203(8): The General Orbital Equation

This equation is obtained from the constrained Minkowski metric as in note 203(7):

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{1}{b^2} - \frac{1}{a^2} - \frac{1}{r^2}\right) \quad -(1)$$

$$= r^4\left(\frac{E^2 - m^2c^4}{c^2L^2} - \frac{1}{r^2}\right)$$

Therefore the general orbital equation is:

$$\boxed{\frac{1}{r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right) = \phi} \quad -(2)$$

where

$$\phi = \frac{E^2 - m^2c^4}{c^2L^2} \quad -(3)$$

is a constant of motion.

We have:

$$E^2 - m^2c^4 = p^2c^2 \quad -(4)$$

so

$$\boxed{\phi = \frac{p}{L}} \quad -(5)$$

and is the ratio of the relativistic linear and angular momenta.

The quantity $dr/d\theta$ is found by observation.

) Therefore for any observed orbit:

$$\boxed{\frac{1}{r^2}\left(1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right)=\frac{p}{L} \\ = \text{constant}} \quad -(6)$$

In the Newtonian limit:

$$\left(\frac{dr}{d\theta}\right)^2=\frac{r^4}{L^2}\left(2m\left(E-V-\frac{L^2}{2mr^2}\right)\right) \quad -(7)$$

$$=\left(\frac{\epsilon}{\alpha}\right)^2 r^4\left(r^2-\frac{1}{\epsilon^2}\left(\alpha-r\right)^2\right)$$

in the notation of previous notes.

In this case:

$$\frac{1}{r^2}\left(1+\frac{r^2}{L^2}\left(2m\left(E-V-\frac{L^2}{2mr^2}\right)\right)\right) \quad -(8)$$

$$=\frac{2m}{L^2}\left(E-V\right)=\frac{2mT}{L^2}$$

where

$$T=E-V \quad -(9)$$

is the kinetic energy. $\quad -(10)$

So: $$\frac{2mT}{L^2}=\frac{E^2-m^2c^4}{c^2L^2}=\frac{p^2}{L^2}$$

and $$\boxed{T=\frac{p^2}{2m}=\text{constant}} \quad -(11)$$

This result has the same format as the classical kinetic energy but should be interpreted as

$$T = \frac{E^2 - m^2c^4}{2mc^2} = \text{constant} \quad -(12)$$

Using the result of previous notes:

$$E = \gamma mc^2 \quad -(13)$$

then

$$\boxed{T = \frac{mc^2}{2}\left(\gamma^2 - 1\right)} \quad -(14)$$

which is the constant relativistic kinetic energy in the Newtonian limit. The latter is:

$$v \ll c. \quad -(15)$$

So:

$$T = \frac{mc^2}{2}\left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right)$$

$$\sim \frac{mc^2}{2}\left(1 + \frac{v^2}{c^2} - 1\right)$$

$$= \frac{1}{2}mv^2 \quad -(16)$$

This is the Newtonian kinetic energy, <u>Q.E.D.</u>

<u>Therefore the analysis is self consistent.</u>

The original equation (1) is obtained

4) directly from the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \qquad -(17)$$

and the lagrangian:

$$\mathcal{L} = \frac{1}{2} mc^2 = \frac{1}{2} mc^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 - \frac{1}{2} m r^2 \left(\frac{d\theta}{d\tau}\right)^2 \qquad -(18)$$

$$= T$$

This is a pure kinetic lagrangian as in the theory of line element general relativity.

Eq. (6) is the direct result of eq. (18). By using eq. (7) in eq. (6), the Newtonian ideas of potential energy V and centrifugal energy $L^2/(2mr^2)$ are transformed into pure kinetic energy and the result is eq. (11). However, the kinetic energy (11) is constant. In the Newtonian dynamics, the hamiltonian is constant:

$$H = T + V \qquad -(19)$$
$$= E$$

The great advantage of eq. (6) is that it is a fully relativistic description valid for all orbits.