1) 171(3): Transverse Components of the ECE Fermion Equation

The transverse components of eqs. (20) to (23) of note 171(2) are:

$$E\phi^1 - c(p_X + ip_Y)\phi^2 = mc^2\phi^3 \quad -(1)$$

$$E\phi^2 - c(p_X - ip_Y)\phi^1 = mc^2\phi^4 \quad -(2)$$

$$E\phi^3 + c(p_X + ip_Y)\phi^4 = mc^2\phi^1 \quad -(3)$$

$$E\phi^4 + c(p_X - ip_Y)\phi^3 = mc^2\phi^2 \quad -(4)$$

If
$$p_Y = 0 \quad -(5)$$
then
$$E\phi^1 - cp_X\phi^2 = mc^2\phi^3 \quad -(6)$$

$$E\phi^2 - cp_X\phi^1 = mc^2\phi^4 \quad -(7)$$

$$E\phi^3 + cp_X\phi^4 = mc^2\phi^1 \quad -(8)$$

$$E\phi^4 + cp_X\phi^3 = mc^2\phi^2 \quad -(9)$$

i.e.
$$E\begin{bmatrix}\phi^1 & \phi^2\\ \phi^3 & \phi^4\end{bmatrix} - cp_X\begin{bmatrix}\phi^2 & \phi^1\\ -\phi^4 & -\phi^3\end{bmatrix} = mc^2\begin{bmatrix}\phi^3 & \phi^4\\ \phi^1 & \phi^2\end{bmatrix} \quad -(10)$$

Now use:
$$\begin{bmatrix}\phi^2 & \phi^1\\ -\phi^4 & -\phi^3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\phi^1 & \phi^2\\ \phi^3 & \phi^4\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad -(11)$$

$$= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\phi^2 & \phi^1\\ \phi^4 & \phi^3\end{bmatrix}$$

where
$$\sigma^1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \sigma^3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad -(12)$$

and
$$\begin{bmatrix}\phi^3 & \phi^4\\ \phi^1 & \phi^2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\phi^1 & \phi^2\\ \phi^3 & \phi^4\end{bmatrix} \quad -(13)$$

2) to find that:

$$\boxed{E\sigma^0\psi - cp_x\sigma^3\psi\sigma^1 = \sigma^1 mc^2\psi} \quad -(14)$$

Similarly, if $p_x = 0$ $\quad -(15)$

then

$$E\psi^1 - icp_y\psi^2 = mc^2\psi^3 \quad -(16)$$
$$E\psi^2 + icp_y\psi^1 = mc^2\psi^4 \quad -(17)$$
$$E\psi^3 + icp_y\psi^4 = mc^2\psi^1 \quad -(18)$$
$$E\psi^4 - icp_y\psi^3 = mc^2\psi^2 \quad -(19)$$

i.e.

$$E\begin{bmatrix}\psi^1 & \psi^2 \\ \psi^3 & \psi^4\end{bmatrix} + icp_y\begin{bmatrix}-\psi^2 & \psi^1 \\ \psi^4 & -\psi^3\end{bmatrix} = mc^2\begin{bmatrix}\psi^3 & \psi^4 \\ \psi^1 & \psi^2\end{bmatrix} \quad -(20)$$

This equation is:

$$\boxed{E\sigma^0\psi + c\sigma^3\psi p_y\sigma^2 = \sigma^1 mc^2\psi} \quad -(21)$$

using:

$$\sigma^3\psi\sigma^2 = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix}\psi^1 & \psi^2 \\ \psi^3 & \psi^4\end{bmatrix}\begin{bmatrix}0 & i \\ -i & 0\end{bmatrix}$$
$$= i\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix}-\psi^2 & \psi^1 \\ -\psi^4 & \psi^3\end{bmatrix} \quad -(22)$$
$$= i\begin{bmatrix}-\psi^2 & \psi^1 \\ \psi^4 & -\psi^3\end{bmatrix}$$

3) So the overall result is:

$$\boxed{\sigma^0 E\phi - c\sigma^3\phi\left(p_x\sigma^1 - p_y\sigma^2 + p_z\sigma^3\right) = \sigma^1 mc^2\phi} \quad -(23)$$

i.e.

