54(2): Tetrads and Metrics of the Electromagnetic Plane Wave.

In paper 153 tetrads were shown to be transformation matrices between various coordinate systems and the Cartesian system. Examples are given below.

1) Complex Circular

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \\ \underline{e}^{(3)} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(1)$$

2) Cylindrical Polar

$$\begin{bmatrix} \underline{e}_r \\ \underline{e}_\phi \\ \underline{e}_Z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(2)$$

3) Spherical Polar

$$\begin{bmatrix} \underline{e}_r \\ \underline{e}_\phi \\ \underline{e}_\theta \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(3)$$

The potential plane wave can be written as:

$$\underline{A}^{(1)} = A^{(0)} \underline{q}^{(1)} \quad -(4)$$

$$\underline{A}^{(2)} = A^{(0)} \underline{q}^{(2)} \quad -(5)$$

where:

$$\underline{q}^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\Phi} \quad -(6)$$

$$\underline{q}^{(2)} = \frac{1}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i\Phi} \quad -(7)$$

2) and

$$\Phi = \omega t - \kappa Z \quad -(8)$$

where ω is the angular frequency at instant t and κ the wavenumber at point Z. So:

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\Phi} & -ie^{i\Phi} \\ e^{-i\Phi} & ie^{-i\Phi} \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(8)$$

The introduction of the phase means that one frame of reference rotates and translates along Z with respect to the other. The electromagnetic field is this frame of reference in the philosophy of relativity. It is shown as follows that the phase enters into the tetrad but not necessarily into the metric. The tetrad contains more information than the metric.

Complex Circular System

This is defined by:

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \\ \underline{e}^{(3)} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(9)$$

$$\begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ i/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \\ \underline{e}^{(3)} \end{bmatrix} \quad -(10)$$

i.e. by:

3)

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k}$$

$$= \frac{1}{\sqrt{2}}(X+iY)\frac{1}{\sqrt{2}}(\underline{i}-i\underline{j}) + \frac{1}{\sqrt{2}}(X-iY)\frac{1}{\sqrt{2}}(\underline{i}+i\underline{j}) + Z\underline{k}$$

$$= r^{(2)}\underline{e}^{(1)} + r^{(1)}\underline{e}^{(2)} + r^{(3)}\underline{e}^{(3)} \quad -(11)$$

therefore

$$\underline{e}^{(1)} = \frac{\partial \underline{r}}{\partial r^{(2)}} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}) \quad -(12)$$

$$\underline{e}^{(2)} = \frac{\partial \underline{r}}{\partial r^{(1)}} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}) \quad -(13)$$

$$\underline{e}^{(3)} = \frac{\partial \underline{r}}{\partial r^{(3)}} = \underline{k} \quad -(14)$$

The scale factors of the complex circular system are

$$h_1 = |\underline{e}^{(1)}| = 1 \quad -(15)$$

$$h_2 = |\underline{e}^{(2)}| = 1 \quad -(16)$$

$$h_3 = |\underline{e}^{(3)}| = 1 \quad -(17)$$

The line element is:

$$ds^2 = h_1^2 dr^{(2)} dr^{(1)} + h_2^2 dr^{(1)} dr^{(2)} + h_3^2 dr^{(3)2} \quad -(18)$$

where:

$$dr^{(1)} = \frac{1}{\sqrt{2}}(dX - i\,dY) \quad -(19)$$

$$dr^{(2)} = \frac{1}{\sqrt{2}}(dX + i\,dY) \quad -(20)$$

so the metric of the complex circular system is:

$$g_{(a)(b)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(21)$$

4) This is the same as the metric of the Cartesian system:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(22)$$

because the two systems are the same by eqn. (11).

In the complex circular system:

$$\underline{e}^{(1)} \times \underline{e}^{(2)} = i\,\underline{e}^{(3)*} \quad -(23)$$

et cyclicum

where $*$ denotes the complex conjugate (i.e. change sign of $i = \sqrt{-1}$). In the Cartesian system:

$$\underline{i} \times \underline{j} = \underline{k}. \quad -(24)$$

et cyclicum

The two systems are related by:

$$\boxed{g_{\mu\nu} = q_\mu^{(a)}\, q_\nu^{(b)*}\, \eta_{(a)(b)}} \quad -(25)$$

Eq. (25) means:

$$g_{11} = 1 = q_1^{(1)} q_1^{(1)*} \eta_{(1)(1)} + q_1^{(2)} q_1^{(2)*} \eta_{(2)(2)}$$

$$g_{22} = 1 = q_2^{(1)} q_2^{(1)*} \eta_{(1)(1)} + q_2^{(2)} q_2^{(2)*} \eta_{(2)(2)}$$

$$g_{33} = 1 = q_3^{(3)} q_3^{(3)*} \eta_{(3)(3)} \quad -(26)$$

where

$$\eta_{(1)(1)} = \eta_{(2)(2)} = \eta_{(3)(3)} = 1 \quad -(27)$$

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From eq. (9):

$$qV^{(a)}_{\mu} = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(22)$$

From eq. (10)

$$qV^{\mu}_{(a)} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ i/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(23)$$

and

$$qV^{(a)}_{\mu}\, qV^{\mu}_{(a)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(24)$$

