1) 154(3): Tetrads and Metrics in the Cylindrical Polar System

In this case the tetrads are defined by superimposing the cylindrical polar coordinates on the Cartesian coordinates. In this note, attention is restricted to three space dimensions for simplicity. It is shown that without consideration of phase, the metric of the cylindrical polar system is:

$$\eta_{(a)(b)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{diag}(1, 1, 1) \quad -(1)$$

and this is the same as the Cartesian metric:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{diag}(1, 1, 1) \quad -(2)$$

The two systems are related by:

$$\boxed{g_{\mu\nu} = q^{(a)}_{\mu} q^{(b)}_{\nu} \eta_{(a)(b)}} \quad -(3)$$

The system is defined by:

$$\begin{bmatrix} \underline{e}_r \\ \underline{e}_\phi \\ \underline{e}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(4)$$

and

$$\begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{e}_r \\ \underline{e}_\phi \\ \underline{e}_z \end{bmatrix} \quad -(5)$$

Therefore:

$$q^{(a)}_{\mu} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(6)$$

$$q^{\mu}_{(a)} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(7)$$

2) Therefore:

$$q^{(a)}_{\mu}\, q^{\mu}_{(a)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(8)$$

From eq. (6):

$$q^{(1)}_{1} = \cos\phi, \quad q^{(1)}_{2} = -\sin\phi,$$

$$q^{(2)}_{1} = \sin\phi, \quad q^{(2)}_{2} = \cos\phi, \quad -(9)$$

$$q^{(3)}_{3} = 1.$$

The Cartesian metric is eq. (2), and from eq. (3) it is defined by:

$$g_{11} = 1 = q^{(1)}_{1} q^{(1)}_{1} \eta_{(1)(1)} + q^{(2)}_{1} q^{(2)}_{1} \eta_{(2)(2)}. \quad -(10)$$

$$g_{22} = 1 = q^{(1)}_{2} q^{(1)}_{2} \eta_{(1)(1)} + q^{(2)}_{2} q^{(2)}_{2} \eta_{(2)(2)} \quad -(11)$$

$$g_{33} = 1 = q^{(3)}_{3} q^{(3)}_{3} \eta_{(3)(3)} \quad -(12)$$

Therefore

$$\cos^2\phi\, \eta_{(1)(1)} + \sin^2\phi\, \eta_{(2)(2)} = 1 \quad -(13)$$

$$\sin^2\phi\, \eta_{(1)(1)} + \cos^2\phi\, \eta_{(2)(2)} = 1 \quad -(14)$$

$$\eta_{(3)(3)} = 1 \quad -(15)$$

and

$$\eta_{(a)(b)} = \mathrm{diag}(1, 1, 1) \quad -(16)$$

Q.E.D., i.e.

$$\boxed{\eta_{(1)(1)} = \eta_{(2)(2)} = \eta_{(3)(3)} = 1} \quad -(17)$$

3) The line element of the cylindrical polar system is (VAPS, p. 1033):

$$ds^2 = dr^2 + r^2 d\phi^2 + dZ^2 \quad -(18)$$

$$= h_1^2 dr^2 + h_2^2 d\phi^2 + h_3^2 dZ^2 \quad -(19)$$

where h_1, h_2 and h_3 are the scale factors:

$$h_1 = \left| \frac{\partial \underline{r}}{\partial r} \right| = 1 \quad -(20)$$

$$h_2 = \left| \frac{\partial \underline{r}}{\partial \phi} \right| = r \quad -(21)$$

$$h_3 = \left| \frac{\partial \underline{r}}{\partial Z} \right| = 1 \quad -(22)$$

with

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(23)$$

$$= r\cos\phi\,\underline{i} + r\sin\phi\,\underline{j} + Z\,\underline{k}$$

The metric form as defined in VAPS p. 1100 is:

$$g_{ij} = g_{ji} = \frac{\partial \underline{r}}{\partial u_i} \cdot \frac{\partial \underline{r}}{\partial u_j} \quad -(24)$$

so for the cylindrical polar system:

$$\eta_{(1)(1)} = \frac{\partial \underline{r}}{\partial r} \cdot \frac{\partial \underline{r}}{\partial r} \quad -(25)$$

$$\eta_{(2)(2)} = \frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \phi} \quad -(26)$$

$$\eta_{(3)(3)} = \frac{\partial \underline{r}}{\partial Z} \cdot \frac{\partial \underline{r}}{\partial Z} \quad -(27)$$

Here, from eq. (23):

4) $$\frac{\partial \underline{r}}{\partial r} = \cos\phi \, \underline{i} + \sin\phi \, \underline{j} \quad -(28)$$

$$\frac{\partial \underline{r}}{\partial \phi} = -r\sin\phi \, \underline{i} + r\cos\phi \, \underline{j} \quad -(29)$$

$$\frac{\partial \underline{r}}{\partial Z} = \underline{k} \quad -(30)$$

So: $$\boxed{\eta_{(1)(1)} = 1, \; \eta_{(2)(2)} = r^2, \; \eta_{(3)(3)} = 1} \quad -(31)$$

It is seen that eqns (17) and (31) are different. Therefore the Cartan definition (3) and the definition (24) lead to different results. Eq. (17) is free of any coordinate parameter, but eq. (31) is not.

In order to resolve this fundamental problem define:

$$\boxed{g_{ij} = g_{ji} = \underline{e}_i \cdot \underline{e}_j} \quad -(32)$$

where the unit vectors of the coordinate system are defined (VAIS 1033) as:

$$\underline{e}_1 = \frac{\partial \underline{r}}{\partial u_1} \Big/ \left| \frac{\partial \underline{r}}{\partial u_1} \right| \quad -(33)$$

$$\underline{e}_2 = \frac{\partial \underline{r}}{\partial u_2} \Big/ \left| \frac{\partial \underline{r}}{\partial u_2} \right| \quad -(34)$$

$$\underline{e}_3 = \frac{\partial \underline{r}}{\partial u_3} \Big/ \left| \frac{\partial \underline{r}}{\partial u_3} \right| \quad -(35)$$

For the cylindrical polar system:

5) $\underline{e}^{(1)} = \underline{e}_r = \cos\phi\,\underline{i} + \sin\phi\,\underline{j}$ — (36)

$\underline{e}^{(2)} = \underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}$ — (37)

$\underline{e}^{(3)} = \underline{e}_Z = \underline{k}$. — (38)

So: $\Lambda_{(a)(b)} = \mathrm{diag}(1, 1, 1)$ — (39)

which is eq. (16), Q.E.D.

Therefore Cartan's definition (3) is equivalent to the new definition (32) of the metric form.

The line element on the other hand is defined by

$$d\underline{r} = \frac{\partial \underline{r}}{\partial r} dr + \frac{\partial \underline{r}}{\partial \phi} d\phi + \frac{\partial \underline{r}}{\partial Z} dZ \quad - (40)$$

and:

$$ds^2 = d\underline{r} \cdot d\underline{r} = |d\underline{r}|^2 \quad - (41)$$

$$= dr^2 + r^2 d\phi^2 + dZ^2$$

The metric may be thought of as follows:

$$ds^2 = 1\,dr^2 + 1\,r^2 d\phi^2 + 1\,dZ^2$$

$$= 1\,dX^2 + 1\,dY^2 + 1\,dZ^2 \quad - (42)$$

— (43)

and $g_{\mu\nu} = \Lambda_{(a)(b)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{diag}(1, 1, 1)$

In the next note the effect of phase will be considered.