1)

192(6): Reduction of the Total Energy to the Newtonian Concept of Total Energy.

Consider the infinitesimal line element:

$$ds^2 = c^2 d\tau^2 = c^2 m(r)\, dt^2 - m(r)^{-1} dr^2 - r^2 d\theta^2 \quad -(1)$$

in the plane:

$$dZ^2 = 0. \quad -(2)$$

The total energy is the constant of motion:

$$E = m(r)\, mc^2 \frac{dt}{d\tau} \quad -(3)$$

The Newtonian total energy is:

$$E_N = \frac{1}{2} mv^2 - \frac{mMG}{r} \quad -(4)$$

$$= T + V$$

where T is the kinetic and V the potential energy.

It is known that a precessing ellipse is given by:

$$m(r) = \frac{A}{\frac{1}{a^2} + \frac{1}{r^2}} \quad -(5)$$

where

$$A = \frac{1}{b^2} - \left(\frac{x\epsilon}{d}\right)^2 + \left(\frac{x}{d}\right)^2 \left(1 - \frac{d}{r}\right)^2$$

and a static ellipse by:

$$x = 1. \quad -(7)$$

In eq. (3):

$$\frac{dt}{d\tau} = \left(m(r) - \frac{v^2}{c^2}\right)^{-1/2}, \quad -(8)$$

2) which follows from:

$$ds^2 = m(r) c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(9)$$

where

$$d\underline{r} \cdot d\underline{r} = v^2 dt^2 \quad -(10)$$

So

$$ds^2 = c^2 d\tau^2 = (m(r)c^2 - v^2) dt^2 \quad -(11)$$

which is eq. (8), QED. So:

$$\boxed{\begin{aligned} E &= mc^2 m(r) \left(m(r) - \frac{v^2}{c^2} \right)^{-1/2} \\ &= \text{constant of motion} \end{aligned}} \quad -(12)$$

Note that the function $m(r)$ contains the classical concept of potential energy. The Newtonian concept of:

$$V = -\frac{mMG}{r} \quad -(13)$$

is replaced by geometry. This is a fundamental idea of general relativity.

If, in eq. (12):

$$m(r) = 1 \quad -(14)$$

then

$$E = \gamma mc^2 \quad -(15)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(16)$$

and E is the total energy of a free particle in special relativity. There is no force of attraction on a free particle. Now consider:

$$E - mc^2 = (\gamma - 1) mc^2 \quad -(17)$$

which is the total energy minus the rest energy. We have:

$$E - mc^2 = \left(\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right) mc^2 . \quad -(18)$$

If:

$$v << c \quad -(19)$$

$$E - mc^2 \sim \left(1 + \frac{1}{2} \frac{v^2}{c^2} - 1 \right) mc^2 \quad -(20)$$

$$= \frac{1}{2} mv^2$$

This is the total Newtonian energy in the absence of a potential energy:

$$E_N = T = \frac{1}{2} mv^2 \quad -(21)$$

In order to reach the Newtonian concept (4) from eq. (12) the rest energy must be subtracted from E, and

$$m(r) \to 1, \quad v << c \quad -(22)$$

It is helpful to illustrate this process first for the old "Schwarzschild" metric (SM):

$$m(r) = 1 - \frac{r_0}{r} \quad -(23)$$

then to apply it to eq. (5). For the SM:

$$E = mc^2 \left(1 - \frac{r_0}{r} \right) \left(1 - \frac{r_0}{r} - \frac{v^2}{c^2} \right)^{-1/2} . \quad -(24)$$

If

$$r_0 << r, \quad v << c \quad -(25)$$

4) $$\left(1-\frac{r_0}{r}-\frac{v^2}{c^2}\right)^{-1/2} \sim \left(1-\frac{r_0}{r}\right)^{-1/2}\left(1-\frac{v^2}{c^2}\right)^{-1/2} \quad -(26)$$

To prove this use:
$$\left(1-\frac{r_0}{r}-\frac{v^2}{c^2}\right)^{-1/2} \sim 1+\frac{1}{2}\frac{r_0}{r}+\frac{1}{2}\frac{v^2}{c^2} \quad -(27)$$

and $$\left(1-\frac{r_0}{r}\right)^{-1/2}\left(1-\frac{v^2}{c^2}\right)^{-1/2} \sim \left(1+\frac{1}{2}\frac{r_0}{r}\right)\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$$
$$= 1+\frac{1}{2}\frac{r_0}{r}+\frac{1}{2}\frac{v^2}{c^2}+\frac{1}{4}\frac{r_0}{r}\frac{v^2}{c^2} \quad -(28)$$

Eqs. (27) and (28) are approximately the same because the product term, the last term RHS of eq. (28), is orders of magnitude smaller.

So:
$$E \sim mc^2\left(1-\frac{r_0}{r}\right)\left(1-\frac{r_0}{r}\right)^{-1/2}\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$
$$= \gamma mc^2\left(1-\frac{r_0}{r}\right)^{1/2} \quad -(29)$$
$$E \sim \gamma mc^2\left(1-\frac{1}{2}\frac{r_0}{r}\right)$$
$$= \gamma mc^2 - \gamma mc^2\frac{MG}{rc^2}$$
$$E = \gamma mc^2 - \frac{\gamma mMG}{r} \quad -(30)$$

and $$E - mc^2 = (\gamma-1)mc^2 - \frac{\gamma mMG}{r} \quad -(31)$$

In the limit:

5)

$$v \ll c \quad -(32)$$

$$\boxed{E - mc^2 \rightarrow E_N = \frac{1}{2} m v^2 - \frac{mMG}{r}} \quad -(33)$$

It is seen that the Newtonian limit is reached after a series of approximations. This appears to be the reason why the function (23) was contrived. It is quite easy to show that this function is totally incorrect, as in previous notes to UFT 192.

Therefore in the next note the condition will be derived under which the Newtonian E_N can be obtained from eq (5) under the condition (7) for a static ellipse.