So:

$$qV^{(1)}_{1} = \frac{1}{\sqrt{2}}, \quad qV^{(1)}_{2} = -\frac{i}{\sqrt{2}} \quad -(25)$$

$$qV^{(2)}_{1} = \frac{1}{\sqrt{2}}, \quad qV^{(2)}_{2} = \frac{i}{\sqrt{2}} \quad -(26)$$

$$qV^{(3)}_{3} = 1 \quad -(27)$$

and

$$qV^{(a)}_{\mu} = \begin{bmatrix} qV^{(1)}_{1} & qV^{(1)}_{2} & qV^{(1)}_{3} \\ qV^{(2)}_{1} & qV^{(2)}_{2} & qV^{(2)}_{3} \\ qV^{(3)}_{1} & qV^{(3)}_{2} & qV^{(3)}_{3} \end{bmatrix} \quad -(28)$$

So

$$qV^{(1)*}_{1} = qV^{(1)}_{1} \quad -(29)$$

$$qV^{(2)*}_{1} = qV^{(2)}_{1} \quad -(30)$$

and

6)

$$q_2^{(1)*} = q_2^{(2)} \quad -(31)$$

$$q_2^{(2)*} = q_2^{(1)}. \quad -(32)$$

Therefore, self consistently: $\quad -(33)$

$$g_{11} = q_1^{(1)} q_1^{(1)*} + q_1^{(2)} q_1^{(2)*} = \frac{1}{2} + \frac{1}{2} = 1$$

$$g_{22} = q_2^{(1)} q_2^{(1)*} + q_2^{(2)} q_2^{(2)*} = \frac{1}{2} + \frac{1}{2} = 1 \quad -(34)$$

$$g_{33} = q_3^{(3)} q_3^{(3)*} = 1$$

Q.E.D.

The Phase Factor

The electromagnetic phase factor enters through the tetrad vectors, as in UFT 1 and previous work:

$$\underline{e}^{(1)} = \underline{q}^{(1)} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j})\, e^{i\Phi} \quad -(35)$$

$$\underline{e}^{(2)} = \underline{q}^{(2)} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j})\, e^{-i\Phi} \quad -(36)$$

$$\underline{e}^{(3)} = \underline{k} \quad -(37)$$

The tetrad vectors (i.e. the unit vectors) propagate in Z as a plane wave.

However, the phase factor cancels out of

7) the scale factors and metrics as follows.

In eq. (11):

$$\underset{\sim}{r} = r^{(2)} e^{i\Phi} \underline{e}^{(1)} + r^{(1)} e^{i\Phi} \underline{e}^{(2)} + r^{(3)} \underline{e}^{(3)} \quad -(38)$$

so

$$\frac{\partial \underline{r}}{\partial r^{(2)}} = \frac{1}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) e^{i\Phi} = \underline{e}^{(1)} = \underline{v}^{(1)}$$

$$\frac{\partial \underline{r}}{\partial r^{(1)}} = \frac{1}{\sqrt{2}} \left(\underline{i} + i \underline{j} \right) e^{-i\Phi} = \underline{e}^{(2)} = \underline{v}^{(2)} \quad -(39)$$

$$\frac{\partial \underline{r}}{\partial r^{(3)}} = \underline{e}^{(3)} = \underline{v}^{(3)} = \underline{k}.$$

$$h_1 = \left(\underline{e}^{(1)} \cdot \underline{e}^{(2)} \right)^{1/2} = 1 \quad -(40)$$

$$h_2 = \left(\underline{e}^{(2)} \cdot \underline{e}^{(1)} \right)^{1/2} = 1$$

$$h_3 = \left(\underline{e}^{(3)} \cdot \underline{e}^{(3)*} \right)^{1/2} = 1$$

so

$$g_{(a)(b)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(41)$$

Conclusion: The tetrad contains the phase factor, the metric does not.

The tetrad is the basis of ECE theory, and the potential is the tetrad within $A^{(0)}$.