155(3): Conservation of Energy / Momentum in Field Particle Interaction in General Relativity

For spherical spacetime, consider the general metric:

$$ds^2 = c^2 d\tau^2 = x(r,t)c^2 dt^2 - y(r,t)dr^2 - r^2 d\phi^2 \quad -(1)$$

in cylindrical polar coordinates in the XY plane. The hamiltonian is conserved under all conditions:

$$H = \frac{1}{2}mc^2 = \frac{1}{2}m\left(xc^2\left(\frac{dt}{d\tau}\right)^2 - y\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2\right) \quad -(2)$$

The total energy is conserved and is:

$$E = mc^2 x \, dt/d\tau. \quad -(3)$$

The radial momentum is conserved and is:

$$\underline{p}_r = my\frac{dr}{d\tau}\underline{e}_r \quad -(4)$$

The angular momentum is conserved and is:

$$\underline{L} = mr^2\frac{d\phi}{d\tau}\underline{e}_\phi \quad -(5)$$

Eq. (2) is therefore:

$$\boxed{E_1^2 = m^2c^4 + c^2 p_1^2} \quad -(6)$$

where

$$E_1^2 = E^2/x, \quad -(7)$$

$$p_1^2 = p_r^2/y + L^2/r^2 \quad -(8)$$

2) Eq. (6) shows that the Einstein energy equation has the same format in general relativity, but the definition of the total energy and of the radial momentum are changed. In special relativity:

$$E_0 = mc^2 \left(\frac{dt}{d\tau}\right)_0 = \gamma mc^2 \quad -(9)$$

$$\underline{p}_{r0} = m\left(\frac{dr}{d\tau}\right)_0 \underline{e}_r = \gamma m \frac{dr}{dt}\underline{e}_r \quad -(10)$$

$$\underline{L}_0 = mr^2\left(\frac{d\phi}{d\tau}\right)_0 \underline{e}_\phi = \gamma m r^2 (d\phi/dt)\, \underline{e}_\phi \quad -(11)$$

so:

$$\boxed{E_0^2 = m^2c^4 + c^2 p_0^2} \quad -(12)$$

where

$$p_0^2 = p_{r0}^2 + L_0^2/r^2 \quad -(13)$$

and the metric of special relativity is the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(14)$$

and:

$$H = \frac{1}{2}mc^2 = \frac{1}{2}m\gamma^2\left(c^2 - \left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\phi}{dt}\right)^2\right) \quad -(15)$$

Note that the hamiltonian H is the same in special and general relativity, being half the rest energy.

The following is a convenient summary of the dynamical quantities involved. These are defined in cylindrical polar coordinates. If the position vector in Cartesian coordinates is:

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(16)$$

Then:

$$\underline{v} = \dot{x}\,\underline{i} + \dot{y}\,\underline{j} + \dot{z}\,\underline{k} \quad -(9)$$

$$= \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi + \dot{z}\,\underline{k}. \quad -(10)$$

Here, the unit vectors of the cylindrical polar system are:

$$\underline{e}_r = \cos\phi\,\underline{i} + \sin\phi\,\underline{j} \quad -(10)$$

$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j} \quad -(11)$$

$$\underline{e}_z = \underline{k} \quad -(12)$$

So:

$$\underline{r} = r\,\underline{e}_r + z\,\underline{e}_z \quad -(13)$$

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r + \dot{z}\,\underline{e}_z$$

$$= \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi + \dot{z}\,\underline{e}_z$$

$$\boxed{\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi} \quad -(14)$$

if

$$\underline{e}_z = \underline{0}. \quad -(15)$$

So from eq. (2), in general relativity:

$$\boxed{\begin{aligned} E_1 &= m c^2 x \frac{dt}{d\tau}, \\ \underline{p}_1 &= m\left(\gamma \frac{dr}{d\tau}\,\underline{e}_r + r\frac{d\phi}{d\tau}\,\underline{e}_\phi\right) \\ p_1^2 &= m^2\left(\gamma^2\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right) \end{aligned}} \quad -(16)$$

4)

In special relativity

$$\boxed{\begin{aligned} E_0 &= \gamma m c^2 \\ \underline{p}_0 &= m\gamma\left(\frac{dr}{dt}\,\underline{e}_r + r\frac{d\phi}{dt}\,\underline{e}_\phi\right) \\ p_0^2 &= m^2\gamma^2\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) \end{aligned}} \quad -(17)$$

Eqs. (17) are for the free particle, eqs. (18) are for field particle interaction.

