1) Notes 157(1): Properties of the Photon with Mass

The de Broglie equation for the rest energy of the photon is:

$$E_0 = \hbar\omega_0 = mc^2 \quad -(1)$$

Note that m is regarded as a universal constant that does not change. Here

$$\omega_0 = \text{rest frequency of photon} \quad -(2)$$

Its equation of motion is:

$$p^\mu p_\mu = m^2c^2 \quad -(3)$$

i.e.

$$E^2 = c^2p^2 + m^2c^4 \quad -(4)$$

Here

$$E = \gamma mc^2, \; p = \gamma mv \quad -(5)$$

are the relativistic total energy and relativistic momentum, and v is the velocity of the photon. Here

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(6)$$

Therefore the Planck law is:

$$E = \hbar\omega = \gamma\hbar\omega_0 \quad -(7)$$

and

$$\hbar\omega = \gamma mc^2 \quad -(8)$$

i.e.

$$\boxed{\frac{\hbar\omega}{mc^2} = \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}} \quad -(9)$$

Also

$$p = \hbar\kappa = \gamma mv \quad -(10)$$

So $$m = \left(1-\frac{v^2}{c^2}\right)^{1/2} \hbar \kappa \quad -(11)$$

In gravitational lensing

$$\frac{\hbar\omega}{mc^2} = \frac{a}{R_0} \quad -(12)$$

is determined experimentally, so:

$$\left(1-\frac{v^2}{c^2}\right) = \left(\frac{R_0}{a}\right)^2 \quad -(13)$$

i.e.

$$\boxed{v^2 = c^2\left(1-\left(\frac{R_0}{a}\right)^2\right)} \quad -(14)$$

By definition:

$$\boxed{\omega_0 = \left(1-\frac{v^2}{c^2}\right)^{1/2}\omega} \quad -(15)$$

In this theory m is always constant.

From eq. (9): It is seen that a change of frequency means a change of velocity v of the photon. The rest energy is conserved:

$$\boxed{mc^2 = \left(1-\frac{v^2}{c^2}\right)^{1/2}\hbar\omega = \hbar\omega_0} \quad -(16)$$

3) The Hamiltonian is:

$$H = \frac{1}{2} m c^2 = \frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \hbar\omega \quad -(17)$$

and is conserved.

From eq. (15),

$$v < c \quad -(18)$$

$$\omega_0 \rightarrow 0. \quad -(19)$$

because

A change of frequency ω means a change of photon velocity, but no change in m and no change in H.

$$\text{As: } v \rightarrow 0 \quad -(20)$$

$$\text{then } \omega \rightarrow \omega_0 \quad -(21)$$

The red shift in cosmology is explained by the photon losing velocity v, not by an expanding universe.

157(2): Literature Search on Light Deflection

I did a literature search and found that the actual experimental value of light deflection is not 1.75 arc seconds. For example, in:

1) D. S. Robertson and W. E. Carter, Nature, 310, 572 (1984) it is given as:

$$\Delta \phi = \frac{2MG}{c^2 R_0}(1 + \gamma) \quad -(1)$$

where

$$1 + \gamma = 2.008 \pm 0.006. \quad -(2)$$

Also I found that the mass of the sun and the G constant of Newton are not known with accuracy. In

2) I. P. Lopes and J. Silk, Monthly Notices R. Astron. Soc., 341(3), 721-728 (2002) it is stated that:

$$MG = 1.32712497(1) \times 10^{20} \, m^3 s^{-2} \quad -(3)$$

So the following value of r_0 should be used:

$$\boxed{r_0 = \frac{2MG}{c^2} = 2953.251 \text{ metres}} \quad -(4)$$

calculated using

$$c = 2.997925 \, m s^{-1} \quad -(5)$$

The experimental result (1) was cited by:

) H. Herold, ESA Space Science and Fundamental Physics, pp 141-149 (1988)

2) The experimental value in eq. (1) is given in terms of the radius of the sun R_0. The latter is given on Wikipedia as the mean radius: $\pm ?$

$$R_0 = 6 \cdot 955 \times 10^8 \text{ metres.} \quad -(6)$$

but this is accurate to only three decimal places, and no uncertainty bounds are given. Also Wikipedia is often unreliable.

The value of r_0 / R_0 from eqs. (4) and (6) is:

$$\frac{r_0}{R_0} = \frac{2MG}{c^2 R_0} = 4 \cdot 246 \text{ microradians} \quad -(7)$$

so
$$\boxed{\Delta\phi = \frac{4MG}{c^2 R_0} = (8 \cdot 49245 \pm ?\,0.025)\ \mu\text{rad}}$$

Now use:

$$1 \text{ arc second} = 4 \cdot 8481368 \text{ microradians} \quad -(8)$$

so
$$\boxed{\Delta\phi = \frac{4MG}{c^2 R_0} = 1 \cdot 75169 \pm ?\,0.0052 \text{ arc seconds.}}$$

However, this depends on eq. (6), and on the accuracy to which the sun's radius is known. From eq. (2) the mass of the sun, M, is not known with accuracy.

From eq. (1), using eq. (6), the

5) experimental deflection is:

$$\Delta\phi\,(\text{exptl.}) = 1.7587 \pm 0.0053 \text{ arc seconds} = 8.5264 \pm 0.0256 \text{ microradians} \qquad -(9)$$

This falls outside the theoretical value.

Table of Results *

Method	Deflection	
	arc seconds	microradians
$\frac{4MG}{c^2 R_0}$	$1.7517 \pm ?$	$8.4925 \pm ?$
1984 Experiment (long baseline r.f. interferometry)	1.7587 ± 0.0053	8.5264 ± 0.0256
Marmet et al. 2009, "limb of sun"	$1.745 \pm ?$	$8.4600 \pm ?$

$$* \quad r_0 = 2953.251 \text{ metres}, \quad R_0 = (6.955 \pm ?) \times 10^8 \text{ metres} \qquad -(10)$$

Einstein's Claim

$$\Delta\phi = 2\int_0^{1/R_0} \left(\frac{R_0 - r_0}{R_0^3} - u^2 + r_0 u^3 \right)^{-1/2} du - \pi = 2r_0/R_0 \qquad -(11)$$

This can be checked by computer using eq. (10)

157(3) Direct Computer Test of Einstein's Integral

In UFT 150 the following integral was evaluated numerically:

$$\Delta\phi = 2\int_0^{1/R_0}\left(\frac{R_0 - r_0}{R_0^3} - u^2 + r_0 u^3\right)^{-1/2} du - \pi \quad -(1)$$

with

$$r_0 = \frac{2mG}{c^2} = 2954.265 \text{ m.} \quad -(2)$$

$$R_0 = 6.955 \times 10^8 \text{ m} \quad -(3)$$

using $m = 1.9891 \times 10^{30}$ kg, $G = 6.67428 \times 10^{-11}$ $m^3 s^{-2} kg^{-1}$

i.e. $mG = 1.327581035\ m^3 s^{-2}$ $-(4)$

The numerical integration gave

$$\boxed{\Delta\phi = \left(8.4934 \pm 10^{-6}\right) \text{ microradians}} \quad -(5)$$

The value claimed by Einstein is $-(6)$

$$\boxed{\Delta\phi = \frac{4mG}{c^2 R_0} = \frac{2r_0}{R_0} = 8.49536 \text{ microrad.}}$$

The 1984 experimental value is: $-(7)$

$$\boxed{\Delta\phi\,(\text{exp.}) = \left(8.5264 \pm 0.0256\right) \text{ microradians}}$$

Therefore the Einstein theory fails.

!) More accurately:

$$MG = 1 \cdot 32712497(1) \times 10^{20} \, m^3 s^{-2}, \quad -(8)$$

$$r_0 = 2953 \cdot 251 \text{ metres} \quad -(9)$$

$$\frac{2r_0}{R_0} = 8 \cdot 49245 \text{ microradians} \quad -(10)$$

We need to recompute eq. (1) with r_0 given by eq. (9).

The discrepancy between eq. (5) and eq. (6) is:

$$\boxed{\Delta\Delta\phi = \left(\frac{8 \cdot 49536 - 8 \cdot 4934}{8 \cdot 4934}\right) \times 100\% = 0 \cdot 023\%}$$

The numerical precision is $< \pm 10^{-4}\%$.

The discrepancy is more than a hundred times larger than the numerical precision.

The discrepancy between eqs. (6) and (7) is:

$$\boxed{\Delta\Delta\phi = \left(\frac{8 \cdot 5264 - 8 \cdot 49536}{8 \cdot 4956}\right) \times 100\% = 0 \cdot 365\%}$$

The discrepancy between eqs. (5) and (7)

is $$\boxed{\Delta\Delta\phi = \left(\frac{8 \cdot 5264 - 8 \cdot 4934}{8 \cdot 4934}\right) \times 100\% = 0 \cdot 389\%}$$

3) The correct theoretical result for integral (1) is eq. (5). The latter correctly carries out the integration. The discrepancy between eqns (5) and (7) can only be due to photon mass within the Einstein theory.

String theory attempts to add terms to the Einstein theory, but this is essentially an empirical exercise with adjustable parameters.

