac{L^2}{2 m x^2 r^2}\right) = \frac{x^2\epsilon^2}{\alpha^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right) \quad -(18)$$

So:

$$\frac{2mx^2}{L^2}\left(E-U-\frac{L^2}{2mx^2r^2}\right)=\frac{\epsilon^2}{\alpha^2}\left(1-\frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2\right)$$

$$=\frac{\epsilon^2}{\alpha^2}\left(1-\frac{1}{\epsilon^2r^2}(\alpha-r)^2\right)$$

$$=\frac{\epsilon^2}{\alpha^2r^2}\left(r^2-\frac{1}{\epsilon^2}(\alpha-r)^2\right). \quad -(19)$$

So:

$$\frac{\epsilon^2}{\alpha^2}\left(r^2-\frac{1}{\epsilon^2}(\alpha-r)^2\right)=\frac{2mx^2r^2}{L^2}\left(E-U-\frac{L^2}{2mx^2r^2}\right)$$

$$=\frac{\epsilon^2}{\alpha^2}\left(r^2-\frac{1}{\epsilon^2}(\alpha^2-2\alpha r+r^2)\right)$$

$$=\frac{2mx^2E}{L^2}r^2+\frac{2mx^2}{L^2}mMGr-1$$

$$-(20)$$

Comparing terms eqn. (20) is:

$$=\frac{\epsilon^2}{\alpha^2}\left(\left(1-\frac{1}{\epsilon^2}\right)r^2+\frac{2\alpha}{\epsilon^2}r\right)-1$$

so:

$$\frac{1}{\alpha}=\frac{x^2m^2MG}{L^2} \quad -(21)$$

$$\boxed{\alpha=\frac{L^2}{x^2m^2MG}} \quad -(22)$$

$$\frac{\epsilon^2}{\alpha^2}\left(1-\frac{1}{\epsilon^2}\right) = \frac{2mx^2E}{L^2} \quad -(23)$$

$$\epsilon^2 - 1 = \frac{2\alpha^2 m x^2 E}{L^2} \quad -(24)$$

$$= \frac{2L^4 m x^2 E}{x^4 m^4 \underline{M}^2 G^2 L^2}$$

$$= \frac{2L^2 E}{x^2 m^3 \underline{M}^2 G^2}$$

So

$$\boxed{\epsilon = \left(1 + \frac{2L^2E}{x^2 m^3 \underline{M}^2 G^2}\right)^{1/2}} \quad -(25)$$

It is clear that the precession constant x can be calculated <u>classically</u>.
