Note 186(2): Relation between Metric Method and Minimal Prescription.

As in note 186(1) the metric method of developing gravitational theory is based on constants of motion. In the absence of gravitation the energy and linear part of the momentum are:

$$E = m_0 c^2 \frac{dt}{d\tau} , \quad -(1)$$

and

$$p_l = m_0 \frac{dr}{d\tau} . \quad -(2)$$

In the presence of gravitation:

$$E' = n(r) m_0 c^2 \frac{dt}{d\tau} \quad -(3)$$

$$p_l' = m(r) m_0 \frac{dr}{dt} \quad -(4)$$

In the gravitational metric:

$$n(r) = 1 - \frac{r_0}{r} \quad -(5)$$

$$m(r) = \left(1 - \frac{r_0}{r}\right)^{-1} \quad -(6)$$

where

$$r_0 = \frac{2MG}{r} \quad -(7)$$

In the minimal prescription the four momentum is changed as follows:

$$p^{\mu\prime} = p^\mu + m_0 \Phi^\mu \quad -(8)$$

2) Define:

$$\Phi^\mu := \left(\frac{\Phi^0}{c}, \underline{\Phi}\right) \quad -(9)$$

Then and

$$p'^\mu = \left(\frac{E'}{c}, \underline{p}'\right) \quad -(10)$$

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad -(11)$$

Then:

$$E' = E + m_0 \Phi^0 \quad -(12)$$

and

$$\Phi^0 = \frac{1}{m_0}\left(E' - E\right) \quad -(13)$$

From eqs. (3) and (4):

$$\boxed{\Phi^0 = c^2\left(n(r) - 1\right)\frac{dt}{d\tau}} \quad -(14)$$

If for example metric (5) is used:

$$\Phi^0 = -c^2 \frac{r_0}{r}\frac{dt}{d\tau} \quad -(15)$$

$$= -\frac{2MG}{r}\frac{dt}{d\tau}$$

In the classical limit:

$$\frac{dt}{d\tau} \to 1 \quad -(16)$$

$$\Phi^0 \to -\frac{2MG}{r} = 2\phi \quad -(17)$$

so where

$$\phi = -\frac{MG}{r} \quad -(18)$$

3) is the classical gravitational potential.

So: $\Phi^{0} = 2\phi$ — (19)

in the classical limit.

We obtain the useful result that a minimal prescription can be used in gravitational theory.

The total momentum p in the absence of gravitation is

$$p^2 = p_l^2 + \frac{L^2}{r^2} \quad \text{— (20)}$$

and in the presence of gravitation it is:

$$p'^2 = p_l'^2 + \frac{L^2}{r^2} \quad \text{— (21)}$$

From eq. (8)

$$p_l' = p_l + m_0 \Phi \quad \text{— (22)}$$

In the absence of angular momentum L:

$$\Phi = \frac{1}{m_0}\left(p_l' - p_l\right) \quad \text{— (23)}$$

i.e.

$$\boxed{\Phi = (n(r) - 1)\frac{dr}{dt}} \quad \text{— (24)}$$

In the classical limit:

4)

$$\frac{d\underline{r}}{d\tau} \rightarrow \frac{d\underline{r}}{dt} = \underline{v} \quad -(25)$$

and

$$n(r) = \left(1 - \frac{r_0}{r}\right)^{-1} \sim 1 + \frac{r_0}{r} \quad -(26)$$

so

$$\underline{\Phi} \rightarrow \left(\frac{2mG}{c^2 r}\right)\underline{v} \quad -(27)$$

and

$$\boxed{\underline{p}' = \underline{p} + \left(\frac{2mG}{c^2 r}\right) m_0 \underline{v}} \quad -(28)$$

For finite L:

$$m_0 \underline{\Phi} = \underline{p}' - \underline{p} \quad -(29)$$

$$m_0^2 \Phi^2 = (\underline{p}' - \underline{p}) \cdot (\underline{p}' - \underline{p}) \quad -(30)$$

$$= p'^2 + p^2 - 2pp' \cos\theta$$

i.e:

$$\Phi^2 = \frac{1}{m_0^2}\left[p_e'^2 + p_e^2 + \frac{2L^2}{r^2} - 2\left(p_e^2 + \frac{L^2}{r^2}\right)^{1/2}\left(p_e'^2 + \frac{L^2}{r^2}\right)^{1/2} \cos\theta\right] \quad -(31)$$

Using Φ^μ, the metrical and field equation method may be related.