165(6) : Development of Phase and Group Velocities

The phase velocity v_p and group velocity v_g are related by

$$v_p v_g = c^2 \quad -(1)$$

Using the equation:

$$\gamma m c^2 = \hbar \omega \quad -(2)$$

it is found that:

$$\gamma = \left(1 - \frac{v_g^2}{c^2}\right)^{-1/2} = \left(1 - \frac{c^2}{v_p^2}\right)^{-1/2} \quad -(3)$$

and

$$\frac{mc^2}{\hbar\omega} = \frac{1}{\gamma} \quad -(4)$$

Here:

$$v_p = \frac{\omega}{\kappa}, \quad v_g = c^2 \frac{\kappa}{\omega} \quad -(5)$$

Now use the covariant mass equation:

$$R = \left(\frac{mc}{\hbar}\right)^2 \quad -(6)$$

to find that:

$$\left(\frac{mc^2}{\hbar\omega}\right)^2 = R\left(\frac{c}{\omega}\right)^2 \quad -(7)$$

Therefore

$$\boxed{\frac{v_g^2}{c^2} = \frac{c^2}{v_p^2} = 1 - \left(\frac{c}{\omega}\right)^2 R} \quad -(8)$$

and

$$\boxed{R = \frac{\omega^2}{c^2}\left(1 - \left(\frac{c}{v_p}\right)^2\right) = \frac{\omega^2}{c^2}\left(1 - \left(\frac{v_g}{c}\right)^2\right)} \quad -(9)$$

2) The group velocity is defined by:

$$v_g = \frac{\partial E}{\partial p} \quad - (10)$$

and so:

$$\boxed{R = \frac{\omega^2}{c^2}\left(1 - \frac{1}{c^2}\left(\frac{\partial E}{\partial p}\right)^2\right)} \quad - (11)$$

this is a general expression for R, which is also defined by:

$$(\Box + R)\, q^a_\mu = 0 \quad - (12)$$

$$R = q^\nu_a \partial^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right) \quad - (13)$$

Therefore:

$$\partial^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right) = q^a_\nu \frac{\omega^2}{c^2}\left(1 - \frac{1}{c^2}\left(\frac{\partial E}{\partial p}\right)^2\right) \quad - (14)$$

Here:

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right), \; p_\mu = \left(\frac{E}{c}, -\underline{p}\right) \quad - (15)$$

and

$$p^\mu p_\mu = \hbar^2 c^2 R \quad - (16)$$

so

$$\boxed{R = \frac{1}{\hbar^2 c^2} p^\mu p_\mu} \quad - (17)$$

From eqns (11) and (17):

$$\boxed{\frac{E^2}{c^2} - p^2 = (\hbar\omega)^2 \left(1 - \frac{1}{c^2}\left(\frac{\partial E}{\partial p}\right)^2\right)} \quad - (18)$$

3) Eq. (18) relates the derivative $\partial E / \partial p$ to E and p individually. From eq. (1):

$$\frac{\partial E}{\partial p} = \frac{c^2}{v_p} = c^2 \frac{\kappa}{\omega} \quad -(19)$$

and $$\frac{\kappa}{\omega} = \frac{1}{c^2} \frac{\partial E}{\partial p} \quad -(20)$$

Special Cases

1) For the hypothetical massless particle moving at c:

$$\frac{\kappa}{\omega} = \frac{1}{c} \quad -(21)$$

so $$\boxed{\frac{\partial E}{\partial p} = c} \quad -(22)$$

and from eq. (18):

$$E^2 = c^2 p^2, \quad -(23)$$

with $$E = \hbar\omega, \quad p = \hbar\kappa \quad -(24)$$

so $$\omega = c\kappa \quad -(25)$$

QED.

In this case eq. (14) gives:

$$\boxed{\omega^a_{\mu\nu} = \Gamma^a_{\mu\nu}} \quad -(26)$$

i.e. $$R = 0 \quad -(27)$$

and

4)

$$\Box q_\mu^a = 0 \quad -(28)$$

using the ECE hypothesis:

$$A_\mu^a = A^{(0)} q_\mu^a \quad -(29)$$

eq. (28) is the d'Alembert equation for each a:

$$\boxed{\Box A_\mu^a = 0} \quad -(30)$$

QED. For the massless particle the R is zero,

i.e.

$$R = \left(\frac{mc}{\hbar}\right)^2 = 0. \quad -(31)$$

QED, and the spin and gamma factors are the same.

2) For the monochromatic plane wave:

$$v_p = \frac{\omega}{\kappa} = \frac{c}{n} = \frac{1}{(\mu\epsilon)^{1/2}} \quad -(32)$$

from the Helmholtz equation, a special case of the ECE field equations of electrodynamics. So:

$$\boxed{R = \frac{\omega^2}{c^2}\left(1 - n^2\right)} \quad -(33)$$

where the refractive index n is given by:

$$\boxed{n = \frac{1}{c}\frac{\partial E}{\partial p} = \left(1 - \frac{c^2}{\omega^2} R\right)^{1/2}} \quad -(34)$$