1)

165(9): Derivation of the Helmholtz Equation for ECE Theory

As in Jackson, the Helmholtz equation is based on:

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0 \quad -(1)$$

$$\underline{\nabla} \times \underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{0} \quad -(2)$$

Eq. (2) is a special case of:

$$\underline{\nabla} \times \underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{J} \quad -(3)$$

Here $\underline{E}$ is the electric field strength, $\underline{B}$ is the magnetic flux density, $\underline{H}$ is the magnetic field strength, $\underline{D}$ the displacement and $\underline{J}$ the current. Eq. (1) is the homogeneous equation in free space, eq. (2) the equation in a dielectric of permittivity ϵ and permeability μ.

To derive this from ECE theory consider the relation based on geometry:

$$T = D \wedge q \quad -(4)$$

$$D \wedge T = R \wedge q \quad -(5)$$

$$D \wedge \tilde{T} = \tilde{R} \wedge q \quad -(6)$$

Eq. (5) is based on:

$$D_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\lambda} V^\lambda \quad -(7)$$

and

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\kappa\mu\nu} V^\kappa - T^\kappa_{\mu\nu} D_\kappa V^\rho \quad -(8)$$

where

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} \quad -(9)$$

$$R^\lambda_{\mu\nu\rho} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho} \quad -(10)$$

together with:

$$\partial_\mu q^a_\nu = \Gamma^\lambda_{\mu\nu} q^a_\lambda - \omega^a_{\mu b} q^b_\nu \quad -(11)$$

$$= \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \quad -(12)$$

Homogeneous Geometry

$$T = D \wedge q$$
$$D \wedge T = R \wedge q$$
$$D_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\lambda} V^\lambda$$
$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\kappa\mu\nu} V^\kappa - T^\kappa_{\mu\nu} D_\kappa V^\rho$$
$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu}$$

$$R^\lambda_{\mu\nu\rho} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho}$$

$$\partial_\mu q^a_\nu = \Gamma^\lambda_{\mu\nu} q^a_\lambda - \omega^a_{\mu b} q^b_\nu = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}$$

$$(\Box + R_0) q^a_\mu = 0$$
$$R_0 = q^\nu_a \partial^\mu (\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu})$$

$$D_\mu T^a_{\nu\rho} + D_\rho T^a_{\mu\nu} + D_\nu T^a_{\rho\mu} = R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu}$$

$$D_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^{a}{}_{\mu}{}^{\mu\nu}$$
$$\partial_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^{a}{}_{\mu}{}^{\mu\nu} - \omega^a_{\mu b} \tilde{T}^{b\mu\nu} = 0 \text{ (experimentally)}$$

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0 \text{ for each } a}$$

$$\boxed{\begin{aligned} \Box A^a_\mu &= 0 \\ R_0 &= 0 \end{aligned}}$$

Inhomogeneous Geometry

$$\tilde{T} = (D \wedge q)_{HD}$$
$$D \wedge \tilde{T} = \tilde{R} \wedge q$$
$$D_\mu V^\rho = \partial_\mu V^\rho + \Lambda^\rho_{\mu\lambda} V^\lambda$$
$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho{}_{\kappa\mu\nu} V^\kappa - \tilde{T}^\kappa_{\mu\nu} D_\kappa V^\rho$$
$$\tilde{T}^\kappa_{\mu\nu} = \tilde{\Gamma}^\kappa_{\mu\nu} - \tilde{\Gamma}^\kappa_{\nu\mu} = \Lambda^\kappa_{\mu\nu} - \Lambda^\kappa_{\nu\mu}$$

$$\tilde{R}^\lambda_{\mu\nu\rho} = \partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\nu \Lambda^\lambda_{\mu\rho} + \Lambda^\lambda_{\mu\sigma} \Lambda^\sigma_{\nu\rho} - \Lambda^\lambda_{\nu\sigma} \Lambda^\sigma_{\mu\rho}$$

$$\partial_\mu q^a_\nu = \Lambda^a_{\mu\nu} - \omega^a_{\mu\nu}$$

$$(\Box + R) q^a_\mu = 0$$
$$R = q^\nu_a \partial^\mu (\omega^a_{\mu\nu} - \Lambda^a_{\mu\nu})$$

$$D_\mu \tilde{T}^a_{\nu\rho} + D_\rho \tilde{T}^a_{\mu\nu} + D_\nu \tilde{T}^a_{\rho\mu} = \tilde{R}^a_{\mu\nu\rho} + \tilde{R}^a_{\rho\mu\nu} + \tilde{R}^a_{\nu\rho\mu}$$

$$D_\mu T^{a\mu\nu} = R^{a}{}_{\mu}{}^{\mu\nu}$$
$$\partial_\mu T^{a\mu\nu} = R^{a}{}_{\mu}{}^{\mu\nu} - \omega^a_{\mu b} T^{b\mu\nu} = J^{a\nu} \text{ (experimentally)}$$

$$\boxed{\begin{aligned} \partial_\mu G^{\mu\nu} &= \mu_0 A^{(0)} J^{a\nu} \\ &= \mu_0 j^{a\nu} \text{ for each } a \end{aligned}}$$

$$\boxed{\begin{aligned} \Box A^a_\mu &= \mu_0 j^a_\mu = -R A^a_\mu \\ R A^a_\mu &= -\mu_0 j^a_\mu \end{aligned}}$$

i)