$$\boxed{E_0^2 = c^2 p_0^2 + m^2 c^4} \quad \text{free particle}$$

$$\downarrow$$

$$\boxed{E_1^2 = c^2 p_1^2 + m^2 c^4} \quad \text{field/particle}$$

In ECE theory the field is defined by the first Cartan structure equation:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \Omega^a_{\mu\nu} \quad -(18)$$

where

$$\Omega^a_{\mu\nu} = \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(19)$$

$$= \omega^a_{\mu\nu} - \omega^a_{\nu\mu} \quad -(20)$$

Summing up over a:

$$T_{\mu\nu} - \Omega_{\mu\nu} = \partial_\mu q_\nu - \partial_\nu q_\mu \quad -(21)$$

$$= \tau_{\mu\nu}$$

Gravitational Field

This is defined by:

$$g_{\mu\nu} = \Phi \, \tau_{\mu\nu} \quad -(22)$$

where Φ is a scalar valued magnitude of the gravitational potential defined by:

$$\Phi_\mu = \Phi \, q_\mu \quad -(23)$$

$$= \left(\Phi_0, - \underline{\Phi} \right)$$

In vector notation:

$$\boxed{\underline{g} = - \underline{\nabla} \, \Phi - \frac{1}{c} \frac{\partial \underline{\Phi}}{\partial t}} \quad -(24)$$

By antisymmetry:

$$\underline{\nabla} \, \Phi = \frac{1}{c} \frac{\partial \underline{\Phi}}{\partial t} \quad -(25)$$

Electromagnetic Field

This is defined by:

$$F_{\mu\nu} = A \, \tau_{\mu\nu} \quad -(26)$$

where A is a scalar valued magnitude of the electromagnetic potential defined by:

6)

$$A_\mu = A q_\mu = \left(\frac{\phi}{c}, -\underline{A} \right) \quad -(27)$$

So:

$$\boxed{\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}} \quad -(28)$$

By antisymmetry:

$$\underline{\nabla}\phi = \frac{\partial \underline{A}}{\partial t} \quad -(29)$$

Thus:

$$\boxed{\begin{aligned} g_{\mu\nu} &= \Phi\left(T_{\mu\nu} - \Omega_{\mu\nu}\right) = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu, \\ F_{\mu\nu} &= A\left(T_{\mu\nu} - \Omega_{\mu\nu}\right) = \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned}} \quad -(30)$$

<u>The potentials are defined by the difference between the Cartan torsion and spin connection.</u>

In special relativity, field particle interaction is described by the minimal prescription:

$$p^\mu \rightarrow p^\mu + eA^\mu \quad -(31)$$

i.e.

$$E \rightarrow E + e\phi \quad -(32)$$

$$\underline{p} \rightarrow \underline{p} + e\underline{A} \quad -(33)$$

In special relativity, eq. (12) is:

$$H = \frac{1}{2m} p^\mu p_\mu = \frac{1}{2} mc^2 \quad -(34)$$

7) where:
$$p^{\mu} = \left(\frac{E_0}{c}, \underline{p_0} \right) \quad -(35)$$
$$p_{\mu} = \left(\frac{E_0}{c}, -\underline{p_0} \right) \quad -(36)$$

for the free particle.

Field particle interaction is therefore described by the Hamilton Jacobi equation:

$$H = \frac{1}{2m}\left(p^{\mu} + eA^{\mu} \right)\left(p_{\mu} + eA_{\mu} \right) = \frac{1}{2} mc^2 \quad -(35)$$
$$= \frac{1}{2m} p_1^{\mu} \, p_{1\mu}$$

Since H is the same in eqns. (34) and (35),

$$\boxed{p^{\mu} p_{\mu} = p_1^{\mu} \, p_{1\mu} = m^2 c^2} \quad -(36)$$

From eqn. (16)

$$p_1^{\mu} = \left(\frac{E_1}{c}, -\underline{p_1} \right) \quad -(37)$$
$$= \left(\frac{1}{c}\left(E + e\phi \right), -\left(\underline{p} + e\underline{A} \right) \right) \quad -(38)$$

$$\boxed{\begin{aligned} \text{i.e.} \quad E_1 &= E + e\phi = mc^2 \times dt/d\tau \\ \underline{p_1} &= \underline{p} + e\underline{A} = m\left(\gamma \frac{dr}{d\tau} \underline{e}_r + r \frac{d\phi}{d\tau} \underline{e}_\phi \right) \end{aligned}}$$