192(2) : New Analytical Equation of the Ellipse

In cylindrical polar coordinates the equation of the ellipse is:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(1)$$

where $2\alpha$ is the latus rectum and $\epsilon$ the eccentricity. The coordinate system is:

$$\left.\begin{array}{llll} X = r\cos\theta & , & r = (X^2 + Y^2)^{1/2} & , \\ Y = r\sin\theta & ; & \theta = \tan^{-1}\frac{Y}{X} & \end{array}\right\} -(2)$$

Therefore:

$$(X^2+Y^2)^{1/2} = \frac{\alpha}{1 + \frac{\epsilon X}{(X^2+Y^2)^{1/2}}} \quad -(3)$$

i.e.

$$(X^2+Y^2)^{1/2} = \frac{(X^2+Y^2)^{1/2}\alpha}{(X^2+Y^2)^{1/2} + \epsilon X} \quad -(4)$$

or

$$(X^2+Y^2)^{1/2} + \epsilon X = \alpha \quad -(5)$$

i.e

$$r = (X^2+Y^2)^{1/2} = \alpha - \epsilon X \quad -(6)$$

$$\boxed{r = \alpha - \epsilon X} \quad -(7)$$

or

$$\boxed{\frac{dr}{dX} = -\epsilon} \quad -(8)$$

These appear to be new equations of the ellipse.

2) A simple and useful orbital equation can be obtained using:

$$\frac{dr}{d\theta} = \frac{\alpha \epsilon \sin\theta}{(1+\epsilon\cos\theta)^2} \quad -(9)$$

$$= \frac{\alpha \epsilon Y / r}{\left(1+\epsilon \frac{X}{r}\right)^2}$$

$$= \frac{\alpha \epsilon Y}{(r+\epsilon X)^2}$$

However, from eq. (7):

$$\alpha = r + \epsilon X \quad -(10)$$

so

$$\boxed{\frac{dr}{d\theta} = \frac{\epsilon}{\alpha} Y} \quad -(11)$$

The orbital equation (11) is very simple and

from eq. (2):

$$\boxed{\frac{dr}{d\theta} = \frac{\epsilon}{\alpha} r \sin\theta} \quad -(12)$$

The following equation is also useful:

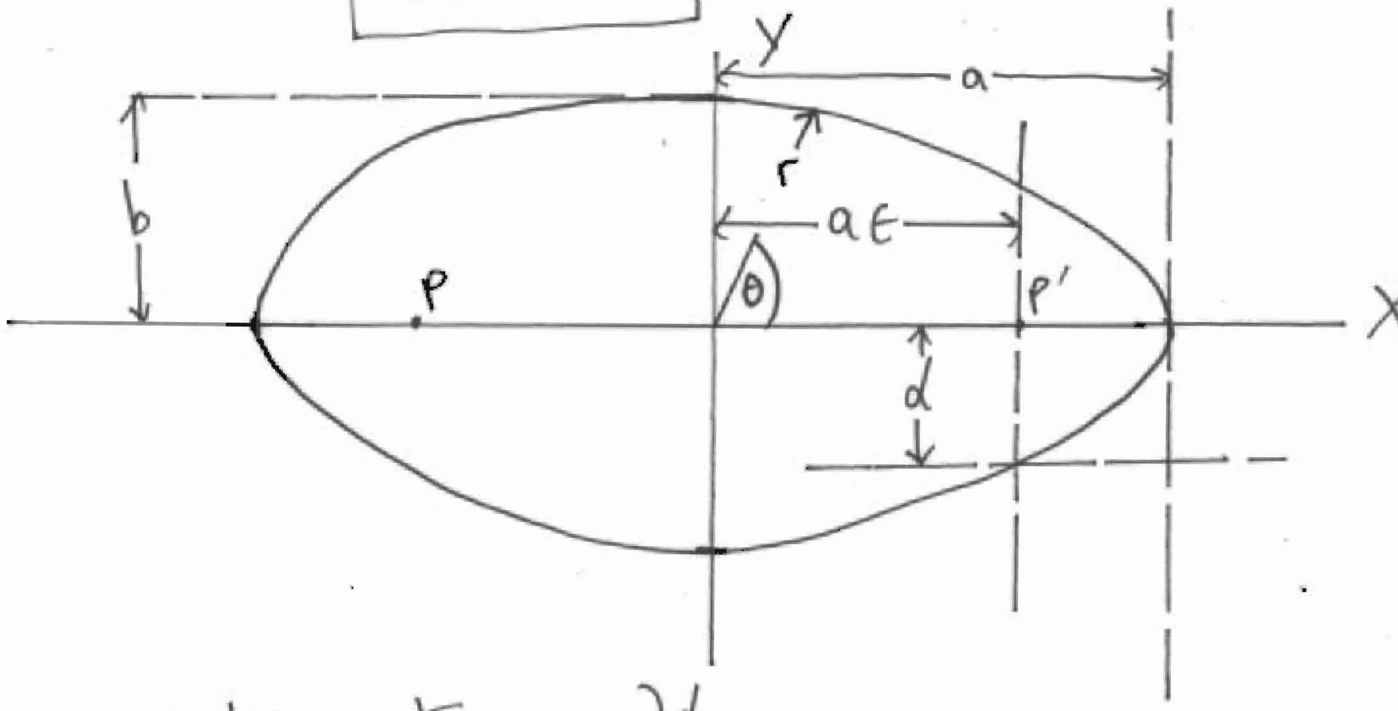
$$\frac{dX}{d\theta} = \frac{dX}{dr}\frac{dr}{d\theta} \quad -(13)$$

