154(1): Tetrads from Metrical Analysis.

The aim of this note is to derive tetrads self-consistently from a metrical analysis, and from the tetrads derive wave and field equations. The general equation linking the metric and tetrad is

$$g_{\mu\nu} = q^a_{\ \mu}\, q^b_{\ \nu}\, \eta_{ab} \quad -(1)$$

where η_{ab} is the Minkowski metric:

$$\eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(2)$$

Restricting attention to the plane XY and to the gravitational metric (the misnamed Schwarzschild metric):

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_0}{r}\right) c^2 dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \quad -(3)$$

then

$$g_{\mu\nu} = \begin{bmatrix} 1 - \frac{r_0}{r} & 0 & 0 \\ 0 & -\left(1 - \frac{r_0}{r}\right)^{-1} & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad -(4)$$

and

$$g_{00} = q^a_{\ 0}\, q^b_{\ 0}\, \eta_{ab}$$
$$= q^0_{\ 0} q^0_{\ 0} \eta_{00} + q^1_{\ 0} q^1_{\ 0} \eta_{11} + q^2_{\ 0} q^2_{\ 0} \eta_{22}, \quad -(5)$$

$$g_{11} = q^a_{\ 1}\, q^b_{\ 1}\, \eta_{ab}$$
$$= q^0_{\ 1} q^0_{\ 1} \eta_{00} + q^1_{\ 1} q^1_{\ 1} \eta_{11} + q^2_{\ 1} q^2_{\ 1} \eta_{22} \quad -(6)$$

$$g_{22} = q^a_{\ 2}\, q^b_{\ 2}\, \eta_{ab}$$
$$= q^0_{\ 2} q^0_{\ 2} \eta_{00} + q^1_{\ 2} q^1_{\ 2} \eta_{11} + q^2_{\ 2} q^2_{\ 2} \eta_{22} \quad -(7)$$

2).

Therefore:

$$g_{00} = 1 - \frac{r_0}{r} = q^0_0 q^0_0 - q^1_0 q^1_0 - q^2_0 q^2_0 \quad -(8)$$

$$g_{11} = -\left(1 - \frac{r_0}{r}\right)^{-1} = q^0_1 q^0_1 - q^1_1 q^1_1 - q^2_1 q^2_1 \quad -(9)$$

$$g_{22} = -1 = q^0_2 q^0_2 - q^1_2 q^1_2 - q^2_2 q^2_2 \quad -(10)$$

There are nine unknowns and three equations. In order to solve for the nine unknowns use the definition of the inverse of the tetrad matrix q^a_μ. If the inverse of the tetrad matrix is denoted by q^μ_a then:

$$q^a_\mu q^\mu_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(11)$$

In definition (11), summation is <u>not</u> implied over repeated indices. The inverse q^μ_a is defined as for any well defined inverse matrix:

$$q^\mu_a = \mathrm{adj}\left(q^a_\mu\right) / \left| q^a_\mu \right| \quad -(12)$$

where adj denotes "adjoint" and $|\ |$ denotes the determinant of the matrix. So:

$$\begin{bmatrix} q^0_0 & q^0_1 & q^0_2 \\ q^1_0 & q^1_1 & q^1_2 \\ q^2_0 & q^2_1 & q^2_2 \end{bmatrix} \begin{bmatrix} q^0_0 & q^1_0 & q^2_0 \\ q^0_1 & q^1_1 & q^2_1 \\ q^0_2 & q^1_2 & q^2_2 \end{bmatrix} = \left| q^a_\mu \right| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(13)$$

in which:

3)

$$|q| = q^0_0\left(q^1_1 q^2_2 - q^1_2 q^2_1\right) - q^0_1\left(q^1_0 q^2_2 - q^1_2 q^2_0\right) + q^0_2\left(q^1_0 q^2_1 - q^1_1 q^2_0\right) \quad -(14)$$

So in addition to eqs. (5) to (7), eqn (14) gives nine other equations. So there are nine unknowns and a total of thirteen equations. Denote $q = |q^a_\mu|$.

This system can be solved by computer algebra.

The nine equations from eq. (14) are:

$$q^0_0 q^0_0 + q^0_1 q^0_1 + q^0_2 q^0_2 = |q| \quad -(15)$$

$$q^1_0 q^1_0 + q^1_1 q^1_1 + q^1_2 q^1_2 = |q| \quad -(16)$$

$$q^2_0 q^2_0 + q^2_1 q^2_1 + q^2_2 q^2_2 = |q| \quad -(17)$$

$$q^0_0 q^1_0 + q^0_1 q^1_1 + q^0_2 q^1_2 = 0 \quad -(18)$$

$$q^0_0 q^2_0 + q^0_1 q^2_1 + q^0_2 q^2_2 = 0 \quad -(19)$$

$$q^1_0 q^0_0 + q^1_1 q^0_1 + q^1_2 q^0_2 = 0 \quad -(20)$$

$$q^1_0 q^2_0 + q^1_1 q^2_1 + q^1_2 q^2_2 = 0 \quad -(21)$$

$$q^2_0 q^1_0 + q^2_1 q^1_1 + q^2_2 q^1_2 = 0 \quad -(22)$$

$$q^2_0 q^0_0 + q^2_1 q^0_1 + q^2_2 q^0_2 = 0 \quad -(23)$$

By solving this system of equations using matrix and computer algebra packages, the nine tetrads can be worked out and used to work out the connections and field equations from the metric.