$$\sigma^0 E\phi - c p_z \sigma^3\phi\sigma^3 = \sigma^1 mc^2\phi \quad \text{if } p_x = p_y = 0$$
$$\sigma^0 E\phi - c p_x \sigma^3\phi\sigma^1 = \sigma^1 mc^2\phi \quad \text{if } p_y = p_z = 0$$
$$\sigma^0 E\phi + c p_y \sigma^3\phi\sigma^2 = \sigma^1 mc^2\phi \quad \text{if } p_x = p_z = 0$$
$$-(24)$$

Eq. (23) is:

$$\boxed{\sigma^0 E\phi - c\sigma^3\phi\, \underline{p}\cdot\underline{\sigma} = \sigma^1 mc^2\phi} \quad -(25)$$

if the following <u>definition</u> is used:

$$\underline{p}\cdot\underline{\sigma} \equiv \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix} \quad -(26)$$

$$= p_x\sigma^1 - p_y\sigma^2 + p_z\sigma^3$$

<u>Note that</u> this is <u>reflected</u> about the <u>XZ plane</u> <u>from the usual</u> definition:

$$\underline{p}\cdot\underline{\sigma} = p_x\sigma^1 + p_y\sigma^2 + p_z\sigma^3 \quad -(27)$$

4) Cross Check

$$\sigma^3 \phi \, \underline{p} \cdot \underline{\sigma} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{bmatrix} \begin{bmatrix} p_Z & p_X - i p_Y \\ p_X + i p_Y & -p_Z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_Z \phi^1 + (p_X + i p_Y)\phi^2 & (p_X - i p_Y)\phi^1 - p_Z \phi^2 \\ p_Z \phi^3 + (p_X + i p_Y)\phi^4 & (p_X - i p_Y)\phi^3 - p_Z \phi^4 \end{bmatrix}$$

$$= \begin{bmatrix} p_Z \phi^1 + (p_X + i p_Y)\phi^2 & (p_X - i p_Y)\phi^1 - p_Z \phi^2 \\ -(p_Z \phi^3 + (p_X + i p_Y)\phi^4) & -((p_X - i p_Y)\phi^3 - p_Z \phi^4) \end{bmatrix}$$

So eq. (23) means:

$$E\phi^1 - c\left(p_Z \phi^1 + (p_X + i p_Y)\phi^2\right) = mc^2 \phi^3$$

$$E\phi^2 - c\left((p_X - i p_Y)\phi^1 - p_Z \phi^2\right) = mc^2 \phi^4$$

$$E\phi^3 + c\left(p_Z \phi^3 + (p_X + i p_Y)\phi^4\right) = mc^2 \phi^1$$

$$E\phi^4 + c\left((p_X - i p_Y)\phi^3 - p_Z \phi^4\right) = mc^2 \phi^2$$

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which are eqs. (20) – (23) of note 171(2), Q.E.D.

In order to transform eq. (25) to the position representation it is written as:

$$\sigma^0 \phi \overleftarrow{E} - c\sigma^3 \phi \, \overleftarrow{\underline{p}} \cdot \underline{\sigma} = \sigma^1 mc^2 \phi \qquad -(28)$$

where $\overleftarrow{E}$ and $\overleftarrow{\underline{p}}$ operate to the left on

5) ϕ and $\sigma^3 \phi$. Here:

$$p^\mu = i\hbar \partial^\mu \quad -(29)$$

so

$$E = i\hbar \frac{\partial}{\partial t}, \quad \underline{p} = -i\hbar \underline{\nabla} \quad -(30)$$

The geration to the left is defined in L. H. Ryder, "Quantum Field Theory" (CUP, 1996).

Eq. (28) is the same as the usual structure:

$$\left(i \gamma^\mu \partial_\mu - mc / \hbar \right) \phi = 0 \quad -(31)$$

where γ^μ are the Dirac matrices and ϕ the Dirac spinor:

$$\phi = \begin{bmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \end{bmatrix} \quad -(32)$$

So the ECE electron and positron eqns. are:

Electron

$$\sigma^0 \phi E - c \sigma^3 \phi \underline{p} \cdot \underline{\sigma} = \sigma^1 mc^2 \phi$$

Positron

$$\sigma^0 \phi E + c \sigma^3 \phi \underline{p} \cdot \underline{\sigma} = \sigma^1 mc^2 \phi$$

$$-(33)$$

which annihilate to give:

$$2 \sigma^0 \phi E = 2 \sigma^1 mc^2 \phi \quad -(34)$$