$$= -\frac{1}{\epsilon}\frac{\epsilon}{\alpha} Y$$

$$\boxed{\frac{dX}{d\theta} = -\frac{Y}{\alpha}} \quad -(14)$$

3) The simplest equation of the ellipse is:

$$\boxed{\frac{dr}{dX} = -\epsilon} \quad -(15)$$




Latus rectum $= 2d$
Eccentricity $= \epsilon$
Semimajor axis $= a$
Semiminor axis $= b$

By definition:

$$a = \frac{d}{1-\epsilon^2} \quad -(15)$$

$$b = \frac{d}{(1-\epsilon^2)^{1/2}} \quad -(16)$$

so

$$\epsilon = \left(1 - \frac{a^2}{b^2}\right)^{1/2} \quad -(17)$$

and

$$\frac{dX}{dr} = -\sin\theta = -\epsilon \quad -(18)$$

$$\boxed{\sin\theta = \epsilon = \left(1 - \frac{a^2}{b^2}\right)^{1/2}} \quad -(19)$$

4) According to general relativity in a spherical spacetime the equation of the ellipse must be obtained as a limit of:

$$\frac{dr}{d\theta} = r^2\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{1/2} \quad -(20)$$

From:

$$\frac{dr}{dX} = \frac{d\theta}{dX}\frac{dr}{d\theta} = -\frac{\alpha}{Y}\frac{dr}{d\theta} \quad -(21)$$

$$\frac{dr}{dX} = -\frac{\alpha r^2}{Y}\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{1/2} \quad -(22)$$

From eq. (10):

$$X = \frac{1}{\epsilon}(\alpha - r) \quad -(23)$$

Also:

$$X^2 + Y^2 = r^2 \quad -(24)$$

so:

$$Y = \left(r^2 - \frac{1}{\epsilon^2}(\alpha - r)^2\right)^{1/2} \quad -(25)$$

Therefore:

$$\alpha r^2\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{1/2} = Y\epsilon \quad -(26)$$

$$\alpha^2 r^2\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right) = Y^2\epsilon^2 \quad -(27)$$

$$\alpha^2 r^2\left(\frac{1}{a^2} + \frac{1}{r^2}\right)m(r) = \left(\frac{\alpha r}{b}\right)^2 - Y^2\epsilon^2 \quad -(28)$$

and

$$m(r) = \frac{\left(\frac{dr}{b}\right)^2 - Y^2\epsilon^2}{d^2 r^2\left(\frac{1}{a^2} + \frac{1}{r^2}\right)} \quad -(29)$$

$$Y^2 = r^2 - \frac{1}{\epsilon^2}(d-r)^2 \quad -(30)$$

It is seen that:

$$m(r) \to 1 \quad -(31)$$

only in the limit of the circle:

$$m(r) \to \left(\frac{a}{b}\right)^2 = 1. \quad -(32)$$

For the circle:

$$d = r \quad -(33)$$

$$\epsilon = 0 \quad -(34)$$

$$a = b \quad -(35)$$

The Newtonian limit should be an ellipse in the limit (31), but the limit is always the circle.