154(4): Detailed Proof of the Tetrad Postulate and ECE Lemma.

The tetrad postulate is the invariance of DX under coordinate transformation:

$$DX = (D_\mu X^\nu)\, dx^\mu \otimes e_\nu = (D_\mu X^a)\, dx^\mu \otimes e_a \quad -(1)$$

where

$$D_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda \quad -(2)$$

$$D_\mu X^a = \partial_\mu X^a + \omega^a_{\mu b} X^b \quad -(3)$$

Therefore:

$$DX \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left(\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda\right) dx^\mu \otimes e_\nu \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(4)$$

Now make the development:

$$(D_\mu X^a)\, dx^\mu \otimes e_a = \left(\partial_\mu X^a + \omega^a_{\mu b} X^b\right) dx^\mu \otimes e_a \quad -(5)$$

$$= \left(\partial_\mu (q^a_\nu X^\nu) + \omega^a_{\mu b} q^b_\lambda X^\lambda\right) dx^\mu \otimes (q^\sigma_a e_\sigma)$$

$$= q^\sigma_a \left(q^a_\nu \partial_\mu X^\nu + X^\nu \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\lambda X^\lambda\right) dx^\mu \otimes e_\sigma$$

$$= q^\nu_a \left(q^a_\nu \partial_\mu X + X^\nu \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\lambda X^\lambda\right) dx^\mu \otimes e_\nu$$

$$= \left(q^\nu_a q^a_\nu \partial_\mu X^\nu + q^\nu_a X^\nu \partial_\mu q^a_\nu + q^\nu_a \omega^a_{\mu b} q^b_\lambda X^\lambda\right) dx^\mu \otimes e_\nu$$

$$= q^\nu_a q^a_\nu \left(\partial_\mu X^\nu + \omega^\nu_{\mu\lambda} X^\lambda + (\partial_\mu q^\nu_\lambda) X^\lambda\right) dx^\mu \otimes e_\nu$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\partial_\mu X^\nu + \left(\partial_\mu q^\nu_\lambda + \omega^\nu_{\mu\lambda}\right) X^\lambda\right) dx^\mu \otimes e_\nu \quad -(6)$$

Compare eq. (4) and eq. (6):

$$\partial_\mu q^{\sim}{}_\lambda + \omega^{\sim}{}_{\mu\lambda} - \Gamma^{\sim}{}_{\mu\lambda} = 0 \quad -(7)$$

where we have used:

$$q^{\sim}_a \, q^a_{\sim} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(8)$$

Eq. (7) is:

$$q^{\sim}_a \left(\partial_\mu q^a_\lambda + \omega^a_{\mu\lambda} - \Gamma^a_{\mu\lambda} \right) = 0 \quad -(9)$$

So:

$$\partial_\mu q^a_\lambda + \omega^a_{\mu\lambda} - \Gamma^a_{\mu\lambda} = 0 \quad -(10)$$

i.e.

$$\boxed{D_\mu q^a_\lambda = \partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - \Gamma^\lambda_{\mu\lambda} q^a_\lambda = 0} \quad -(11)$$

which is the tetrad postulate, QED.

Proof of the ECE Lemma

Write eq. (10) as:

$$\partial_\mu q^a_\lambda = \alpha^a_{\mu\lambda} \quad -(12)$$

where:

$$\alpha^a_{\mu\lambda} = \Gamma^a_{\mu\lambda} - \omega^a_{\mu\lambda} \quad -(13)$$

Thus:

$$\partial^\mu \partial_\mu q^a_\lambda = \partial^\mu \alpha^a_{\mu\lambda} \quad -(14)$$

i.e.

$$\square q^a_\lambda = \partial^\mu \alpha^a_{\mu\lambda} \quad -(15)$$

) Define the scalar curvature R by:

$$R\, q^{a}_{\lambda} := \partial^{\mu} \omega^{a}_{\mu\lambda} \quad -(16)$$

so:

$$\boxed{\Box\, q^{a}_{\lambda} = R\, q^{a}_{\lambda}} \quad -(17)$$

which is the ECE Lemma, Q.E.D.

From eq. (16):

$$R\, q^{a}_{\lambda}\, q^{\lambda}_{a} = q^{\lambda}_{a}\, \partial^{\mu} \omega^{a}_{\mu\lambda} \quad -(18)$$

i.e.

$$R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = q^{\lambda}_{a}\, \partial^{\mu} \omega^{a}_{\mu\lambda} \quad -(19)$$

On the right hand side of eq. (19) define R again by eq. (16), to obtain:

$$R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(20)$$

Q.E.D.

These proofs make clear that the correct definition of $q^{a}_{\mu}\, q^{\mu}_{a}$ is eq. (8).

The tetrad q^{a}_{μ} is a mixed index tensor, and an $n \times n$ invertible matrix. The inverse matrix of q^{a}_{μ} is denoted q^{μ}_{a}.