Note 192(3) : m(r) Function for a Precessing Ellipse

In this case:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(1)$$

so

$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha} r^2 \sin(x\theta) \quad -(2)$$

$$= \frac{x\epsilon}{\alpha} r^2 \left(1-\cos^2(x\theta)\right)^{1/2} \quad -(3)$$

$$= r^2\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2}+\frac{1}{r^2}\right)\right)^{1/2}$$

So:

$$\boxed{m(r) = \frac{\frac{1}{b^2} - \left(\frac{x\epsilon}{\alpha}\right)^2\left(1-\cos^2(x\theta)\right)}{\frac{1}{a^2}+\frac{1}{r^2}}} \quad -(4)$$

where:

$$\cos^2(x\theta) = \frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2 \quad -(5)$$

so

$$\boxed{m(r) = \frac{\frac{1}{b^2} - \left(\frac{x\epsilon}{\alpha}\right)^2 + \left(\frac{x}{\alpha}\right)^2\left(1-\frac{\alpha}{r}\right)^2}{\frac{1}{a^2}+\frac{1}{r^2}}}$$

$$-(6)$$

2) The Correct Method

The orbit in the plane $dZ = 0$ in spherical spacetime is:

$$\frac{dr}{d\theta} = r^2\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{1/2} \quad -(9)$$

where

$$m(r) = 2 - \exp\left(2\exp\left(-r/R\right)\right) \quad -(10)$$

The angular velocity of the orbit is:

$$\omega = \frac{d\theta}{dt} = cb\,\frac{m(r)}{r^2} \quad -(11)$$

and is the easiest to measure experimentally.

The inverse square laws of Hooke, communicated to Newton, must be replaced by these concepts. A new limiting method is needed to reduce eqns. (9) and (10) to observed orbits. It is now known that in the limit:

$$r \to \infty \quad -(12)$$

neither the ellipse nor the precessing ellipse is observed. In the limit:

$$m(r) \to 1 \quad -(13)$$

the Minkowski spacetime is observed:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(14)$$

and Newtonian dynamics is a limit of eq. (14) when

$$v \ll c \quad -(15)$$

3) but only for a free particle. This is because eq. (14) is

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(16)$$

in which the total energy is:

$$E = \gamma mc^2 \quad -(17)$$

and the relativistic kinetic energy is:

$$T = (\gamma - 1) mc^2 \quad -(18)$$

$$\rightarrow \frac{1}{2} mv^2$$

when:

$$v << c.$$

The Newtonian limit of eq. (11) is:

$$\omega = \frac{L}{mr^2} \quad -(19)$$

where L is the angular momentum, a constant of motion. Therefore from eqs. (11) and (19):

$$m(r) \rightarrow \frac{L}{mcb} = \frac{E}{mc^2} \quad -(20)$$

So:

$$E \rightarrow mc^2 \quad -(21)$$

if:

$$m(r) \rightarrow 1 \quad -(22)$$

The limiting orbit is therefore obtained from eqs. (9) and (10) when:

$$\boxed{R \rightarrow \infty} \quad -(23)$$