165(5): The Group and Phase Velocities of the de Broglie Einstein Equations

The group velocity as defined by de Broglie for any wave/particle is:

$$v_g = \frac{\partial \omega}{\partial \kappa} = \frac{\partial (E/\hbar)}{\partial (p/\hbar)} = \frac{\partial E}{\partial p} \quad -(1)$$

Therefore:

$$v_g = \frac{\partial E}{\partial p} = \frac{\partial}{\partial p}\left(p^2c^2 + m^2c^4\right)^{1/2} \quad -(2)$$

$$= pc^2\left(p^2c^2 + m^2c^4\right)^{-1/2}$$

$$= \frac{p}{m}\left(\left(\frac{p}{mc}\right)^2 + 1\right)^{-1/2}$$

$$= \frac{1}{\gamma}\frac{p}{m} = \frac{mv\gamma}{m\gamma} = v \quad -(3)$$

Therefore:

$$\boxed{v_g = v} \quad -(4)$$

The phase velocity is:

$$\boxed{v_p = f\lambda = \frac{\omega}{\kappa}} \quad -(5)$$

From eq. (3):

$$\boxed{v_g = \frac{\gamma m v}{\gamma m} = c^2\frac{\kappa}{\omega}} \quad -(6)$$

Therefore

$$\boxed{v_g v_p = c^2} \quad -(7)$$

Therefore the Einstein / de Broglie equations are expressed in terms of the group velocity because it is equal to the particle velocity:

$$E = \hbar\omega = \gamma m c^2 \quad -(8)$$

$$\underline{p} = \hbar\underline{\kappa} = \gamma m \underline{v} \quad -(9)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad -(10)$$

$$v = v_g. \quad -(11)$$

The generalization of these equations is based on the ECE wave equation:

$$(\Box + R)\, q^a_\mu = 0 \quad -(12)$$

whose classical equivalent is:

$$p^\mu p_\mu = \hbar^2 c^2 R \quad -(13)$$

i.e.

$$\boxed{E^2 = p^2 c^2 + \hbar^2 c^2 R} \quad -(14)$$

Therefore the group velocity is:

$$v_g = \frac{\partial E}{\partial p} = \frac{\partial}{\partial p}\left(p^2 c^2 + \hbar^2 c^2 R\right)^{1/2} \quad -(15)$$

If it is assumed for simplicity that R is not an intrinsic function of p, then:

$$v_g = \frac{\partial}{\partial p}\left(p^2c^2 + \hbar^2 c^2 R\right)^{1/2} \quad - (16)$$

$$= c\,\frac{\partial}{\partial p}\left(p^2 + \hbar^2 R\right)^{1/2}$$

$$v_g = \frac{cp}{\left(p^2 + \hbar^2 R\right)^{1/2}} \quad - (17)$$

Now use:

$$\underline{p} = \gamma m \underline{v} \quad - (18)$$

$$R = \left(\frac{mc}{\hbar}\right)^2 \quad - (19)$$

Therefore

$$\boxed{\underline{p} = \hbar\gamma \frac{R^{1/2}}{c}\,\underline{v}} \quad - (20)$$

where $\underline{v}$ is the particle velocity.

So:

$$v_g = \frac{\hbar\gamma R^{1/2} v}{\left(\hbar^2 R \gamma^2 \frac{v^2}{c^2} + \hbar^2 R\right)^{1/2}}$$

$$= \frac{\gamma v}{\left(\gamma^2 \frac{v^2}{c^2} + 1\right)^{1/2}} = v \quad - (21)$$

<u>So if R is independent of p</u>, the group velocity <u>is the particle velocity.</u>

If

$$\frac{\partial R}{\partial p} \neq 0 \quad - (22)$$

then:

$$v_g = \frac{c}{2}\,\frac{\left(2p + \hbar^2\, \partial R/\partial p\right)}{\left(p^2 + \hbar^2 R\right)^{1/2}} \quad - (23)$$

i.e.

$$v_g = v + \frac{1}{2}\hbar^2 c \frac{\partial R}{\partial p}\left(p^2 + \hbar^2 R\right)^{-1/2}$$

$$\boxed{v_g = v + \frac{1}{2}\frac{\hbar c}{\gamma R^{1/2}}\frac{\partial R}{\partial p}} \quad -(24)$$

Therefore the group velocity becomes different from the particle velocity and is no longer constant. Eq (24) is the result of general relativity.