## The Line Element for a Precessing Ellipse

Consider the infinitesimal line element of relativity theory in a plane:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

for the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(2)$$

or any conical section.

From eq. (2):

$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha} r^2 \sin(x\theta) \quad -(3)$$

so

$$d\theta^2 = dr^2 \left( \frac{\alpha^2}{x^2 \epsilon^2 r^4 \sin^2(x\theta)} \right) \quad -(4)$$

where

$$\sin^2(x\theta) = 1 - \frac{1}{\epsilon^2} \left( \frac{\alpha}{r} - 1 \right)^2 \quad -(5)$$

Therefore:

$$\boxed{ds^2 = c^2 dt^2 - A\, dr^2} \quad -(6)$$

where:

$$A = 1 + \left( \frac{\alpha}{x \epsilon r \sin(x\theta)} \right)^2 \quad -(7)$$

In eq. (6):

$$d\underline{r} \cdot d\underline{r} = A\, dr^2 = v^2 dt^2 \quad -(8)$$

so:

2)

The total linear velocity is therefore:

$$\boxed{v = A^{1/2}\,\frac{dr}{dt}} \quad -(9)$$

Let this be the velocity of a particle of mass $m$, then the lagrangian in relativity theory is, from eq. (6):

$$\mathcal{L} = \frac{1}{2}mc^2 = \frac{1}{2}mc^2\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2}mA\left(\frac{dr}{d\tau}\right)^2. \quad -(10)$$

The total energy is:

$$E = mc^2\,\frac{dt}{d\tau}. \quad -(11)$$

From eqs. (6) and (8):

$$c^2 d\tau^2 = c^2 dt^2 - v^2 dt^2 \quad -(12)$$

so

$$\frac{d\tau}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \quad -(13)$$

and

$$\frac{dt}{d\tau} = \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad -(14)$$

Therefore

$$\boxed{E = \gamma mc^2} \quad -(15)$$

Eq. (10) is:

$$mA\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - mc^2. \quad -(16)$$

Define the relativistic momentum as:

$$p = \gamma m\,\frac{dr}{dt} = m\,\frac{dr}{d\tau} \quad -(17)$$

3.

Then:

$$\boxed{E^2 = Ac^2p^2 + m^2c^4} \quad -(18)$$

Define:

$$\left.\begin{aligned} p^\mu &= \left(\frac{E}{c}, A^{1/2}\underline{p}\right) \\ p_\mu &= \left(\frac{E}{c}, -A^{1/2}\underline{p}\right) \end{aligned}\right\} \quad -(19)$$

Then

$$\boxed{p^\mu p_\mu = m^2c^2} \quad -(20)$$

The lagrangian (10) is:

$$\mathcal{L} = \frac{1}{2}mc^2 = \frac{1}{2}m\left(c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2\right) \quad -(21)$$

from which the angular momentum is:

$$L = mr^2\frac{d\theta}{d\tau} \quad -(22)$$

In eq. (16), use:

$$\frac{dr}{d\tau} = \frac{dr}{d\theta}\frac{d\theta}{d\tau} \quad -(23)$$

so

$$\frac{AL^2}{mr^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{E^2}{mc^2} - mc^2 \quad -(24)$$

i.e.

$$\boxed{\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{A}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)} \quad -(25)$$

where

$$a = \frac{L}{mc}, \quad b = \frac{cL}{E} \qquad (26)$$

Eqs. (24) and (25) are the correct relativistic description of the precessing elliptical orbit.

These equations follow directly from the observation (2). These equations are equivalent to the Newtonian description:

$$H = \frac{1}{2} m v^2 - \frac{m M G}{r} \qquad (27)$$

$$\mathcal{L} = \frac{1}{2} m v^2 + \frac{m M G}{r} \qquad (28)$$

if

$$v^2 = \frac{1}{2} m \left( \left( \frac{dr}{dt} \right)^2 + x^2 r^2 \left( \frac{d\theta}{dt} \right)^2 \right) \qquad (29)$$

$$L = x^2 m r^2 \frac{d\theta}{dt} \qquad (30)$$

and if:

$$\alpha = \frac{L^2}{x^2 m^2 M G} \qquad (31)$$

$$\epsilon = \left( 1 + \frac{2 L^2 E}{x^2 m^3 M^2 G^2} \right)^{1/2} \qquad (32)$$

For a self consistent description of the solar system these equations are adequate, and the Einsteinian general relativity is rejected as incorrect and obsolete.