1) 165(8): Development of Group Velocity

Using the covariant mass equation:

$$R = (mc / \hbar)^2 \quad -(1)$$

the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(2)$$

is generalized to:

$$E^2 = c^2 p^2 + \hbar^2 c^2 R. \quad -(3)$$

Using:

$$E = \hbar\omega, \quad p = \hbar\kappa \quad -(4)$$

eq. (3) is:

$$\boxed{R = \frac{\omega^2}{c^2} - \kappa^2}. \quad -(5)$$

The group velocity is:

$$v_g = \frac{\partial E}{\partial p} = \frac{\partial \omega}{\partial \kappa}. \quad -(6)$$

From eq. (5)

$$\omega = c\left(R + \kappa^2\right)^{1/2} \quad -(7)$$

so

$$\frac{\partial \omega}{\partial \kappa} = \frac{\kappa c}{(R+\kappa^2)^{1/2}} + \frac{c}{2}\frac{\partial R}{\partial \kappa} \cdot \frac{1}{(R+\kappa^2)^{1/2}} \quad -(8)$$

$$= \frac{c^2}{\omega}\left(\kappa + \frac{1}{2}\frac{\partial R}{\partial \kappa}\right)$$

$$\boxed{v_g = \frac{c^2}{\omega}\left(\kappa + \frac{1}{2}\frac{\partial R}{\partial \kappa}\right)} \quad -(9)$$

as derived in previous notes, QED.

By definition,

$$\gamma m c^2 = \hbar\omega \quad -(10)$$

so:

$$\therefore \frac{1}{\gamma^2} = 1 - \frac{v_g^2}{c^2} = \left(\frac{mc^2}{\hbar\omega}\right)^2 \quad - (11)$$

using eqn (1):

$$\boxed{v_g = c\left(1 - R\left(\frac{c}{\omega}\right)^2\right)^{1/2}} \quad - (12)$$

It is observed experimentally that v_g can be zero, greater than c or negative.

, Zero Group Velocity

In this case:

$$R = \left(\frac{\omega}{c}\right)^2 \quad - (13)$$

from eq. (12). Therefore from eq. (5):

$$\kappa = 0 \quad - (14)$$

and so:

$$p = 0. \quad - (15)$$

From eq. (9):

$$\frac{\partial R}{\partial \kappa} = 0. \quad - (16)$$

The phase velocity is:

$$v_p = \frac{\omega}{\kappa} \to \infty \quad - (17)$$

A standing wave with no momentum can have this characteristic.

Superluminal Case

This is given by

$$v_g > c \quad - (18)$$

) In this case, from eq. (12), R is negative. This means:

$$\boxed{R = q^{\nu}_{a} \partial^{\mu} \left(\omega^{a}_{\mu\nu} - \Gamma^{a}_{\mu\nu} \right) < 0} \quad - (19)$$

From eq. (1):

$$R^{1/2} = \pm \left(\frac{mc}{\hbar} \right)^{1/2} \quad - (20)$$

If the negative root is taken it is possible for $R^{1/2}$ to be negative, and the quantity R can be positive or negative geometrically. From eq. (5), R is negative if:

$$\boxed{\kappa^2 > \frac{\omega^2}{c^2}} \quad - (21)$$

In special relativity, the condition (21) is not allowed, because in special relativity:

$$\omega^2 = c^2 \kappa^2 + \left(\frac{m_0 c^2}{\hbar} \right)^2 \quad - (22)$$

and

$$\kappa^2 = \frac{\omega^2}{c^2} - \left(\frac{m_0 c}{\hbar} \right)^2 . \quad - (23)$$

The mass m_0 in special relativity is always greater than zero, so eq. (21) can never apply. In ECE theory and general relativity, m^2 in eq. (1) is a proportionality factor between R and $(c/\hbar)^2$. So it is possible ~~that~~:

$$R := -\left(\frac{mc}{\hbar} \right)^2 . \quad - (24)$$

From eq. (9) the superluminal condition is:

$$\frac{c}{\omega}\left(\kappa + \frac{1}{2}\frac{\partial R}{\partial \kappa}\right) > 1 \quad - (25)$$

i.e.

$$\boxed{\kappa > \frac{\omega}{c} - \frac{1}{2}\frac{\partial R}{\partial \kappa}} \quad - (26).$$

) Condition for Negative Group Velocity

From eq. (12), the negative group velocity is:

$$v_g = -c\left(1 - R\left(\frac{c}{\omega}\right)^2\right)^{1/2} \quad - (27)$$

and from eq. (9) it is required that:

$$\kappa + \frac{1}{2}\frac{\partial R}{\partial \kappa} < 0 \quad - (28)$$

i.e $\partial R / \partial \kappa$ must be negative.