1) 157(4): Basic Concepts of Photon Mass Theory

These concepts are based on the Planck theory of the photon and the special relativity of Einstein. The Planck theory is:

$$E = \hbar\omega \quad -(1)$$

where E is the quantum of energy contained in one photon. Here $\hbar$ is the reduced Planck constant and ω the angular frequency of the photon. Later eq. (1) was augmented by the de Broglie wave particle duality:

$$p = \hbar \kappa \quad -(2)$$

where κ is the wavenumber and p the momentum. In the special theory of relativity of Einstein:

$$E = \gamma m c^2, \quad -(3)$$

$$p = \gamma m v. \quad -(4)$$

These quantities are known respectively as the relativistic total energy E, and the relativistic momentum p. Here m is mass, which is the constant, unchanging mass of an elementary particle, v is the velocity of the particle, c is a fixed universal constant. The Lorentz factor is:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad -(5)$$

These quantities are related by the Einstein energy equation of 1905:

$$E^2 = p^2c^2 + m^2c^4. \quad -(6)$$

This equation can be expressed as:

$$p^\mu p_\mu = m^2c^2 \quad -(7)$$

2) where

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad -(8)$$

$$p_\mu = \left(\frac{E}{c}, -\underline{p}\right) \quad -(9)$$

in contravariant - covariant notation. Eq. (6) is a direct consequence of the definitions (3) and (4).

Photon mass theory uses a combination of eqns. (1), (2) and (6) together with the definitions:

$$\boxed{\begin{aligned} E &= \hbar\omega = \gamma mc^2 \quad &-(10) \\ p &= \hbar\kappa = \gamma mv \quad &-(11) \end{aligned}}$$

So the Planck quantum $\hbar\omega$ is the conserved and total relativistic energy of one photon, and the de Broglie quantum $\hbar\kappa$ is the conserved and total relativistic linear momentum of one photon.

Note carefully that as ω changes, v changes, but both m and c are constant. The hamiltonian is:

$$H = \frac{1}{2}mc^2 = \frac{1}{2m}p^\mu p_\mu \quad -(12)$$

and does not change.

There are two ways in which eq. (6) can be quantized:

1) Use eqs (1) and (2) in eq. (6):

$$\hbar^2\omega^2 = \hbar^2 c^2\kappa^2 + m^2c^4 \quad -(13)$$

so:

3) $$\boxed{\omega^2 = \kappa^2 c^2 + \left(\frac{mc}{\hbar}\right)^2} \quad - (14)$$

If it were possible to measure ω and κ in two independent experiments, m could be found experimentally.

In the old physics:

$$m = 0, \quad \omega = \kappa c \quad - (15)$$

which is the relation between ω and κ for light travelling at c.

Note carefully that eq. (14) is valid for all elementary particles, for example the electron.

2) Use the operator definition of quantum mechanics:

$$p^\mu = i\hbar \partial^\mu \quad - (16)$$

where $$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) \quad - (17)$$

So $$\boxed{\begin{aligned} E &= i\hbar\frac{\partial}{\partial t}, & \underline{p} &= -i\hbar\underline{\nabla} \\ &= \hbar\omega & &= \hbar\underline{\kappa} \end{aligned}} \quad - (18)$$

Therefore: $$\omega \rightarrow i\frac{\partial}{\partial t}, \quad \underline{\kappa} \rightarrow -i\underline{\nabla} \quad - (19)$$

$$\omega^2 \rightarrow -\frac{\partial^2}{\partial t^2}, \quad \kappa^2 \rightarrow -\nabla^2 \quad - (20)$$

4) Use eq. (20) in eq. (14) and use the fact that the operators act on the wave function ϕ. So:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0. \quad -(21)$$

Finally define the d'Alembertian operator:

$$\Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2, \quad -(22)$$

to obtain the wave equation of relativistic quantum mechanics

$$\boxed{(\Box + \kappa^2)\phi = 0} \quad -(23)$$

where

$$\kappa = \frac{mc}{\hbar} \quad (24)$$

is the Compton wavenumber.

Eq. (23) is now known to be a limit of the ECE wave equation of spacetime:

$$(\Box + R)q^a_{\ \mu} = 0 \quad -(25)$$

where $q^a_{\ \mu}$ is the Cartan tetrad. Eq. (25) is the tetrad postulate:

$$\partial_\mu q^a_{\ \nu} = \Omega^a_{\ \mu\nu} := \Gamma^a_{\ \mu\nu} - \omega^a_{\ \mu\nu}. \quad -(26)$$

In eq. (26):

5)

$$\Gamma^a_{\mu\nu} = \Gamma^\lambda_{\mu\nu} q^a_\lambda, \quad (27)$$

$$\omega^a_{\mu\nu} = \omega^a_{\mu b} q^b_\nu. \quad (28)$$

Here Γ denotes the general connection of Riemann geometry and ω the spin connection of Cartan.

To obtain eq. (25) from eq. (26) use:

$$\partial^\mu \partial_\mu q^a_\nu = \Box q^a_\nu = \partial^\mu \Omega^a_{\mu\nu}. \quad (29)$$

Define:

$$q^a_\nu R := -\partial^\mu \Omega^a_{\mu\nu} \quad (30)$$

to obtain eq. (25). So:

$$\boxed{R := -q^\nu_a \partial^\mu \Omega^a_{\mu\nu}}. \quad (31)$$

In the limit:

$$R \rightarrow \kappa^2 \quad (32)$$

the general space in four dimensions (the physical space-time) approaches the Minkowski spacetime, and R approaches

$$R_0 \rightarrow \left(\frac{mc}{\hbar}\right)^2 \quad (33)$$

which is the quantum of curvature.

The Proca equation for the spin-one bosons is found using the ECE hypothesis:

$$A^a_\mu = A^{(0)} q^a_\mu \quad (34)$$

to give:

6)

$$(\Box + R) A^a_\mu = 0 \quad -(35)$$

the Proca equation of 1934 is the limit:

$$R \rightarrow \left(\frac{mc}{\hbar}\right)^2 \quad -(36)$$

$$A_\mu = \sum_a A^a_\mu \quad -(37)$$

i.e.

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) A_\mu = 0 \quad -(38)$$

In eq. (37):

$$a = (0), (1), (2), (3) \quad -(39)$$

In the old physics:

$$m = ?\ 0, \quad a = ?\ (1), (2) \quad -(40)$$

and eq. (38) was replaced by the d'Alembert equation:

$$\Box A_\mu = 0 \quad -(41)$$

The Proca equation (38) is:

$$\Box A_\mu = -\left(\frac{mc}{\hbar}\right)^2 A_\mu \quad -(42)$$

and in photon mass theory the right hand side is [illegible] in terms of the four current of ~~the~~ spacetime:

$$\Box A_\mu = -\left(\frac{mc}{\hbar}\right)^2 A_\mu \quad -(43)$$

$$\boxed{\Box A_\mu = -\left(\frac{mc}{\hbar}\right)^2 \frac{J_\mu}{\epsilon_0 c^2}} \quad -(44)$$

7) Therefore spacetime behaves as an absorbing dielectric and Higgs is replaced by photon mass theory.

Definitions and Units

$$J^\mu = (c\rho, \underline{J}), \quad A^\mu = \left(\frac{\phi}{c}, \underline{A}\right) \qquad -(45)$$

Charge density $= \rho = C\,m^{-3}$

Current density $= \underline{J} = A\,m^{-2} = C\,s^{-1}\,m^{-2}$

Scalar potential $= \phi = \text{volt} = J\,C^{-1}$

Vector potential $= \underline{A} = J\,s\,C^{-1}\,m^{-1}$

Vacuum permittivity $= \epsilon_0 = 8.854188 \times 10^{-12}\,J^{-1}\,C^{2}\,m^{-1}$

Therefore

$$\Box A_\mu = \mu_0 J_\mu = -\left(\frac{mc}{\hbar}\right)^2 A_\mu \qquad -(46)$$

where

$$\mu_0 = 4\pi \times 10^{-7}\,J\,s^2\,C^{-2}\,m^{-1}$$

and

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

The spacetime current is:

$$\boxed{J_\mu = -\frac{1}{\mu_0}\left(\frac{mc}{\hbar}\right)^2 A_\mu} \qquad -(47)$$

157(5): The Relativistic Compton Scattering of One Photon and One Electron

If p_1^μ is the initial four momentum of the photon, and p_2^μ the initial four-momentum of the electron, and if p_3^μ is the final four momentum of the photon and p_4^μ the final four momentum of the electron, then

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu . \qquad (1)$$

In the traditional theory a photon with no mass collides with an initially stationary electron, as in Fig (1)

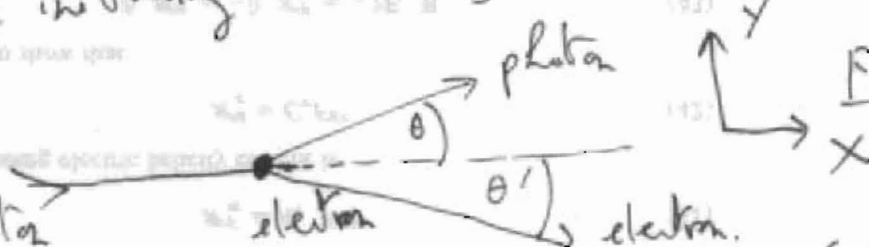

Fig (1).

