1)

155(12): Binomial Approximation for Photon Mass from Light Deflection due to Gravitation.

The fundamental integral is:

$$\phi = \int \frac{1}{r^2}\left(\frac{1}{b^2} - \left(1-\frac{r_0}{r}\right)\left(\frac{1}{a^2}+\frac{1}{r^2}\right)\right)^{-1/2} dr \qquad -(1)$$

where

$$a = \frac{L}{mc}, \quad b = \frac{cL}{E} \qquad -(2)$$

are constants of motion. Here:

$$\left.\begin{aligned} m &= \text{mass of the photon}, \\ r_0 &= \frac{2MG}{c^2} \end{aligned}\right\} \qquad -(3)$$

where M is the mass of the sun.

The conserved photon total energy is:

$$E = mc^2\left(1-\frac{r_0}{r}\right)\left(\frac{dt}{d\tau}\right). \qquad -(4)$$

The conserved orbital angular momentum of the photon is:

$$L = mr^2\frac{d\phi}{d\tau}. \qquad -(5)$$

The constant c is the S.I. universal constant of the standards laboratories:

$$c = 2.997925 \times 10^{8}\ m\,s^{-1} \qquad -(6)$$

and is the standard laboratories is taken as fixed. In eq. (6) it is given only to its first six decimal places. Finally G is Newton's constant.

2) In the case of

$$m = 0, \quad M = 0 \qquad -(7)$$

$$\phi_0 = 2\int_{R_0}^{\infty} \frac{1}{r^2}\left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{-1/2} dr \qquad -(8)$$

$$= \pi$$

and there is no deflection because $\pi = 180^\circ$, and this may be regarded as a straight line. In eq. (8) R_0 is the distance of closest approach. Making the change of variable:

$$u = \frac{1}{r} \qquad -(9)$$

$$\phi_0 = 2\int_{0}^{1/R_0} \left(\frac{1}{b^2} - u^2\right)^{-1/2} du \qquad -(10)$$

$$= \pi = 2\sin^{-1}\frac{b}{R_0}.$$

So:

$$\boxed{b = R_0} \qquad -(11)$$

Eq. (10) is Wald's eq. (6.3.39) with, in his notation:

$$M = 0. \qquad -(12)$$

Now define the light deflection from the straight line as:

$$\Delta\phi = \phi_0 - \phi \qquad -(13)$$

3) where:

$$\phi = 2\int_{R_0}^{\infty} \frac{1}{r^2}\left(\frac{1}{b^2} - \left(1 - \frac{r_0}{r}\right)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{-1/2} dr \quad -(14)$$

So:

$$\Delta\phi = 2\int_{R_0}^{\infty} \frac{1}{r^2}\left[\left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{-1/2} - \left(\frac{1}{b^2} - \left(1 - \frac{r_0}{r}\right)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{-1/2}\right] dr \quad -(15)$$

$$= 2\int_{R_0}^{\infty} \frac{1}{r^2}\left[\frac{1}{\left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{1/2}} - \frac{1}{\left(\frac{1}{b^2} - \frac{1}{r^2} - \left(1 - \frac{r_0}{r}\right)\frac{1}{a^2} + \frac{r_0}{r^3}\right)^{1/2}}\right] dr \quad -(16)$$

Let

$$x^2 := \frac{1}{b^2} - \frac{1}{r^2} \quad -(17)$$

$$y^2 := \frac{r_0}{r^3} - \left(1 - \frac{r_0}{r}\right)\frac{1}{a^2}.$$

So:

$$\Delta\phi = 2\int_{R_0}^{\infty} \frac{1}{r^2}\left(\frac{1}{x} - \frac{1}{(x^2+y^2)^{1/2}}\right) dr$$

$$= 2\int_{R_0}^{\infty} \frac{1}{r^2 x}\left(1 - \left(\frac{x^2}{x^2+y^2}\right)^{1/2}\right) dr$$

$$= 2\int_{R_0}^{\infty} \frac{1}{r^2 x}\left(1 - \left(1 + \left(\frac{y}{x}\right)^2\right)^{-1/2}\right) dr$$

$$\boxed{\Delta\phi = 2\int_{R_0}^{\infty} \frac{1}{r^2 x}\left(1 - \left(1 + \alpha\right)^{-1/2}\right) dr}, \quad -(18)$$

where

$$\alpha = \left(\frac{y}{x}\right)^2 = \left(\frac{r_0}{r^3} - \left(1 - \frac{r_0}{r}\right)\frac{1}{a^2}\right)\left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{-1}. \quad -(19)$$

4) Experimentally: $\alpha \ll 1$. $-(20)$

Using the binomial expansion:

$$(1+\alpha)^{-1/2} = 1 - \frac{\alpha}{2} + \frac{3}{8}\alpha^2 - \frac{5}{16}\alpha^3 + \dots \quad -(21)$$

so

$$\Delta\phi = 2\int_{R_0}^{\infty} \frac{1}{r^2 x}\left(\frac{\alpha}{2} - \frac{3}{8}\alpha^2 + \frac{5}{16}\alpha^3 - \dots\right) dr \quad -(22)$$

To first order in α:

$$\boxed{\Delta\phi \sim \int_{R_0}^{\infty} \frac{\alpha}{r^2 x}\, dr} \quad -(23)$$

where:

$$\frac{\alpha}{x} = \left(\frac{r_0}{r^3} - \left(1 - \frac{r_0}{r}\right)\frac{1}{a^2}\right)\left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{-3/2} \quad -(24)$$

In this calculation, b is fixed by eq. (11):

$$b = \frac{cL}{E} = R_0, \quad -(25)$$

because we are not using Einstein's assumption of a circular photon orbit.

The photon energy is:

$$E = \hbar\omega \quad -(26)$$

So:

$$L = \frac{\hbar\omega}{c} R_0 \qquad -(27)$$

and

$$q = \frac{L}{mc} = \left(\frac{\hbar\omega}{mc^2}\right) R_0 \qquad -(28)$$

The photon mass may therefore be determined from eq. (23) with:

$$\Delta\phi = 1.75 \text{ arc seconds} = 8.484 \times 10^{-6} \text{ radians}$$

$$R_0 = 6.955 \times 10^{8} \text{ metres}$$

$$M = 1.989 \times 10^{30} \text{ kg}$$

$$G = 6.6726 \times 10^{-11} \, m^3 \, kg^{-1} \, s^{-2}$$

$$\hbar = 1.05459 \times 10^{-34} \text{ Js.}$$

Assume

$$\omega \sim 10^{16} \text{ rad. } s^{-1} \text{ (visible range).}$$

If it is assumed to an order of magnitude approximation only that eqn. (23) gives:

$$\Delta\phi \sim \left(\frac{mc^2}{\hbar\omega}\right)^2 \qquad -(29)$$

then

$$\boxed{m \sim 10^{-38} \text{ kilograms}} \qquad -(30)$$