1) 202(4): The Correct Expression for Light Deflection due to Gravity

The orbit of a photon of mass m around the sun is given by a precessing ellipse, as for any object of mass m. The precessing ellipse is:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(1)$$

as in previous notes. So:

$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha} r^2 \sin(x\theta) \quad -(2)$$

$$= \frac{x\epsilon}{\alpha} r^2 \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2}$$

$$= \frac{x\epsilon}{\alpha} r^2 \left(1 - \frac{1}{\epsilon^2 r^2}(\alpha - r)^2\right)^{1/2}$$

$$= \frac{x\epsilon}{\alpha} r \left(r^2 - \frac{1}{\epsilon^2}(\alpha - r)^2\right)^{1/2} \quad -(3)$$

Therefore:

$$\frac{d\theta}{dr} = \frac{\alpha}{x\epsilon} \frac{1}{r} \left(r^2 - \frac{1}{\epsilon^2}(\alpha - r)^2\right)^{-1/2}$$

The light deflection due to gravity is:

$$\boxed{\Delta\theta = \frac{2\alpha}{x\epsilon} \int_{R_0}^{\infty} \frac{dr}{r\left(r^2 - \frac{1}{\epsilon^2}(\alpha - r)^2\right)^{1/2}}} \quad -(4)$$

... previous notation.

2) This expression is a standard integral which may be evaluated analytically. It is:

$$\Delta\theta = \frac{2\alpha}{x\epsilon}\int_{R_0}^{\infty} \frac{dr}{r\left(r^2\left(1-\frac{1}{\epsilon^2}\right)+\frac{2\alpha}{\epsilon^2}r-\frac{\alpha^2}{\epsilon^2}\right)^{1/2}} \qquad -(5)$$

This is the integral:

$$\int \frac{dx}{x(ax^2+bx+c)^{1/2}} = \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx+2c}{|x|(b^2-4ac)^{1/2}}\right)$$

with $c<0$, $b^2>4ac$,

$$a = 1-\frac{1}{\epsilon^2}, \quad b = \frac{2\alpha}{\epsilon^2}, \quad c = -\frac{\alpha^2}{\epsilon^2},$$

$$b^2-4ac = \frac{4\alpha^2}{\epsilon^4} + \frac{4\alpha^2}{\epsilon^2}\left(1-\frac{1}{\epsilon^2}\right) \qquad -(6)$$

$$= \frac{4\alpha^2}{\epsilon^2}$$

So:

$$\Delta\theta = \frac{2\alpha}{x\epsilon}\frac{\epsilon}{\alpha}\left(\sin^{-1}\left(\frac{\frac{2\alpha r}{\epsilon^2}-\frac{2\alpha^2}{\epsilon^2}}{|r|\,\frac{2\alpha}{\epsilon}}\right)\right)\Bigg|_{R_0}^{\infty}$$

$$\boxed{\begin{aligned}\Delta\theta &= \frac{2}{x}\sin^{-1}\left(\frac{r-\alpha}{\epsilon|r|}\right)\Bigg|_{R_0}^{\infty} = \frac{2}{x}\sin^{-1}\left(\frac{1}{\epsilon}\left(1-\frac{\alpha}{r}\right)\right)\Bigg|_{R_0}^{\infty} \\ &= \frac{2}{x}\left(\sin^{-1}\frac{1}{\epsilon} - \sin^{-1}\left(\frac{R_0-\alpha}{\epsilon|R_0|}\right)\right) \qquad -(7)\end{aligned}}$$

Therefore for any object of mass m orbiting an object of mass M is a precessing ellipse:

$$\boxed{\Delta\theta = \frac{\alpha}{x}\left(\sin^{-1}\frac{1}{\epsilon} - \sin^{-1}\left(\frac{1}{\epsilon}\left(1-\frac{\alpha}{R_0}\right)\right)\right)}$$

— (8)

Here:

Right latitude (latus rectum) $= 2\alpha$
Eccentricity $= \epsilon$
Precession constant $= x$
Distance of closest approach $= R_0$

The Newtonian result is given by:

$$x = 1 \quad — (9)$$

and for a planet in the solar system:

$$x - 1 \sim 10^{-5} \quad — (10)$$

There is no way in which Einstein could have correctly calculated $\Delta\theta$ because he used the incorrect:

$$\left(\frac{x\epsilon}{\alpha}\right)^2 \sin^2(x\theta) = ? \ \frac{1}{b^2} - \left(1-\frac{r_0}{r}\right)\left(\frac{1}{a^2}+\frac{1}{r^2}\right)$$

— (11)

From NASA Cassini for the photon:

$$\Delta\theta = (8.4848 \pm 0.003) \times 10^{-6} \text{ radians} \quad — (12)$$
$$= 1.75 \text{ arc seconds.}$$

If the orbit of the photon is an open orbit then

$$\epsilon > 1, \quad -(13)$$

but if it is a precessing ellipse:

$$\epsilon < 1 \quad -(14)$$

The only thing that is known experimentally is the result (11). Nothing is actually known about the complete orbit of light around the sun.

In the Newtonian limit:

$$\alpha = \frac{L^2}{m^2 M G} \quad -(15)$$

$$\epsilon = \left(1 + \frac{2EL^2}{m^3 M^2 G^2}\right)^{1/2} \quad -(16)$$

where E is the total energy, L is the total angular momentum, m the mass of the orbiting object, M the mass of the sun and G is Newton's constant.

So for a comet such as Halley's comet, the deflection $\Delta\theta$ depends on all the variables of eq. (15) and (16). Note carefully that the deflection depends on m, the mass of the orbiting object. In Einstein's theory:

$$\Delta\theta \overset{?}{=} \frac{4MG}{c^2 R_0}, \quad -(17)$$

and the dependence on m is missing. Also, the so-called "Newtonian" result cannot be:

$$\Delta\theta \overset{?}{=} \frac{2MG}{c^2 R_0} \quad -(18)$$