We have:

$$p_1^\mu = \left(\frac{\hbar\omega_1}{c}, \hbar \underline{\kappa}_1 \right) \qquad (2)$$

with

$$\omega_1 = \kappa_1 c, \qquad (3)$$

$$p_2^\mu = (mc, \underline{0}) \qquad (4)$$

$$p_3^\mu = \left(\frac{\hbar\omega_2}{c}, \hbar \underline{\kappa}_2 \right) \qquad (5)$$

$$p_4^\mu = \left(\frac{1}{c}\left(c^2 p^2 + m^2 c^4\right)^{1/2}, \underline{p} \right) \qquad (6)$$

Therefore,

Initial momentum of photon $= \hbar \underline{\kappa}_1$

Final momentum of photon $= \hbar \underline{\kappa}_2$

Initial momentum of electron $= \underline{0}$

Final momentum of electron $= \underline{p}$

Initial energy of photon $= \hbar\omega_1 = \hbar c \kappa_1$
Final energy of photon $= \hbar\omega_2 = \hbar c \omega_2$
Initial energy of electron $= E_0 = mc^2$
Final energy of electron $= E = (c^2 p^2 + m^2 c^4)^{1/2}$
Here m is the mass of the electron. The photon mass is identically zero in the traditional theory.
Initial velocity of photon $= c$
Final velocity of photon $= c$
Initial velocity of electron $= 0$
Final velocity of electron $= v_2$.
The relativistic velocity addition rule is:

$$\boxed{v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}} \quad -(7)$$

Initially, $v_1 = c, \; v_2 = 0, \; v = c \quad -(8)$
After collision, $v_1 = c, \; v_2 = v_2, \; v = c \quad -(9)$
the total velocity v of electron and photon is always c, and the velocity of the photon is always c.
This is counter intuitive and is the result of special relativity, confirmed in experiments on Compton scattering. This is why

3) The speed of light c is the same upon entering and leaving a sample in physical optics. Inside the sample the speed of light is still c, not v. The index of refraction is:

$$n = \left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)^{1/2} = \frac{c}{v_p} \quad -(10)$$

where v_p is the phase velocity of the wave in the medium (Jackson 3rd ed., p. 296).

In the standard theory (massless photon), the phase velocity of the light wave in the medium is not the same as the speed of the photon in the medium. This is a fundamental conceptual difficulty of the standard model.

Conservation of Energy

$$\hbar\omega_1 + mc^2 = \hbar\omega_2 + (c^2p^2 + m^2c^4)^{1/2} \quad -(11)$$

In the quantum theory there is a frequency shift:

$$\hbar(\omega_1 - \omega_2) = (c^2p^2 + m^2c^4)^{1/2} - mc^2 \quad -(12)$$

Conservation of Momentum

In the X Axis

$$\hbar \underline{\kappa_1} = \hbar \underline{\kappa_2}\cos\theta + \underline{p}\cos\theta' \quad -(13)$$

In the Y Axis

$$\underline{0} = \hbar \underline{\kappa_2}\sin\theta - \underline{p}\sin\theta' \quad -(14)$$

4) So:

$$p^2 = (\hbar \kappa_2 \sin\theta)^2 + (\hbar\kappa_1 - \hbar\kappa_2\cos\theta)^2 \quad -(15)$$

$$p^2 + m^2c^4 = (\hbar\omega_1 - \hbar\omega_2 + mc^2)^2$$
$$= c^2(\hbar\kappa_1 - \hbar\kappa_2 + mc)^2 \quad -(16)$$

Here θ' has been eliminated using:

$$\hbar\kappa_1 - \hbar\kappa_2\cos\theta = p\cos\theta' \quad -(17)$$

$$\hbar\kappa_2\sin\theta = p\sin\theta' \quad -(18)$$

$$p^2(\cos^2\theta' + \sin^2\theta') = p^2 \quad -(19)$$

From (15) and (16):

$$p^2 = \hbar^2(\kappa_1-\kappa_2)^2 + 2mc\hbar(\kappa_1-\kappa_2)$$
$$= \hbar^2\kappa_1^2 + \hbar^2\kappa_2^2 - 2\hbar^2\kappa_1\kappa_2 + 2mc\hbar(\kappa_1-\kappa_2)$$
$$= \hbar^2\kappa_2^2\sin^2\theta + \hbar^2\kappa_1^2 + \hbar^2\kappa_2^2\cos^2\theta - 2\hbar^2\kappa_1\kappa_2\cos\theta$$
$$= \hbar^2\kappa_1^2 + \hbar^2\kappa_2^2 - 2\hbar^2\kappa_1\kappa_2\cos\theta \quad -(20)$$

So:

$$\boxed{\frac{\kappa_1-\kappa_2}{\kappa_1\kappa_2} = \frac{\hbar}{mc}(1+\cos\theta) = \frac{2\hbar}{mc}\sin^2\frac{\theta}{2}} \quad -(21)$$

Eq. (21) is

$$\frac{1}{\kappa_2} - \frac{1}{\kappa_1} = 2\left(\frac{\hbar}{mc}\right)\sin^2\frac{\theta}{2} \quad -(22)$$
$$= \lambda_2 - \lambda_1$$

5) The Compton wavelength of the electron is:

$$\lambda_c = \frac{\hbar}{mc} \quad - (23)$$

and this verified de Broglie's wave-particle dualism. It is also one of the verifications of the special theory of relativity, but the photon mass is left out of consideration. As pointed out by Dr. Gareth J. Evans recently, there is a fundamental difficulty in that the phase velocity in a medium in physical optics is not c, while in particle physics the photon velocity is always c. The latter is the phase velocity in the vacuum. As Dr. G. J. Evans points out, there is no mechanism for the phase velocity to regain its value of c after emerging from the medium. In the next note it will be shown how photon mass changes this theory completely.

157(6): Some concepts of Photon Mass

: Relativistic momentum of particulate photon $= \gamma m \underline{v} = \underline{p}$ — (1)

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad \text{— (2)}$$

Relativistic total energy:

$$E = \gamma m c^2 . \quad \text{— (3)}$$

Relativistic kinetic energy of particulate photon:

$$T = mc^2(\gamma - 1) \quad \text{— (4)}$$

If

$$v \ll c \quad \text{— (5)}$$

$$T \rightarrow mc^2\left(\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) - 1\right) = \frac{1}{2}mv^2 . \quad \text{— (6)}$$

Hamiltonian:

$$H = \frac{1}{2}mc^2 \quad \text{— (7)}$$

$$= \frac{1}{2m} p^{\mu} p_{\mu}$$

Four momentum

$$p^{\mu} = \left(\frac{E}{c}, \underline{p}\right), \quad p_{\mu} = \left(\frac{E}{c}, -\underline{p}\right) \quad \text{— (8)}$$

Wave Particle Duality

$$p^{\mu} = \underbrace{\left(\frac{\hbar\omega}{c}, \hbar \underline{\kappa}\right)}_{\text{wave}} = \underbrace{\left(\frac{E}{c}, \underline{p}\right)}_{\text{particle}} \quad \text{— (9)}$$

i.e.

$$E = \hbar\omega = \gamma m c^2 \quad \text{— (10)}$$

$$\underline{p} = \hbar \underline{\kappa} = \gamma m \underline{v} \quad \text{— (11)}$$

$$E_0 = \hbar\omega_0 = mc^2 \quad \text{— (12)}$$

2)

$$p_0 = 0 \quad -(13)$$

Eq. (12) is the de Broglie equation, in which the rest frequency ω_0 is defined: the photon cannot have a lower frequency than:

$$\omega_0 = \frac{mc^2}{\hbar} \quad -(14)$$

If as in latest work:

$$m = 2.42 \times 10^{-38} \text{ kgm} \quad -(15)$$

then

$$\boxed{\begin{aligned}\omega_0 &= 3.28 \times 10^{12} \text{ radian per second} \\ f_0 &= 52.2 \times 10^{10} \text{ Hz} = \frac{\omega_0}{2\pi}\end{aligned}} \quad -(16)$$

Kinetic Energy of Photon

$$T = \hbar\omega - mc^2 \quad -(17)$$

Absorption Theory

The Planck distribution for energy density U:

$$\frac{dU}{d\omega} = \frac{8\pi\hbar\omega^3}{c^3}\left(\frac{\exp(-\hbar\omega/kT)}{1-\exp(-\hbar\omega/kT)}\right) \quad -(18)$$

where:

$$\int_0^\infty \left(\frac{x^{2n-1}}{e^x - 1}\right) dx = \frac{(2\pi)^{2n} B_n}{4n}, \quad -(19)$$

$$B_1 = \frac{1}{6}, \; B_2 = \frac{1}{30}, \; B_3 = \frac{1}{42} \ldots$$

3) Total energy density at temperature T is

$$U = \frac{8\pi\hbar}{c^3}\int \omega^3 \left(\frac{\exp(-\hbar\omega/kT)}{1-\exp(-\hbar\omega/kT)} \right) d\omega$$

$$\boxed{U = \left(\frac{\pi^2}{15}\right)\left(\frac{k^4}{c^3\hbar^3}\right) T^4}$$

which is the Stefan-Boltzmann Law. Here:

Boltzmann constant $k = 1.38066 \times 10^{-23}\ JK^{-1}$

$c = 2.997925 \times 10^{8}\ ms^{-1}$

$\hbar = 1.05459 \times 10^{-34}\ J\,s$

Units of U are $\dfrac{J^4}{m^3 s^{-3} J^3 s^3} = J m^{-3}$

Energy density of electromagnetic radiation = joules per cubic metre.