As shown on the previous page the homogeneous and inhomogeneous geometries are defined by various Hodge duals in the general space. The Hodge dual of the field tensor is:

$$\tilde{G}^{\mu\nu} := \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu}{}_{\alpha\beta} G^{\alpha\beta} \quad -(13)$$

where

$$\epsilon^{\mu\nu}{}_{\alpha\beta} = g^{\mu\rho} g^{\nu\sigma} \epsilon_{\rho\sigma\alpha\beta} \quad -(14)$$

where $\epsilon_{\rho\sigma\alpha\beta}$ is the totally antisymmetric unit tensor in four dimensions and where $g^{\mu\nu}$ is the inverse metric. Here $\|g\|^{1/2}$ is the square root of the absolute value of the determinant of the metric. If:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(15)$$

then:

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu}{}_{\alpha\beta} F^{\alpha\beta} \quad -(16)$$

Here:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \quad -(17)$$

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{bmatrix} \quad -(18)$$

$$G^{\alpha\beta} = \begin{bmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{bmatrix} \quad -(19)$$

$$\tilde{G}^{\mu\nu} = \begin{bmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & D_z & -D_y \\ H_y & -D_z & 0 & D_x \\ H_z & D_y & -D_x & 0 \end{bmatrix} \quad -(20)$$

So, for each a:

$$\left.\begin{array}{ll} & \underline{\nabla} \cdot \underline{D} = \rho \\ \underline{\nabla} \cdot \underline{B} = 0 & \\ \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} & \underline{\nabla} \times \underline{H} + \frac{\partial \underline{D}}{\partial t} = \underline{J} \\ A_\mu = \left(\frac{\phi}{c}, -\underline{A} \right) & j_\mu = \left(c\rho, -\underline{J} \right) \\ & \underline{B} = \mu_0 (\underline{H} + \underline{M}) \end{array}\right] -(21)$$

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}$$

$\underline{E}$ = electric field strength (volt m^{-1}), volt = $J C^{-1}$
$\underline{D}$ = displacement ($C m^{-2}$)
$\underline{B}$ = magnetic flux density (tesla)
$\underline{H}$ = magnetic field strength ($A m^{-1}$)
$\underline{J}$ = current density ($A m^{-2}$)
ρ = charge density ($C m^{-3}$)
$\underline{P}$ = polarization, $\underline{M}$ = magnetization,
ϵ_0 = vacuum permittivity, μ_0 = vacuum permeability.
the Helmholtz equation is obtained with:

$$j_\mu = 0 \quad -(22)$$

5) and

$$\underline{D} = \epsilon \underline{E} \quad , \quad \underline{B} = \mu \underline{H} \quad -(23)$$

so:

$$\left(\nabla^2 + \epsilon\mu\omega^2\right)\underline{E} = 0 \quad -(24)$$

$$\left(\nabla^2 + \epsilon\mu\omega^2\right)\underline{B} = 0 \quad -(25)$$

assuming solutions of type $\exp(-i\omega t)$. This gives

$$\kappa = (\mu\epsilon)^{1/2}\omega \quad -(26)$$

and the phase velocity:

$$v = \frac{\omega}{\kappa} = (\mu\epsilon)^{-1/2} = \frac{c}{n} \quad -(27)$$

It is seen that the phase velocity uses:

$$R = 0 \quad -(28)$$

However, as in previous notes the group velocity is based on a finite R. The Helmholtz wave equation uses zero momentum and finite energy.

The condition (28) means:

$$\boxed{\omega^a_{\mu\nu} = \Lambda^a_{\mu\nu}} \quad -(29)$$

in ECE theory. It does not mean a Minkowski spacetime because the electromagnetic field tensor is defined by spacetime torsion.

$$\tilde{G}^a_{\mu\nu} = A^{(0)}\tilde{T}^a_{\mu\nu} \quad -(30)$$

$$F^a_{\mu\nu} = A^{(0)} T^a_{\mu\nu} \quad -(31)$$

where, for each a:

6)

$$\tilde{T}^{a}_{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\mu\nu}{}^{\alpha\beta} T^{a}_{\alpha\beta} \quad -(32)$$

where:

$$\epsilon_{\mu\nu}{}^{\alpha\beta} = g_{\mu\rho} g_{\nu\kappa} \epsilon^{\rho\kappa\alpha\beta} \quad -(33)$$

Similarly:

$$\Lambda^{\kappa}_{\mu\nu} = \tilde{\Gamma}^{\kappa}_{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\mu\nu}{}^{\alpha\beta} \Gamma^{\kappa}_{\alpha\beta} \quad -(34)$$

Note carefully that $\tilde{T}^{a}_{\mu\nu}$ is defined by $\Lambda^{\kappa}_{\mu\nu}$, but $T^{a}_{\mu\nu}$ is defined by $\Gamma^{\kappa}_{\mu\nu}$. The connections are both antisymmetric. Λ is generated from Γ by use of the metrics, so have characteristics defined by the spacetime being used. These considerations give an ab initio geometrical basis for D and H.

The metric is defined by:

$$g_{\mu\nu} = q^{a}_{\mu} q^{b}_{\nu} \eta_{ab} \quad -(35)$$

where η_{ab} is the Minkowski metric diag $(1, -1, -1, -1)$.
As shown in previous work it is possible to have:

$$g_{\mu\nu} = (1, -1, -1, -1) \quad -(36)$$

with complex valued, phase dependent tetrad components, such as those of a plane wave.