151(6): A General Metric for Cosmology

The metric for a spherically symmetric spacetime is:

$$ds^2 = e^{2\alpha} c^2 dt^2 - e^{2\beta} dr^2 - r^2 d\phi^2 \quad -(1)$$

in the XY plane. Other more general metrics may be used. In general α and β are functions of r and t. However, for simplicity assume that α and β are function of r.

The lagrangian is then:

$$\mathcal{L} = T = \frac{1}{2} mc^2 = \frac{1}{2} m \left(e^{2\alpha} c^2 \left(\frac{dt}{d\tau}\right)^2 - e^{2\beta}\left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \right) \quad -(2)$$

The constants of motion are:

$$E = e^{2\alpha} mc^2 \frac{dt}{d\tau} \quad -(3) \quad \text{(energy)}$$

$$L = mr^2 \frac{d\phi}{d\tau} \quad -(4) \quad \text{(ang. mom.)}$$

$$p = e^{2\beta} m \frac{dr}{d\tau} \quad -(5) \quad \text{(linear mom.)}$$

The Orbital Theorem states that:

$$e^{2\alpha} = e^{-2\beta} \quad -(6)$$

Thus:

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - e^{-2\beta}\left(mc^2 + \frac{L^2}{mr^2}\right) \quad -(7)$$

where

$$\frac{dr}{d\tau} = \frac{d\phi}{d\tau}\frac{dr}{d\phi} = \left(\frac{L}{mr^2}\right)\frac{dr}{d\phi} \quad -(8)$$

2)

So:

$$\left(\frac{dr}{d\phi}\right)^2 = r^4\left(\frac{1}{b^2} - e^{-2\beta}\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right) \quad -(9)$$

where

$$a = \frac{L}{mc}, \quad b = c\frac{L}{E} \quad -(10)$$

The orbital equation is:

$$\boxed{\frac{d\phi}{dr} = \frac{1}{r^2}\left(\frac{1}{b^2} - e^{-2\beta}\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{-1/2}}$$

$$-(11)$$

for any spherically symmetric spacetime.

Minkowski Spacetime

$$e^{2\alpha} = e^{-2\beta} = 1 \quad -(12)$$

$$\frac{d\phi}{dr} = \frac{1}{r^2}\left(\frac{1}{b^2} - \frac{1}{a^2} - \frac{1}{r^2}\right)^{-1/2} \quad -(13)$$

Gravitational Metric

$$e^{2\alpha} = e^{-2\beta} = 1 - \frac{r_0}{r}, \quad -(14)$$

$$r_0 = \frac{2GM}{c^2} \quad -(15)$$

$$\frac{d\phi}{dr} = \frac{1}{r^2}\left(\frac{1}{b^2} - \left(1 - \frac{r_0}{r}\right)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{-1/2}$$

$$-(15)$$

3)

Binary Pulsar

$$e^{2\alpha} = e^{-2\beta} = 1 - \frac{r_0}{r} - \frac{\xi}{r^2} \quad -(16)$$

$$\frac{d\phi}{dr} = \frac{1}{r^2}\left(\frac{1}{b^2} - \left(1 - \frac{r_0}{r} - \frac{\xi}{r^2}\right)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{-1/2} \quad -(17)$$

Whirlpool Galaxy

The stars are arranged a the log spiral

$$r = r_0 e^{\xi\phi} \quad -(18)$$

so

$$\frac{dr}{d\phi} = \xi r \quad -(19)$$

and

$$\xi = r\left(\frac{1}{b^2} - e^{-2\beta}\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{1/2} \quad -(20)$$

so:

$$\boxed{e^{2\alpha} = e^{-2\beta} = \left(\frac{1}{b^2} - \frac{\xi^2}{r^2}\right)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)^{-1}} \quad -(21)$$

The metric is:

$$ds^2 = e^{2\alpha} c^2 dt^2 - e^{-2\alpha} dr^2 - r^2 d\phi^2 \quad -(22)$$

<u>in all cases.</u>