1) 147(7) : Kinetic and Potential Energy

In note 147(6), it was found that the effect of gravitation on the electron Sagnac effect ~~was~~ is:

$$\omega = \frac{v_1}{r} \quad -(1)$$

where

$$v_1 = \left(v^2 - 2\frac{GM}{R}\right)^{1/2} \quad -(2)$$

Here v is the tangential velocity of the electron and

$$\Phi = -\frac{GM}{R} \quad -(3)$$

is the classical gravitational potential. The classical potential energy is

$$U = m\Phi \quad -(4)$$

where m is the mass of the electron. From eq. (2):

$$v_1^2 = v^2 - 2\frac{GM}{R} \quad -(5)$$

and so

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 - \frac{GmM}{R}. \quad -(6)$$

The hamiltonian is:

$$H = T + U \quad -(7)$$

where

$$T = \frac{1}{2}mv^2, \quad -(8)$$

$$U = -\frac{GmM}{R}. \quad -(9)$$

The classical gravitational force between m and

2) on attracting mass M is:

$$\underline{F} = -m \underline{\nabla} \Phi = -\underline{\nabla} U \quad -(10)$$

$$\boxed{\underline{F} = -\frac{GmM}{R^2} \underline{k}} \quad -(11)$$

in the Z axis. This is the Newton inverse square law.

The kinetic energy (T) and work done (W) is related to the force by:

$$W = T = \int \frac{d\underline{F}}{dt} \cdot \underline{v} \, dt \quad -(12)$$

So

$$\boxed{\underline{F} = m \frac{d\underline{v}}{dt}} \quad -(13)$$

To see this use the fact that the work done from 1 to 2 is:

$$W_{12} = \int_1^2 \underline{F} \cdot d\underline{r} \quad -(14)$$

$$\underline{F} \cdot d\underline{r} = m \frac{d\underline{v}}{dt} \cdot \frac{d\underline{r}}{dt} dt = m \frac{d\underline{v}}{dt} \cdot \underline{v} \, dt$$

$$= \frac{m}{2} \frac{d}{dt} (\underline{v} \cdot \underline{v}) \, dt$$

$$= d\left(\frac{1}{2} m v^2\right) \quad -(15)$$

So $W_{12} = \frac{1}{2} m v^2 \Big|_1^2 = \frac{1}{2} m (v_2^2 - v_1^2)$

$$= T_2 - T_1 \quad -(16)$$

If:

3)
$$W_{12} < 0 \quad -(17)$$

The particle of mass m does work, and loses kinetic energy.

If:
$$T_1 = 0 \quad -(18)$$

the result (14) is obtained.

The potential energy is the capacity to do work. If the force $\underline{F}$ transports m from 1 to 2 it does work on the particle. If there is no change in kinetic energy, the work required to move a particle from 1 to 2 is independent of the path. So

$$\int_1^2 \underline{F} \cdot d\underline{r} = U_1 - U_2 \quad -(19)$$

If
$$\underline{F} = -\underline{\nabla} U \quad -(20)$$

then
$$\underline{\nabla} \times \underline{F} = -\underline{\nabla} \times \underline{\nabla} U = 0 \quad -(21)$$

In this case:
$$\int_1^2 \underline{F} \cdot d\underline{r} = -\int_1^2 \underline{\nabla} U \cdot d\underline{r}$$
$$= -\int_1^2 dU = U_1 - U_2 \quad -(22)$$

So if m is raised to a height h by any path, an amount of work is done on it. In a constant $\underline{g}$ this is:

$$W_{12} = m \int_1^2 \underline{g} \cdot d\underline{r} = mg \Big|_0^h$$
$$= mgh = U_1 - U_2 \quad -(23)$$

4) and if:
$$U_2 = 0 \quad -(24)$$
then
$$W_{12} = mgh \quad -(25)$$
From eqs. (16) and (19)
$$T_2 - T_1 = U_1 - U_2 \quad -(26)$$
So
$$\boxed{U_1 + T_1 = U_2 + T_2} \quad -(27)$$
This means that the Hamiltonian is constant. The equivalence principle is:
$$\underline{F} = m\underline{g} = -\frac{mMG}{R^2}\underline{k} \quad -(28)$$
So all the features of classical dynamics are obtained from eq. (1). This proves that the method of deriving eq. (1) is correct.

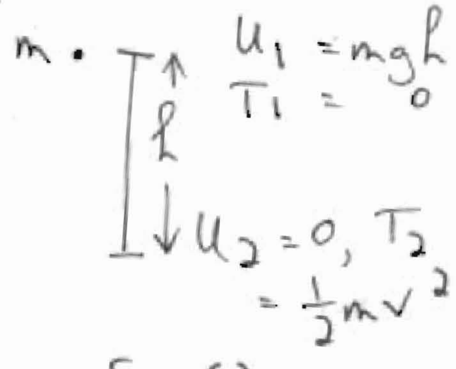

Fig (1).

Finally we:
$$\boxed{g = -\frac{MG}{R^2}} \quad -(29)$$
to obtain:
$$\boxed{\begin{aligned} v_1 &= \left(v^2 + 2Rg\right)^{1/2} \quad -(30)\\ \omega &= v_1 / r \quad -(31)\\ \alpha &= \omega t \quad -(32)\end{aligned}}$$

5) In the case of the photon of mass m:

$$v_1 = \left(c^2 + 2Rg\right)^{1/2} \quad -(31)$$

It is seen that:

$$\omega_0 = \frac{c}{r} \rightarrow \frac{1}{r}\left(c^2 + 2Rg\right)^{1/2} \quad -(32)$$

where r is the radius of the platform. Here R is the Earth's mean radius, g the magnitude of the acceleration due to gravity at the Earth's surface. Here:

$$c \sim 3 \times 10^8 \ m s^{-1}$$

$$R = 6.37 \times 10^6 \ m$$

$$g = 9.8 \ m s^{-2}$$

So for the photon:

$$c^2 = 9 \times 10^{16} \ (m/s)^2$$

$$2Rg = 1.25 \times 10^8 \ (m/s)^2$$

The instrument has to have a resolution of one part in 10^8 to see an effect. However, the instrument is already in the Earth's gravitational field, so for a Sagnac interferometer on the Earth's surface, eq. (33) is observed directly. The e/m frequency shift is

$$\alpha = \omega t = \frac{t}{r}\left(c^2 + 2Rg\right)^{1/2}. \quad -(33)$$

The time taken for the light beam to go around 2π radians is:

6)

$$\boxed{t = \frac{2\pi r}{(c^2 + 2Rg)^{1/2}}} \quad -(34)$$

for the photon (ordinary Sagnac effect with static platform) and for the electron:

$$\boxed{t = \frac{2\pi r}{(v^2 + 2Rg)^{1/2}}} \quad -(35)$$

These times can be measured directly and measure the effect of gravitation on a photon and electron, ~~so~~ respectively, going around a circle or loop of any shape.

Therefore gravitation affects electromagnetism and electronic trajectories, for example an electron going around the rim of a spinning Faraday disk.

1) 157(8): Compton Scattering of Photon with Mass

As per previous note the energy conservation law gives:

$$\hbar(\omega_1-\omega_2) = (c^2p^2+m^2c^4)^{1/2} - Mc^2 \quad — (1)$$

In photon mass ~~theory~~ (Louis de Broglie):

$$\hbar\omega_1 = \gamma_1 mc^2, \quad \hbar\omega_2 = \gamma_2 mc^2, \quad — (2)$$

$$\gamma_1 = \left(1-\left(\frac{v_1}{c}\right)^2\right)^{-1/2}, \quad \gamma_2 = \left(1-\left(\frac{v_2}{c}\right)^2\right)^{-1/2}. \quad — (3)$$

The momentum conservation law gives:

$$\hbar\kappa_1 = \hbar\kappa_2\cos\theta + p\cos\theta' \quad — (4)$$

$$0 = \hbar\kappa_2\sin\theta - p\sin\theta' \quad — (5)$$

so

$$\boxed{p^2 = \hbar^2(\kappa_1^2+\kappa_2^2-2\kappa_1\kappa_2\cos\theta)} \quad — (6)$$

From eq. (1):

$$c^2p^2 + M^2c^4 = (\hbar(\omega_1-\omega_2)+Mc^2)^2 \quad — (7)$$

$$= \hbar^2(\omega_1-\omega_2)^2 + 2\hbar(\omega_1-\omega_2)Mc^2 + M^2c^4$$

$$\boxed{p^2 = \frac{1}{c^2}\left(\hbar^2(\omega_1-\omega_2)^2 + 2\hbar(\omega_1-\omega_2)Mc^2\right)} \quad — (8)$$

Here $\boxed{\begin{aligned} m &= \text{mass of photon} \\ M &= \text{mass of electron}\end{aligned}}$ — (9)

From eqs. (6) and (8):

$$\kappa_1^2+\kappa_2^2-2\kappa_1\kappa_2\cos\theta = \frac{1}{c^2}\left((\omega_1-\omega_2)^2 + 2(\omega_1-\omega_2)\frac{Mc^2}{\hbar}\right)$$

2)

$$= \frac{1}{c^2}\left(\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2 + 2(\omega_1 - \omega_2)\frac{Mc^2}{\hbar}\right)$$

Let: $\underline{m} = \underline{\text{mass of photon}}$.

