) 155(7): Relativistic Hamilton Jacobi Equation

In the space with torsion and curvature this is:

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = m^2 c^2 \quad -(1)$$

where S is the action and where:

$$p_\mu = \frac{\partial S}{\partial x^\mu}, \quad p_\nu = \frac{\partial S}{\partial x^\nu} \quad -(2)$$

Therefore $g^{\mu\nu} p_\mu p_\nu = m^2 c^2 \quad -(3)$

i.e. $p^\mu p_\mu = m^2 c^2 \quad -(4)$

because $p^\mu = g^{\mu\nu} p_\nu \quad -(5)$

by definition.

The Hamiltonian is conserved under all conditions and is:

$$\boxed{H = \frac{1}{2m} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = \frac{1}{2} mc^2} \quad -(6)$$

Eq. (6) is a relation between the inverse metric $g^{\mu\nu}$ and the Hamiltonian. The metric $g_{\mu\nu}$ and inverse metric are defined by:

2)

$$g^{\mu\nu} g_{\mu\nu} = 4. \quad -(7)$$

The Hamiltonian is also defined in terms of the line element ds^2 as in previous work. If the line element of spherical spacetime is chosen:

$$ds^2 = c^2 d\tau^2 = c^2 x(r,t)\,dt^2 - y(r,t)\,dr^2 - r^2 d\phi^2 \quad -(8)$$

then:

$$H = \frac{1}{2} mc^2 = \frac{1}{2} m \left(x c^2 \left(\frac{dt}{d\tau}\right)^2 - y \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \right). \quad -(9)$$

so

$$\boxed{p_1^{\mu} p_{1\mu} = m^2 \left(x c^2 \left(\frac{dt}{d\tau}\right)^2 - y \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \right)} \quad -(10)$$

By definition:

$$p_1^{\mu} p_{1\mu} = \frac{E_1^2}{c^2} - \underline{p_1} \cdot \underline{p_1}, \quad -(11)$$

where

$$E_1^2 = E_0^2 / x, \quad -(12)$$

$$p_1^2 = p_r^2 / y + L^2 / r^2, \quad -(13)$$

$$E_0 = x m c^2 \, dt / d\tau, \quad -(14)$$

$$\underline{p_r} = m y \frac{dr}{d\tau} \underline{e}_r. \quad -(15)$$

) For the free particle of mass m in Minkowski spacetime:

$$p^\mu p_\mu = m^2\gamma^2\left(c^2 - \left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right)\right)$$

$$= m^2\gamma^2\left(c^2 - v^2\right) \quad -(16)$$

where

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad -(17)$$

Here

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 \quad -(18)$$

so

$$E = \gamma m c^2, \quad p = \gamma m v. \quad -(19)$$

If the particle interacts with the electromagnetic field:

$$H = \frac{1}{2m} p_0^\mu p_{0\mu} \rightarrow \frac{1}{2m}\left(p^\mu + eA^\mu\right)\left(p_\mu + eA^\mu\right) \quad -(20)$$

$$= \frac{1}{2} m c^2$$

This equation means that the Hamiltonian is conserved and is an invariant, so the interaction with the electromagnetic field changes the four momentum p_0^μ to p^μ according to the law of conservation of energy / momentum. This same interaction is described in general relativity by eqn. (9), i.e. by changing the Minkowski spacetime to a spacetime with torsion and curvature.

So:

$$H = \frac{1}{2}mc^2 = \frac{1}{2}m\left(xc^2\left(\frac{dt}{d\tau}\right)^2 - y\left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\phi}{dt}\right)^2\right)$$

$$= \frac{1}{2m}\left(\frac{(E+e\phi)^2}{c^2} - (p+eA)^2\right) \quad -(21)$$

A possible solution is:

$$(E+e\phi)^2 = m^2c^4x\left(\frac{dt}{d\tau}\right)^2 = \frac{E_1^2}{x} \quad -(22)$$

$$(p+eA)^2 = m^2\left(y\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right) \quad -(23)$$

Here:

$$\left.\begin{array}{l} E_1 = mc^2x\left(\frac{dt}{d\tau}\right), \quad p_1 = my\left(\frac{dr}{d\tau}\right) \\ L_1 = mr^2\left(\frac{d\phi}{d\tau}\right) \end{array}\right\} \quad -(24)$$

So:

$$\boxed{\begin{array}{l} (E+e\phi)^2 = E_1^2/x \\ (p+eA)^2 = p_1^2/y + \frac{L_1^2}{mr^2} \end{array}} \quad -(25)$$

Deduction

The minimal prescription and general relativity can be made to be an equivalent procedure using eq. (25). This procedure can be extended to the gravitational potential and combined potential of the four fundamental forces