1) 192(10): Limit Calculation of $m(r)$.

Consider the correct $m(r)$ function for a precessing ellipse:

$$m(r) = \frac{\frac{1}{b^2} - \left(\frac{x\epsilon}{\alpha}\right)^2 + \left(\frac{x}{\alpha}\right)^2\left(1-\frac{\alpha}{r}\right)^2}{\frac{1}{a^2} + \frac{1}{r^2}} \quad -(1)$$

In the limit:

$$r \to \infty \quad -(2)$$

$$m(r) \xrightarrow[r\to\infty]{} a^2\left(\frac{1}{b^2} - \left(\frac{x\epsilon}{\alpha}\right)^2 + \left(\frac{x}{\alpha}\right)^2\right) \quad -(3)$$

For Earth:

1) Rate of precession of perihelion $= 3.84$ arcsec / century, so to an excellent approximation:

$$x = 1 \quad -(4)$$

2) Eccentricity $\epsilon = 0.01671123. \quad -(5)$

3) Aphelion $r_{max} = 1.52098232 \times 10^{12}$ m $-(6)$

4) Perihelion $r_{min} = 1.47098290 \times 10^{12}$ m $-(7)$

Therefore:

$$\alpha = (1+\epsilon) r_{min} = (1-\epsilon) r_{max} = 1.4956 \times 10^{12} \text{ m} \quad -(8)$$

So:

$$m(r) \xrightarrow[r\to\infty]{} \left(\frac{a}{b}\right)^2 = \left(\frac{E}{mc^2}\right)^2 \quad -(9)$$

to an excellent approximation.

ii) Also to an excellent approximation for the Earth:

$$n(r) = \frac{1}{b^2}\left(\frac{1}{a^2} + \frac{1}{r^2}\right)^{-1} \quad -(10)$$

$$n(r) \doteq \left(\frac{a}{b}\right)^2 \left(\frac{r^2}{a^2 + r^2}\right) \quad -(11)$$

because:

$$\frac{xE}{d} << 1, \quad -(12)$$

$$\frac{x}{d} << 1, \quad -(13)$$

$$x = 1. \quad -(14)$$

Therefore for all practical purposes in the solar system:

$$n(r) = \left(\frac{E}{mc^2}\right)^2 \left(\frac{r^2}{a^2 + r^2}\right) \quad -(15)$$

where

$$a = \frac{L}{mc}, \quad b = \frac{Lc}{E}. \quad -(16)$$

so

$$\boxed{n(r) = \left(\frac{E}{mc^2}\right)^2 \left(\frac{r^2}{\left(\frac{L}{mc}\right)^2 + r^2}\right)} \quad -(17)$$

where $E$ = total energy = constant

$L$ = angular momentum = constant

3)

The total angular momentum is:

$$L = mr^2\omega \quad -(18)$$

where

$$\omega = \frac{d\theta}{dt} \quad -(19)$$

is the angular velocity. Therefore:

$$n(r) = \left(\frac{E}{mc^2}\right)^2 \left(\frac{r^2}{\left(r^2\frac{\omega}{c}\right)^2 + r^2}\right)$$

$$n(r) = \left(\frac{E}{mc^2}\right)^2 \left(\frac{1}{1+\left(\frac{\omega}{c}r\right)^2}\right)$$

$$\boxed{n(r) \sim \left(\frac{E}{mc^2}\right)^2 \left(1 - \frac{1}{2}\left(\frac{\omega r}{c}\right)^2\right)} \quad -(20)$$

because:

$$\frac{\omega}{c} << 1 \quad -(21)$$

The angular velocity can be estimated from the fact that the year is about 365.25 days.

$$\omega \sim \frac{2\pi}{365.25 \times 24 \times 3600} \text{ radians per second}$$

$$\Rightarrow \omega \sim \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{365.25 \times 24 \times 3600}$$

$$= 2 \times 10^{-7} \text{ radians per second}$$