Then:

$$\omega_1^2 = c^2\kappa_1^2 + \left(\frac{mc^2}{\hbar}\right)^2 \quad -(10)$$

$$\omega_2^2 = c^2\kappa_2^2 + \left(\frac{mc^2}{\hbar}\right)^2 \quad -(11)$$

Eq. (9) is:

$$\frac{1}{c^2}\left(\omega_1^2 + \omega_2^2\right) - \left(\kappa_1^2 + \kappa_2^2\right) \quad -(12)$$

$$= \frac{2\omega_1\omega_2}{c^2} - 2(\omega_1 - \omega_2)\frac{M}{\hbar} - 2\kappa_1\kappa_2\cos\theta$$

$$= 2\left(\frac{mc}{\hbar}\right)^2 \quad -(13)$$

Therefore:

$$\omega_1 - \omega_2 = \frac{\hbar}{Mc^2}\omega_1\omega_2 - \frac{\hbar}{M}\kappa_1\kappa_2\cos\theta + \frac{\hbar}{M}\left(\frac{mc}{\hbar}\right)^2$$

$$= \frac{\hbar}{Mc^2}\left(\omega_1\omega_2 - c^2\kappa_1\kappa_2\cos\theta + \frac{m^2c^2}{\hbar M}\right) \quad -(14)$$

Define

$$\omega_0 = \frac{mc^2}{\hbar} \quad -(15)$$

3) then

$$\kappa_1 \kappa_2 = \frac{1}{c^2}\left((\omega_1^2 - \omega_0^2)(\omega_2^2 - \omega_0^2) \right)^{1/2} \quad - (16).$$

So:

$$\boxed{\omega_1 - \omega_2 = \frac{\hbar}{Mc^2}\left[\omega_1 \omega_2 - \left((\omega_1^2 - \omega_0^2)(\omega_2^2 - \omega_0^2) \right)^{1/2} \cos\theta \right] + \frac{m^2 c^2}{\hbar M}} \quad - (17)$$

This is the photon - electron Compton effect for a photon of mass m.

ω_1 = initial angular frequency of photon

ω_2 = final angular frequency of photon

θ = scattering angle

ω_0 = $mc^2 / \hbar$

If

$$m = 0 \quad - (18)$$

eq. (17) reduces to:

$$\omega_1 - \omega_2 = \frac{\hbar}{Mc^2}\, \omega_1 \omega_2 (1 - \cos\theta)$$

$$= \frac{2\hbar}{Mc^2} \sin^2 \frac{\theta}{2} \quad - (19)$$

where

$$\omega_1 = c\kappa_1 = \frac{c}{\lambda_1} \quad - (20)$$

$$\omega_2 = c\kappa_2 = \frac{c}{\lambda_2} \quad - (21)$$

4) So

$$\kappa_1 - \kappa_2 = \frac{\hbar}{Mc}\kappa_1\kappa_2(1-\cos\theta)$$

$$\frac{1}{\kappa_2} - \frac{1}{\kappa_1} = \frac{2\hbar}{Mc}\sin^2\frac{\theta}{2},$$

or

$$\lambda_2 - \lambda_1 = \frac{2\hbar}{Mc}\sin^2\frac{\theta}{2} \qquad (22)$$

which is the usual expression for Compton scatterings, (P.W. Atkins, "Molecular Quantum Mechanics" (OUP, 2nd ed., 1983), eq. (1.4.1)).

Here:

ω = angular frequency (radians per second)

κ = wavenumber (m^{-1})

λ = wavelength (m)

The Compton effect with finite photon mass, eq. (17), is different from the Compton effect with no photon mass, eq. (19).

Suggested Computer Work

1) Plot $\omega_1 - \omega_2$ as a function of photon mass m for given ω_1 and ω_2 from eq. (17).

2) Compare with the $m = 0$ result, eq. (19).

157(9): Basic Equations of Photon Mass Theory

The basic equations are a combination of special relativity and quantum theory. The relativistic total energy of one photon is:

$$E = \hbar\omega = \gamma m c^2 \quad — (1)$$

and the relativistic momentum of one photon is:

$$p = \gamma m v = \hbar \kappa \quad — (2)$$

Here:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad — (3)$$

where v is the velocity of the photon and m its mass. The angular frequency of the photon is ω and its wavenumber is κ.

From eqs (1) and (2)

$$\boxed{\omega = c^2 \frac{\kappa}{v}} \quad — (4)$$

Eq. (4) eliminates the Lorentz factor γ. The velocity of the photon can be expressed as

$$\boxed{v = c^2 \frac{\kappa}{\omega}} \quad — (5)$$

The Hamiltonian is:

$$\boxed{H = \frac{1}{2} m c^2 = \frac{\hbar\omega}{2}\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \quad — (6)$$

and for a constant photon mass is a universal constant.

2) Therefore:

$$\boxed{\begin{aligned} H &= \frac{1}{2} mc^2 = \frac{\hbar\omega}{2}\left(1 - \left(\frac{\kappa c}{\omega}\right)^2\right)^{1/2} \quad -(7) \\ &= \frac{1}{2m} p^\mu p_\mu \end{aligned}}$$

Here:

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad -(8)$$

$$p_\mu = \left(\frac{E}{c}, -\underline{p}\right) \quad -(10)$$

The mass of the photon can be expressed as follows in terms of ω and κ of the photon:

$$\boxed{m = \frac{\hbar\omega}{c^2}\left(1 - \left(\frac{\kappa c}{\omega}\right)^2\right)^{1/2}} \quad -(11)$$

Therefore in photon mass theory the frequency ω and the wavenumber κ are related as in eq. (11). If it were possible to measure ω and κ independently the photon mass could be found. Eq. (11) shows that if

$$\omega = \kappa c \quad -(12)$$

then

$$m = 0 \quad -(13)$$

for all ω. This is the usual theory of light in a vacuum. If eq. (12) is true, then eq. (5) implies

$$v = c. \quad -(14)$$

Eq. (5) can be written as:

$$\boxed{\frac{v}{c} = c\left(\frac{\kappa}{\omega}\right)} \quad (15)$$

Eq. (15) has some similarities with the concept of refractive index in physical optics. However in physical optics the refractive index is used to define the phase velocity, v_p, of a wave. In eq. (15) v is the velocity of the photon thought of as a particle.

Eq. (1) can be written as:

$$E^2 = c^2 p^2 + m^2 c^4 \quad (16)$$

i.e.

$$\omega^2 = c^2 \kappa^2 + \left(\frac{mc}{\hbar}\right)^2 . \quad (17)$$

Using

$$p^\mu = i\hbar \partial^\mu \quad (18)$$

eq. (16) quantizes to:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\psi = 0 . \quad (19)$$

If the wave function ψ is identified as the four-potential A_μ^a for each sense of polarization, a, the Proca equation is obtained:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) A_\mu = 0 \quad (20)$$

This equation shows that the photon mass affects the wave properties of e/m radiation in a manner that is conceptually the same as the theory of e/m radiation in the presence of a four current density.

4) Thus:

$$\Box A_\mu = \mu_0 . J_\mu \quad -(21)$$

where

$$\boxed{J_\mu = -\frac{1}{\mu_0}\left(\frac{mc}{\hbar}\right)^2 A_\mu} \quad (22)$$

The solutions of eq (21) are the Liénard-Wiechert potentials. Here:

$$J_\mu = \left(c\rho, -\underline{J}\right), \quad -(23)$$

$$A_\mu = \left(\frac{\phi}{c}, -\underline{A}\right). \quad -(24)$$

Eq. (21) developed by Proca in 1934, can be regarded as an equation of the new quantum theory. In the old quantum theory the most ~~or~~ fundamental quantization is:

$$\boxed{E = \hbar\omega = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}} \quad -(25)$$

so any frequency ω is a photon velocity, v. The entire structure of the old quantum theory can be developed in terms of photon velocity. For example, absorption is described by the acquisition of precisely $\hbar\omega$ of energy by an atom or molecule. According to eq. (25), a quantum of velocity is transferred to the atom.

In the standard physics, v is always fixed at c, and is not quantized. In order to define a finite $\hbar\omega$, the mass m is zero and at $v = c$ the denominator is zero. Therefore:

5)

$$E(\text{standard model}) = \hbar\omega = ?\ \frac{0}{0} \quad -(26)$$

This is incompatible with special relativity, and the Planck constant is mathematically indeterminate for all ω. Both the quantum theory and the theory of special relativity are very precisely tested experimentally. The standard model is forced to make the unphysical and absurd suggestion of a particle (the photon) whose mass is identically zero in order that $\hbar\omega$ be finite.

On the other hand, ECE theory produces the Proca wave equation as a limit of the ECE wave equation, and experiments should be interpreted in terms of a constant and fundamental m of the photon. This means that experiments should be interpreted in terms of the velocity v with constant m.

Looked at in another way, the ratio of κ to ω for one photon is not fixed. Both κ and ω are well known to be quantized, so from eq. (5), v is quantized. This means that the Lorentz factor γ is also quantized. From eq. (1) it is seen that the quantum of energy E is the γ quantum.

