186(3): Relation between Metric and Connection.

In this note the first calculations are made on the reduction of the antisymmetric connection from the symmetric metric. In this first calculation it is assumed that the metric is diagonal and defined by spherically symmetric spacetime.

The conventional calculation is based on the metric compatibility condition:

$$\partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0 \quad -(1)$$

$$\partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \quad -(2)$$

$$\partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0 \quad -(3)$$

from which:

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda}$$
$$+ \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} + \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} + \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} + \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0$$
$$-(4)$$

Levi Civita and Ricci assumed a symmetric connection, in which case eq (4) reduces to:

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0$$
$$-(5)$$

2)

Multiply by $g^{\sigma\rho}$ and use:

$$2 g^{\sigma\rho} \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} = 2\Gamma^{\sigma}_{\mu\nu} \quad -(6)$$

This result follows by using:

$$\lambda \to \sigma \quad -(7)$$

so
$$g^{\sigma\rho} \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} = g^{\sigma\rho} \Gamma^{\sigma}_{\mu\nu} g_{\sigma\rho} \quad -(8)$$
$$= g^{\sigma\rho} g_{\sigma\rho} \Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu}$$

so
$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \quad -(9)$$

This is the usual result that appeared in the twentieth century textbooks.

However, it is now known that the connection is antisymmetric, so in eq. (4):

$$\Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} = \Gamma^{\lambda}_{\rho\mu} g_{\nu\lambda} = - \Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda} \quad -(10)$$

and
$$\Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} + \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} = 0 \quad -(11)$$

Therefore:

3)

$$2\left(\Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} + \Gamma^{\lambda}_{\rho\nu} g_{\lambda\mu}\right) = \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} \quad -(12)$$

For a diagonal metric:

$$\mu = \nu \quad -(13)$$

so $$4\Gamma^{\lambda}_{\rho\mu} g_{\lambda\mu} = \partial_\rho g_{\mu\mu} - \partial_\mu g_{\mu\rho} - \partial_\mu g_{\rho\mu}. \quad -(14)$$

The connection is antisymmetric, so:

$$\rho \neq \mu \quad -(15)$$

and $$4\Gamma^{\lambda}_{\rho\mu} g_{\lambda\mu} = \partial_\rho g_{\mu\mu}. \quad -(16)$$

Multiply by $g^{\sigma\mu}$:

$$4 g^{\sigma\mu} \Gamma^{\sigma}_{\rho\mu} g_{\sigma\mu} = g^{\sigma\mu} \partial_\rho g_{\mu\mu} \quad -(17)$$

$$\Gamma^{\sigma}_{\rho\mu} = \frac{1}{4} g^{\sigma\alpha} \partial_\rho g_{\mu\alpha} \quad -(18)$$

in which summation is over the α indices,

with

$$\alpha = \mu \quad -(19)$$

4)

From antisymmetry:

$$\Gamma^{\sigma}_{\rho\mu} = -\Gamma^{\sigma}_{\mu\rho}, \quad -(20)$$

and

$$g^{\sigma\alpha}\partial_{\rho} g_{\mu\alpha} = -g^{\sigma\alpha}\partial_{\mu} g_{\rho\alpha} \quad -(21)$$

So:

$$\boxed{\Gamma^{\sigma}_{\rho\mu} = \frac{1}{4} g^{\sigma\alpha}\partial_{\rho} g_{\mu\alpha}} \quad -(22)$$

in which

$$\left.\begin{aligned} g^{\sigma\alpha} &= 0 \quad \text{if } \sigma \neq \alpha \\ g_{\mu\alpha} &= 0 \quad \text{if } \mu \neq \alpha \end{aligned}\right\} \quad -(23).$$
