111(2): Electron Positron Annihilation.

The simplistic view of electron positron annihilation is that an electron and its anti-particle the positron collide to give two photons moving in opposite directions.

Before Collision

$$e^{-} \longrightarrow \longleftarrow e^{+}$$

After Collision

$$\hbar\omega \longleftarrow \longrightarrow \hbar\omega$$

However the textbook description asserts that only the rest mass mc^2 is transformed into $\hbar\omega$. The rest mass energy, mc^2, of the electron and positron is 0.511 MeV, so the two photons (γ rays) have a combined energy of 1.022 MeV. If the electron and positron have very high energies it is known experimentally that there are other products of the collision.

However, if attention is restricted to the γ rays in order to begin the analysis, it is seen immediately that a purely classical description fails. This is because the total energy of the electron and positron is

$$E(\text{initial}) = 2\gamma mc^2 \quad -(1)$$

assuming that the velocities of the electron and positron are equal and opposite. So:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(2)$$

The de Broglie Einstein relation is:

$$E = \hbar\omega = \gamma mc^2 \quad -(3)$$

so the energy after collision cannot be:

$$E(\text{final}) = ?\ 2mc^2, \quad -(4)$$

because there is violation of conservation of energy.

Now we can add to this the major problems encountered in UFT 158 onwards to UFT 166. The classical momentum conservation is:

$$\underline{p}_1 + \underline{p}_2 = \hbar \underline{\kappa}_1 + \hbar \underline{\kappa}_2 \quad -(5)$$

and the Einstein de Broglie equation is:

$$\underline{p} = \hbar \underline{\kappa} = \gamma m \underline{v} \quad -(6)$$

This classical description is gone through only for the sake of illustration, because it is well known that the Dirac equation is needed for the description of an anti-particle. However it remains true that the classical or Compton like description of particle interaction has failed totally after UFT 158 to UFT 166.

Before launching into the Dirac equation, we stay at the classical level for the time being and factorize the Einstein energy equation:

$$E^2 = p^2c^2 + m^2c^4 \quad -(7)$$

using the Pauli matrices. Eq. (7) can be factorized into

$$(E - ic\underline{\sigma}\cdot\underline{p})(E + ic\underline{\sigma}\cdot\underline{p}) = m^2c^4 \quad -(8)$$

and possible solutions of this equation are:

$$E - ic\underline{\sigma}\cdot\underline{p} = mc^2 \quad -(9)$$

$$E + ic\underline{\sigma}\cdot\underline{p} = mc^2 \quad -(10)$$

in which

$$h = \underline{\sigma}\cdot\underline{p} \quad -(11)$$

is the helicity. The annihilation process is therefore described by adding eqns. (9) and (10) to give:

$$2E = 2mc^2 = 2\hbar\omega \quad -(12)$$

Eqs. (9) and (10) are the basis of the Dirac equation:

$$(E - c\underline{\sigma}\cdot\underline{p})\begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix} = mc^2\begin{bmatrix}\phi^3\\ \phi^4\end{bmatrix} \quad -(13)$$

$$(E + c\underline{\sigma}\cdot\underline{p})\begin{bmatrix}\phi^3\\ \phi^4\end{bmatrix} = mc^2\begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix}. \quad -(14)$$

Here:

$$\underline{\sigma}\cdot\underline{p} = \begin{bmatrix} p_Z & p_X + ip_Y \\ p_X - ip_Y & -p_Z \end{bmatrix}. \quad -(15)$$

The Dirac equation is the basis of anti particles as is well known, but initially, Dirac thought that the anti-particle of the electron was the proton. Annihilation was first predicted by Oppenheimer, not Dirac. The anti-particle of the electron was the positron, not the proton. The positron was first demonstrated experimentally by Anderson. The anti-particle has opposite charge and opposite helicity to

) the particle. The helicity is:

$$h = \underline{\sigma} \cdot \underline{p} \quad -(16)$$

The four Pauli matrices are:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad -(17)$$

It is instructive to write out the structure of the Dirac equation in full detail, because it is the basis of all annihilation theory. Usually it is expressed in terms of 4×4 matrices, but in UFT 130 it was expressed in terms of a 2×2 matrix structure, which was related to Cartan geometry.