4) If the system is considered to be radially symmetric and the angular parts averaged out, we may consider:

$$\underline{dr} \cdot \underline{dr} = dr^2 \quad -(24)$$

$$ds^2 = c^2 dt^2 - dr^2 \quad -(25)$$

for the Minkowski metric, and:

$$\underline{dr} \cdot \underline{dr} = \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 \quad -(26)$$

$$ds^2 = \left(1 - \frac{r_0}{r}\right) c^2 dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 \quad -(27)$$

for the gravitational metric. Then:

$$q^a_\mu = \begin{bmatrix} q^0_0 & q^0_1 \\ q^1_0 & q^1_1 \end{bmatrix}, \quad \mathrm{adj}(q^a_\mu) = \begin{bmatrix} q^0_0 & q^1_0 \\ q^0_1 & q^1_1 \end{bmatrix}, \quad -(28)$$

$$|q^a_\mu| = q^0_0 q^1_1 - q^1_0 q^0_1 .$$

So:

$$\begin{bmatrix} q^0_0 & q^0_1 \\ q^1_0 & q^1_1 \end{bmatrix} \begin{bmatrix} q^0_0 & q^1_0 \\ q^0_1 & q^1_1 \end{bmatrix} = |q^a_\mu| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \quad -(29)$$

If

$$q := |q^a_\mu| \quad -(30)$$

then

$$q^0_0 q^0_0 + q^0_1 q^0_1 = q \quad -(31)$$

$$q^1_0 q^1_0 + q^1_1 q^1_1 = q \quad -(32)$$

$$q^0_0 q^1_0 + q^0_1 q^1_1 = 0 \quad -(33)$$

$$q^1_0 q^0_0 + q^1_1 q^0_1 = 0 \quad -(34)$$

$$q = q^0_0 q^1_1 - q^1_0 q^0_1 \quad -(35)$$

5)

$$g_{00} = 1 - \frac{r_0}{r} = q^0_0 q^0_0 - q^1_0 q^1_0 \quad -(36)$$

$$g_{11} = \left(1 - \frac{r_0}{r}\right)^{-1} = q^0_1 q^0_1 - q^1_1 q^1_1 \quad -(37)$$

A possible solution is:

$$g_{00} = q^0_0 q^0_0 \quad -(38)$$

$$g_{11} = -q^1_1 q^1_1 \quad -(39)$$

$$q^0_1 = 0 \quad -(40)$$

$$q^1_0 = 0 \quad -(41)$$

So: $$q^0_0 = \left(1 - \frac{r_0}{r}\right)^{1/2} \quad -(42)$$

$$q^1_1 = \left(1 - \frac{r_0}{r}\right)^{-1/2} \quad -(43)$$

as in previous ECE papers

If the general spherical metric is used:

$$ds^2 = e^{-r_0/r} c^2 dt^2 - e^{r_0/r} dr^2 \quad -(44)$$

if $$\langle d\phi^2 \rangle = 0, \quad -(45)$$

so $$g_{00} = e^{-r_0/r}, \quad g_{11} = -e^{r_0/r} \quad -(46)$$

$$q^0_0 = e^{-r_0/(2r)}, \quad q^1_1 = e^{r_0/(2r)} \quad -(47)$$

6) and as in note 153(2):

$$\Gamma^0_{10} - \omega^0_{10} = \frac{r_0}{2r^2} \exp\left(-\frac{r_0}{2r}\right) \quad -(36)$$

$$\Gamma^1_{11} - \omega^1_{11} = -\frac{r_0}{2r^2} \exp\left(\frac{r_0}{2r}\right) \quad -(37)$$

$$\left(\Box + R_0\right) q^0_0 = 0 \quad -(38)$$

$$\left(\Box + R_1\right) q^1_1 = 0 \quad -(39)$$

$$R_0 = \frac{r_0}{r^3}\left(\frac{r_0}{4r} - 1\right) \quad (40)$$

$$R_1 = \frac{r_0}{r^3}\left(\frac{r_0}{4r} + 1\right) \quad -(41)$$

$$R = R_0 + R_1 = \frac{1}{2}\left(\frac{r_0}{r^2}\right)^2 \quad -(42)$$

One may also consider <u>phase dependent</u>

<u>solutions</u>:

$$q^0_0 = \exp\left(-\frac{r_0}{2r}\right) \exp\left(i\left(\omega t - \kappa r\right)\right) \quad -(43)$$

$$q^{0\,*}_0 = \exp\left(-\frac{r_0}{2r}\right) \exp\left(-i\left(\omega t - \kappa r\right)\right) \quad -(44)$$

$$q^1_1 = q^{1\,*}_1 = \exp\left(\frac{r_0}{2r}\right) \exp\left(i\left(\omega t - \kappa r\right)\right) \quad -(45)$$

leading to <u>wave equations</u> of ~~ff~~ <u>spherical spacetime</u>