The analogy with the physical optics of e/m radiation interacting with a four-current produces the Proca field equation:

6)

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad -(27)$$

where

$$J^\nu = -\frac{1}{\mu_0}\left(\frac{mc}{\hbar}\right)^2 A^\nu \quad -(28)$$

Eq. (27) produces the analogues of the Coulomb law:

$$\underline{\nabla} \cdot \underline{D} = \rho \quad -(29)$$

and the Ampère - Maxwell law:

$$\underline{\nabla} \times \underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{J}. \quad -(30)$$

Here:

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P} \quad -(31)$$

$$\underline{B} = \mu_0 \underline{H} + \mu_0 \underline{M} \quad -(32)$$

where:

$\underline{E}$ = electric field strength ($V m^{-1}$)

$\underline{D}$ = electric displacement ($C m^{-2}$)

ρ = charge density ($C m^{-3}$)

$\underline{H}$ = magnetic field strength ($A m^{-1}$)

$\underline{B}$ = magnetic flux density (T)

$\underline{P}$ = polarization ($C m^{-2}$)

$\underline{M}$ = magnetization ($A m^{-1}$)

From eq. (28) the charge density of spacetime itself is:

$$\boxed{\rho = -\epsilon_0\left(\frac{mc}{\hbar}\right)^2 \phi} \quad -(33)$$

where the units of the scalar potential ϕ are volts

7) (i.e. joules per coulomb). The units of ϵ_0, the permittivity of spacetime, are $J^{-1} C^2 m^{-1}$, and the units of $mc/\hbar$ are m^{-1}. The units of ρ are coulombs per cubic metre.

Similarly the current density generated by spacetime is:

$$\boxed{\underline{J} = \frac{1}{\mu_0}\left(\frac{mc}{\hbar}\right)^2 \underline{A}} \quad -(34)$$

where $\underline{J}$ has the units of $C s^{-1} m^{-2} = A m^{-2}$, the permeability of spacetime is $J s^2 C^{-2} m^{-1}$ and where the vector potential $\underline{A}$ has the units of amps per square metre.

The Coulomb Law is therefore:

$$\boxed{\underline{\nabla} \cdot \underline{D} = -\epsilon_0 \left(\frac{mc}{\hbar}\right)^2 \phi = \rho} \quad -(35)$$

and the Ampère Maxwell law is:

$$\boxed{\underline{\nabla} \times \underline{H} - \frac{\partial \underline{D}}{\partial t} = -\frac{1}{\mu_0}\left(\frac{mc}{\hbar}\right)^2 \underline{A} = \underline{J}} \quad -(36)$$

The Proca equations are counter examples to gauge theory

1) 157(10): Combination of Experiments to Measure Photon Mass

1) Direct Measurement of Velocity

The basic equations of photon mass theory are:

$$E = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \hbar\omega \quad -(1)$$

and

$$p = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} mv = \hbar\kappa. \quad -(2)$$

Therefore if ω, κ and v can be measured experimentally the mass m can be found. Such an experiment would also test the self-consistency of eqns. (1) and (2).

In the standard model:

$$\omega = \kappa c \quad -(3)$$

and the velocity of light in the absence of matter is regarded as c. However it is now known that light deflection and time delay experiments fail if the speed of light is c. This is because the null geodesic condition:

$$ds^2 = 0 \quad -(4)$$

cannot be applied theoretically.

2) Compton Scattering

In theory, the Compton scattering of a photon from an electron gives the photon mass m directly, because the effect is described by:

$$\omega_1 - \omega_2 = \frac{\hbar}{Mc^2}\left[\omega_1\omega_2 - \left(\omega_1^2 - \omega_0^2\right)^{1/2}\left(\omega_2^2 - \omega_0^2\right)^{1/2}\cos\theta\right] + \frac{m^2c^2}{\hbar M} \quad -(5)$$

2) where $\omega_0 = \frac{mc^2}{\hbar}$ — (6).

Here m is the mass of the photon and M the mass of the electron. Eq. (5) is true for the scattering of one photon from one electron. It is also true for the scattering of one photon from one proton, neutron, or one nucleus, or any particle of mass M.

In the Compton scattering theory of the standard model:

$$m^2 = 0 \quad — (7)$$

so:

$$\omega_1 - \omega_2 = \frac{\hbar}{Mc^2}(1 - \cos\theta)\,\omega_1\omega_2 \quad — (8).$$

$$\text{i.e.}\quad \frac{1}{\omega_2} - \frac{1}{\omega_1} = \frac{\hbar}{Mc^2}(1 - \cos\theta)$$

$$\text{or}\quad \frac{1}{\kappa_2} - \frac{1}{\kappa_1} = \left(\frac{\hbar}{Mc}\right)(1 - \cos\theta) \quad — (9)$$

$$= \lambda_2 - \lambda_1$$

3) Light Deflection and Time Delay due to Gravitation

These experiments depend on photon mass and energy. A consistent interpretation of both experiments is needed to give the fundamental photon mass. The latter should be the same as in experiments (1) and (2).

3)
4)

Photoelectric Effect

This is the emission of electrons from metals when they are irradiated with ultraviolet light. There is no emission if the frequency of the light is below a threshold value typical of each element. In the old quantum theory of Planck and Einstein:

$$\hbar\omega = \frac{1}{2} m v^2 + \Phi, \quad -(10)$$

$$\frac{1}{2} m v^2 = \hbar\omega - \Phi \quad -(11)$$

$$\geq 0$$

so

$$\boxed{\hbar\omega \geq \Phi} \quad -(12)$$

Here, $\frac{1}{2} m v^2$ is the kinetic energy of an emitted electron in the non-relativistic limit, and Φ is the work function. Eq. (10) is the non-relativistic limit of a more complete theory. This can be seen from the fact that the relativistic kinetic energy of the electron is

$$T = (\gamma - 1) m c^2 \quad -(13)$$

and its relativistic total energy is:

$$E = \gamma m c^2, \quad -(14)$$

so

$$\boxed{T = E - E_0} \quad -(15)$$

where

$$E_0 = m c^2. \quad -(16)$$

The relativistic version of eq. (11) is therefore:

4)

$$\boxed{T = E - E_0 = \hbar\omega - \Phi} \quad - (17)$$

This equation can be compared with the equation of conservation of energy when a photon collides with an initially static electron in the Compton effect:

$$\hbar(\omega_1 - \omega_2) = E - mc^2 = E - E_0 \quad - (18)$$

in which ω_1 is the initial angular frequency of the photon, and ω_2 its final angular frequency. From eqs. (17) and (18) it is seen that both the Compton effect and the photoelectric effect involve consideration of $E - E_0$, the relativistic kinetic energy of one electron. In eq. (18) the electron is considered to be initially static ($p = \gamma m v = 0$), so its energy is its rest energy $mc^2 = E_0$. In a realistic experiment the electron is initially static in a material, so it is reasonable to assume

$$\boxed{\Phi = \alpha m c^2} \quad - (19)$$

where α is characteristic of the material. So a more realistic theory of the photoelectric effect is:

$$\boxed{\begin{aligned} \hbar(\omega_1 - \omega_2) &= E - \alpha m c^2, \quad - (20) \\ E &= (c^2 p^2 + m^2 c^4)^{1/2} \quad - (21) \end{aligned}}$$

This will be worked out in the next note.

157(11): Literature Search on Photon Mass

: Upper limits on photon mass are traditionally set by using experimental measurements on the Coulomb law, assumed to have a form:

$$F = \frac{e_1 e_2}{4\pi\epsilon_0} \cdot \frac{1}{r^{2+\epsilon}} \quad -(1)$$

It is often assumed that the potential has the "Yukawa" format $r^{-1}\exp(-\mu r)$ where:

$$\mu = \frac{mc}{\hbar} \quad -(2)$$

An upper limit on m is set by laboratory and gyromagnetic experiments. These give a limit on photon mass of

$$m < 1.6 \times 10^{-50} \text{ kgm} \quad -(3)$$

References

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4) J. D. Jackson, "Classical Electrodynamics", (Wiley, 3rd ed., 1999), page 7.
5) L.-C. Tu, J. Luo and G. T. Gillies, Rep. Prog. Phys., 68, 77 (2005).

Jackson suggests that at each resonance frequency.

2)

$$\boxed{m < \frac{\hbar\omega_0}{mc^2}} \quad -(4)$$

Presumably this means that when a photon is absorbed, its velocity and momentum are zero, so its energy is given by its rest energy. From study of Schumann resonances:

$$m \underset{\sim}{<} 6 \times 10^{-50}\ \mathrm{kg} \quad -(5)$$

and from surface measurements of the Earth's magnetic field:

$$m \underset{\sim}{<} 4 \times 10^{-51}\ \mathrm{kg} \quad -(6)$$

However, ref. (5) gives a photon mass of order 10^{-43} kg, while the theory of light deflection and time delay gives a photon mass of order 10^{-38} kg, if the energy of the light is $\hbar\omega$.

The latest estimate of photon mass from ref. (5) is about 10^{-52} kgm. Using this in the ECE theory of light deflection gives an effective energy of the light beam of

$$\langle E \rangle = 1.85 \times 10^{-38}\ \mathrm{J} \quad -(7)$$

much less than the energy of one photon at a frequency of 10^{16} radians per second (visible):

$$E = \hbar\omega \sim 1 \times 10^{-18}\ \mathrm{J}. \quad -(8)$$

There is also a lot of material and references to photon mass theory in "The Enigmatic Photon".

The effective energy $\langle E \rangle$ of the light beam could be calculated from the Planck distribution.