The Dirac equation consists of both eqs (13) and (14) simultaneously. Eq. (13) is:

$$\left(E\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - c\begin{bmatrix} p_Z & p_X + ip_Y \\ p_X - ip_Y & -p_Z \end{bmatrix}\right)\begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} = mc^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \phi^3 \\ \phi^4 \end{bmatrix} \quad -(18)$$

Eq. (14) is:

$$\left(E\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} p_Z & p_X + ip_Y \\ p_X - ip_Y & -p_Z \end{bmatrix}\right)\begin{bmatrix} \phi^3 \\ \phi^4 \end{bmatrix} = mc^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \quad -(19)$$

Eqs. (18) and (19) are the following four equations:

$$E\phi^1 - c\left(p_Z\phi^1 + (p_X + ip_Y)\phi^2\right) = mc^2\phi^3 \quad -(20)$$

$$E\phi^2 - c\left((p_X - ip_Y)\phi^1 - p_Z\phi^2\right) = mc^2\phi^4 \quad -(21)$$

$$E\phi^3 + c\left(p_Z\phi^3 + (p_X + ip_Y)\phi^4\right) = mc^2\phi^1 \quad -(22)$$

$$E\phi^4 + c\left((p_X - ip_Y)\phi^3 - p_Z\phi^4\right) = mc^2\phi^2 \quad -(23)$$

Z Component of Momentum

If attention is restricted to the Z component eqns. (20) to (23) simplify to:

$$E\phi^1 - cp_Z\phi^1 = mc^2\phi^3 \quad -(24)$$

$$E\phi^2 + cp_Z\phi^2 = mc^2\phi^4 \quad -(25)$$

$$E\phi^3 + cp_Z\phi^3 = mc^2\phi^1 \quad -(26)$$

$$E\phi^4 - cp_Z\phi^4 = mc^2\phi^2 \quad -(27)$$

Eqs. (24) to (27) can be written as a single equation in 2 x 2 matrices:

$$E\begin{bmatrix}\phi^1 & \phi^2\\ \phi^3 & \phi^4\end{bmatrix} + c\begin{bmatrix}p_Z & 0\\ 0 & -p_Z\end{bmatrix}\begin{bmatrix}-\phi^1 & \phi^2\\ -\phi^3 & \phi^4\end{bmatrix} = mc^2\begin{bmatrix}\phi^3 & \phi^4\\ \phi^1 & \phi^2\end{bmatrix}.$$

$$-(28)$$

Now use:

$$\begin{bmatrix}\phi^3 & \phi^4\\ \phi^1 & \phi^2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\phi^1 & \phi^2\\ \phi^3 & \phi^4\end{bmatrix} \quad -(29)$$

$$\begin{bmatrix}-\phi^1 & \phi^2\\ -\phi^3 & \phi^4\end{bmatrix} = \begin{bmatrix}\phi^1 & \phi^2\\ \phi^3 & \phi^4\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \quad -(30)$$

Define:

$$\phi = \begin{bmatrix}\phi^1 & \phi^2\\ \phi^3 & \phi^4\end{bmatrix} \quad -(31)$$

Eq. (28) is therefore an equation in ϕ:

$$\boxed{\sigma^0 E\phi - cp_Z\sigma^3\phi\sigma^3 = \sigma^1 mc^2\phi}$$

$$-(32)$$

Eq. (32) is a major advance in mathematics because it shows that the Dirac equation can be written in terms of 2 x 2 matrices.

The eigenfunction ϕ is a tetrad generated by

$$\begin{bmatrix} V^a \\ V^b \end{bmatrix} = \begin{bmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad -(32)$$

in SU(2) representation space.

The antiparticle equation is generated by:

$$\hat{P}\left(-c p_Z \sigma^3\right) = c p_Z \sigma^3 \quad -(33)$$

and is

$$\boxed{E \sigma^0 \phi + c p_Z \sigma^3 \phi \sigma^3 = \sigma^1 m c^2 \phi} \quad -(34)$$

Eqs. (32) and (34) are the quantum versions of eqs. (9) and (10). Annihilation is described by adding eqs. (32) and (34) to give:

$$2 E \sigma^0 \phi = 2 \sigma^1 m c^2 \phi \quad -(35)$$

which is the quantum version of:

$$2E = 2mc^2 \quad -(36)$$