3) The energy of states is needed. The mean energy is:

$$\epsilon = \frac{dU}{dN} = \frac{8\pi h \nu^3}{c^3}\left(\frac{\exp(-h\nu/kT)}{1-\exp(-h\nu/kT)}\right)\left(\frac{d\nu}{dN}\right) \quad -(9)$$

where ν is the wavenumber, k is Boltzmann's constant, T the temperature, dN the number density of photons in the range ν to $\nu + d\nu$. The mean energy is

$$\epsilon = \frac{h\nu\, e^{-h\nu/kT}}{1-e^{-h\nu/kT}} \quad -(10)$$

$$\boxed{\langle E \rangle = \epsilon = \hbar\omega\left(\frac{e^{-\hbar\omega/kT}}{1-e^{-\hbar\omega/kT}}\right)} \quad -(11)$$

The Boltzmann constant is:

$$k = 1.38066 \times 10^{-23}\ \mathrm{J\,K^{-1}} \quad -(12)$$

If

$$\exp\left(-\frac{\hbar\omega}{kT}\right) \ll 1 \quad -(13)$$

then:

$$\langle E \rangle = \hbar\omega \exp\left(-\frac{\hbar\omega}{kT}\right) \quad -(14)$$

and:

$$\boxed{\frac{\langle E \rangle}{\hbar\omega} = \exp\left(-\frac{\hbar\omega}{kT}\right)} \quad -(15)$$

From eqs. (7) and (8):

4) $$\frac{\langle E \rangle}{\hbar\omega} = 1.85 \times 10^{-20} \quad - (16)$$

if $$m \sim 10^{-52} \text{ kgm.} \quad - (17)$$

So: $$\frac{\hbar\omega}{kT} = \log_e \left(0.54 \times 10^{20}\right)$$

$$= 2.3026 \log_{10} \left(0.54 \times 10^{20}\right)$$

$$= 2.3026 \left(\log_{10} 0.54 + \log_{10} 10^{20}\right)$$

$$\frac{\hbar\omega}{kT} = 46.668 \quad - (18)$$

So $$\boxed{T = 76{,}340 \text{ K}} \quad - (19)$$

This is in the experimental range between the temperature of the sun's photosphere, which is 5,800 K, and that of the sun's corona, which is $1 - 3 \times 10^{6}$ K.

$$\boxed{\text{So: } \langle E \rangle = \hbar\omega \left(\frac{e^{-\hbar\omega/(kT)}}{1 - e^{-\hbar\omega/(kT)}} \right) \quad \text{in the light deflection calculation.}} \quad - (20)$$

This gives a photon mass from light deflection of order 10^{-52} kg for $T \doteq 76{,}340$ K

157(12): Calculation of Photon Mass with $\langle E \rangle$

We have:

$$\langle E \rangle = \hbar\omega \left(\frac{e^{-\hbar\omega/kT}}{1 - e^{-\hbar\omega/kT}} \right) \quad -(1)$$

The photon mass (note 155(20)) is:

$$m = \frac{\langle E \rangle R_0}{c^2 a} \quad -(2)$$

where $R_0 = 6.955 \times 10^8$ metres.

The mass is smaller than the one calculated for $\hbar\omega$ by a factor of $\exp(-\hbar\omega/kT)$ to an excellent approximation. This gives:

T / K	$x = \hbar\omega/kT$	e^{-x}	m / kg
5,778	12.35	3.40×10^{-6}	8.22×10^{-44}
3,000	23.79	4.66×10^{-11}	1.13×10^{-48}
2,500	28.54	4.03×10^{-13}	9.74×10^{-52}
2,000	35.68	3.19×10^{-16}	7.72×10^{-54}

$\hbar\omega \sim 10^{-18}$ joules

$\omega \sim 10^{16}$ radians per second.

The temperature of 5,778 K is that of

2) the surface of the sun. The latest estimate of photon mass is $\lesssim 10^{-52}$ kg from:

L.-C. Tu, J. Luo and G. T. Gillies, Rep. Prog. Phys., 68, 77 (2005).

The estimate of photon mass from light deflection is obtained from:

$$\Delta\phi = 2\int_{R_0}^{\infty} \frac{1}{r^2}\left(\frac{1}{R_0^2} - \left(1 - \frac{r_0}{r}\right)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{-1/2} dr - \pi \quad -(3)$$

Using the experimental $\Delta\phi$, numerical evaluation of eq. (3) gives:

$$a = 3.37654478 22 \times 10^{11} \text{ metres} \quad -(4)$$

J. D. Jackson, "Classical Electrodynamics" gives:

$$m \lesssim 6 \times 10^{-50} \text{ kg} \quad -(4)$$

and from Schumann resonance,

$$m \lesssim 4 \times 10^{-51} \text{ kg} \quad -(5)$$

from surface measurements of the earth's magnetic field.

This range is given if the light beam is heated up to r_0 about 2,500 K.

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1)

157(13): Calculation of Photon Velocity from Light Deflection due to Gravitation.

Use the formula:

$$\frac{a}{R_0} = \frac{\langle \hbar\omega \rangle}{mc^2} = \gamma \quad -(1)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(2)$$

and

$$\langle \hbar\omega \rangle = \hbar\omega \left(\frac{e^{-x}}{1 - e^{-x}} \right) \quad -(3)$$

$$x = \frac{\hbar\omega}{kT}. \quad -(4)$$

From numerical evaluation:

$$a = 3 \cdot 3765447822 \times 10^{11} \text{ m} \quad -(5)$$

$$R_0 = 6 \cdot 955 \times 10^{8} \text{ m}. \quad -(6)$$

Use.

$$\omega = 10^{16} \text{ rad s}^{-1} \text{ (visible)}, \quad -(7)$$

$$\hbar = 1 \cdot 054 \times 10^{-34} \text{ Js}. \quad -(8)$$

For

$$T = 2,500 \text{ K} \quad -(9)$$

$$m = 9 \cdot 74 \times 10^{-52} \text{ kg} \quad -(10)$$

So using

$$c = 2 \cdot 9979 \times 10^{8} \text{ m s}^{-1}. \quad -(11)$$

then

$$\boxed{v = 2 \cdot 99757 \times 10^{8} \text{ m s}^{-1}} \quad -(12)$$

d)

The ratio:

$$\frac{\langle \hbar\omega \rangle}{mc^2} = 4 \cdot 855 \times 10^8 \quad -(13)$$

and it has been assumed that:

$$\langle \hbar\omega \rangle = \gamma mc^2. \quad -(14)$$

Eq. (14) means that the mean energy of one photon at frequency ω is γmc^2. The Planck postulate is that the energy of an oscillator at angular frequency ω is not continuously variable, but is restricted to $n\hbar\omega$ where n is an integer. This postulate leads to the Planck distribution (3).

The mean energy $\langle \hbar\omega \rangle$ is related to the intensity I of the beam in joules per square metre as follows:

$$\boxed{I = 8\pi \left(\frac{\nu}{c} \right)^2 \langle \hbar\omega \rangle} \quad -(15)$$

where

$$\omega = 2\pi\nu. \quad -(16)$$

So if the intensity can be measured experimentally, so can $\langle \hbar\omega \rangle$. From the Rayleigh / Jeans / Planck theory:

$$I = c \frac{dU}{d\nu} \quad -(17)$$

3) where the energy density (joules per cubic metre) in the range ω to $\omega + d\omega$ is:

$$dU = \rho(\omega)\, d\omega \qquad -(18)$$

and the density of states is:

$$\rho(\omega) = 8\pi\hbar \left(\frac{\omega}{c}\right)^3 \left(\frac{e^{-x}}{1-e^{-x}}\right) \qquad -(19)$$

We have:

$$\frac{dU}{d\omega} = \frac{8\pi\omega^2}{c^3}\langle \hbar\omega \rangle = 8\pi\hbar\left(\frac{\omega}{c}\right)^3 \left(\frac{e^{-x}}{1-e^{-x}}\right) \qquad -(20)$$

$$= \frac{8\pi\omega^2}{c^3}\langle \hbar\omega \rangle$$

From eqs. (17) and (20), eq. (15) follows.

Finally, the flux density in the range ω to $\omega + d\omega$ is

$$d\Phi = c\rho(\omega)\, d\omega \qquad -(21)$$

and is measured in watts per square metre $= J\, s^{-1}\, m^{-2}$ $-(22)$

The number density of photons in the range ω to $\omega + d\omega$ is:

$$dN = N(\omega + d\omega) - N(\omega) \qquad -(22)$$

in photons per cubic metre. Thus:

$$\boxed{\langle \hbar\omega \rangle = \frac{dU}{dN}} \qquad -(23)$$

4)

Therefore:

$$\boxed{\langle \hbar\omega \rangle = \gamma m c^2 = \frac{dU}{dN} = \frac{1}{8\pi}\left(\frac{c}{\omega}\right)^2 I} \quad - (24)$$

Eq. (24) relates the intensity I of a light beam in joules per square metre to $\gamma m c^2$:

$$I = 8\pi\omega^2\gamma m \quad - (25)$$

i.e.

$$\boxed{I = 8\pi\omega^2 m\left(1-\frac{v^2}{c^2}\right)^{-1/2}} \quad - (26)$$

The total energy density of the light beam is ($J m^{-3}$):

$$U = \int \frac{8\pi\hbar}{c^3}\left(\frac{\omega^3 \, d\omega}{e^{\hbar\omega/kT}-1}\right) \quad - (27)$$

and the total number of photons per cubic metre is:

$$N = \int \frac{8\pi}{c^3}\left(\frac{\omega^2 \, d\omega}{e^{\hbar\omega/kT}-1}\right) \quad - (28)$$

From eq. (17):

$$\boxed{U = I\frac{\omega}{c}} \quad - (29)$$

From eqs. (18) and (21) the power density or flux density of the light beam in watts per square metre

5) is

$$\Phi = cU = \omega I \quad \text{— (30)}$$

Therefore:

$$\boxed{\Phi = 8\pi \omega^3 m \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad \text{— (31)}}$$

watts per square metre or $J s^{-1} m^{-2}$

The power density Φ of the beam in watts per square metre is related to the magnetic flux density of the beam in tesla, $B^{(0)}$ and electric field strength of the beam in volts per metre, $E^{(0)}$:

$$\boxed{\Phi = \frac{c}{\mu_0} B^{(0)2} = \epsilon_0 c E^{(0)2}} \quad \text{— (32)}$$

S.I. Units

$B^{(0)}$ = tesla = $J s C^{-1} m^{-2}$

$E^{(0)}$ = volt m^{-1} = $J C^{-1} m^{-1}$

$\epsilon_0 = 8.854188 \times 10^{-12} J^{-1} C^2 m^{-1}$

$\mu_0 = 4\pi \times 10^{-7} J s^2 C^{-2} m^{-1}$

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \quad \text{— (33)}$$

$$E^{(0)} = c B^{(0)} \quad \text{— (34)}$$

$$\boxed{\Phi = \epsilon_0 c E^{(0)2} = \frac{c}{\mu_0} B^{(0)2} = 8\pi \omega^3 m \left(1 - \frac{v^2}{c^2}\right)^{-1/2}} \quad \text{— (35)}$$

1)

157(14): Photon Mass and de Broglie Einstein Equations

References

1) L. de Broglie, Compt. Rendus, 177, 507 (1923).
2) L. de Broglie, Phil. Mag., 47, 446 (1924).
3) www.aias.net / physics / umml2.pdf.
4) http: // th-www.if.uj.edu.pl / acta / vol37 / pdf / v37p0565.pdf
5) M. Planck, Ann. Phys., 309, 533 (1901)
6) ~~L. de~~ A. Einstein, ibid., 322, 132 (1905).

The de Broglie Einstein equations are:

$$E = \hbar\omega = \gamma mc^2 , \quad -(1)$$

$$p = \hbar\kappa = \gamma m v_g , \quad -(2)$$

$$\gamma = \left(1 - \frac{v_g^2}{c^2}\right)^{-1/2} . \quad -(3)$$

Where v_g is the group velocity. The latter is the velocity of the envelope of two or more waves. For two waves:

$$v_g = \frac{\Delta\omega}{\Delta\kappa} = \frac{\omega_2 - \omega_1}{\kappa_2 - \kappa_1} \quad -(4)$$

For many waves:

$$v_g = \frac{\partial\omega}{\partial\kappa} \quad -(5)$$

The phase velocity v_p is the average velocity of the waves in a wave packet:

$$v = \frac{\omega_{av}}{\kappa_{av}} . \quad -(6)$$

For n waves:

2) $$\omega_{av} = (\omega_1 + \omega_2 + \dots + \omega_n)/n \quad -(7)$$

$$\kappa_{av} = (\kappa_1 + \kappa_2 + \dots + \kappa_n)/n \quad -(8)$$

For one wave:

$$v_p = \frac{E}{p} = \frac{\omega}{\kappa}, \quad -(9)$$

so

$$\boxed{v_g v_p = c^2} \quad -(10)$$

This is discussed by J.D. Jackson "Classical Electrodynamics" (Wiley, 3rd ed 1999), pp. 324 ff.

The phase velocity is:

$$v_p = \frac{\omega(\kappa)}{\kappa} = \frac{c}{n(\kappa)} \quad -(11)$$

where $n(\kappa)$ is the index of refraction.

The group velocity is:

$$v_g = \frac{c}{n(\omega) + \omega\left(\frac{dn}{d\omega}\right)} \quad -(12)$$

From eqns. (10) (11) and (12):

$$v_p v_g = c^2 = \frac{c^2}{n\left(n + \omega\left(\frac{dn}{d\omega}\right)\right)} \quad -(13)$$

so

$$n\left(n + \omega\frac{dn}{d\omega}\right) = 1 \quad -(14)$$

3) In the non-relativistic limit:

$$v \ll v_p \quad -(15)$$

In this approximation:

$$\hbar\omega = mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} \quad -(16)$$

$$\sim mc^2 + \frac{1}{2} m v_g^2 ,$$

$$\hbar\kappa = \left(1-\frac{v_g^2}{c^2}\right)^{-1/2} m v_g$$

$$\sim m v_g\left(1+\frac{1}{2}\frac{v_g^2}{c^2}\right)$$

$$\hbar\kappa \sim m v_g + \frac{m}{2}\frac{v_g^3}{c^2} \quad -(17)$$

The <u>rest</u> frequency ω_0 and kinetic frequency <u>ω_K</u> were defined by de Broglie as:

$$\boxed{\begin{aligned} \hbar\omega_0 &= mc^2 \quad -(18) \\ \hbar\omega_K &\sim \frac{1}{2} m v_g^2 \quad -(19) \end{aligned}}$$

So

$$\boxed{\omega = \omega_0 + \omega_K} \quad -(20)$$

The rest wavenumber κ_0 and kinetic wavenumber κ_K were defined by de Broglie as:

$$\boxed{\begin{aligned} \hbar\kappa_0 &= m v_g \quad -(21) \\ \hbar\kappa_K &= \frac{1}{2} m \frac{v_g^3}{c^2} \quad -(22) \end{aligned}}$$

4) The kinetic energy of the photon was defined by omitting the rest frequency:

$$T = \hbar\omega_\kappa \sim \frac{1}{2} m v_g^2 \quad -(23)$$

$$= \frac{p^2}{2m}$$

where

$$p = m v_g \quad -(24)$$

From eqns (10) and (23):

$$T = \hbar \kappa_\kappa v_p = \frac{1}{2} m v_g^2 \quad -(25)$$

In the non-relativistic approximation:

$$\hbar\omega_0 = mc^2, \quad \hbar\kappa_0 \sim m v_g \quad -(26)$$

so

$$\boxed{\frac{\omega_0}{\kappa_0} \sim \frac{c^2}{v_g} = v_p} \quad -(27)$$

and

$$\hbar\omega_\kappa \sim \frac{1}{2} m v_g^2$$

$$\hbar\kappa_\kappa \sim \frac{1}{2} \frac{m v_g^3}{c^2}$$

$$\boxed{\frac{\omega_\kappa}{\kappa_\kappa} = \frac{c^2}{v_g} = v_p} \quad -(28)$$

So

$$\boxed{\frac{\omega_0}{\kappa_0} = \frac{\omega_\kappa}{\kappa_\kappa} = v_p} \quad -(29)$$

5) It follows that

$$\boxed{\frac{\omega}{\kappa} = \frac{\omega_o + \omega_\kappa}{\kappa_o + \kappa_\kappa} = v_p} \quad -(30)$$

The phase velocity is the same.

From eqns. (21) and (23):

$$\left.\begin{aligned} \hbar\kappa_o &= m v_g \\ \hbar\omega_\kappa &= \frac{1}{2} m v_g^2 \end{aligned}\right\} \quad -(31)$$

and

$$\frac{\kappa_o}{\omega_\kappa} = \frac{2}{v_g} \quad -(32)$$

so

$$\frac{\omega_\kappa}{\kappa_o} = \frac{v_g}{2} \quad -(33)$$

A possible solution of eq. (30) is:

$$\frac{\omega_\kappa}{\kappa_o} = \frac{\omega_o}{\kappa_\kappa} = v_p \quad (34)$$

so

$$\boxed{v_g = 2 v_p} \quad -(35)$$

If eq. (35) is used in eqs. (11) and (12) it is found that:

$$\frac{c}{n(\omega) + \omega \frac{dn}{d\omega}} = 2\frac{c}{n(\omega)}, \quad -(36)$$

i.e.

$$\boxed{n(\omega) = -2\omega \frac{dn(\omega)}{d\omega}} \quad -(37)$$

6) The refractive index obeys the differential equation:

$$\boxed{n(\omega) + 2\omega \frac{dn(\omega)}{d\omega} = 0} \quad -(38)$$

This means that the refractive index is a function of frequency, even when light travels in free space. Eq. (38) is

$$\frac{dn(\omega)}{d\omega} = -\frac{n(\omega)}{2\omega} \quad -(39)$$

i.e.

$$\int \frac{dn}{n} = -\int \frac{d\omega}{2\omega} \quad -(40)$$

or

$$\log_e n = -\frac{1}{2}\log_e \omega + \log_e C \quad -(41)$$

where C is a constant of integration. Therefore:

$$n(\omega) = \frac{C}{\omega^{1/2}} \quad -(42)$$

If

$$C = \omega_c^{1/2} \quad -(43)$$

then

$$\boxed{n(\omega) = \left(\frac{\omega_c}{\omega}\right)^{1/2}} \quad -(44)$$

where ω_c is a characteristic frequency. If

$$\omega < \omega_c \quad -(45)$$

there is a red shift, and

$$n(\omega) > 1 \quad -(46)$$