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# A Rigorous Proof of the Hodge Dual of the Bianchi Identity of Cartan

by

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## Abstract

Using the action of the commutator of covariant derivatives on the four vector in a four dimensional space-time, it is proven that there exists a well defined Hodge dual of the Bianchi identity of Cartan for any metric and any connection.

## 11.1 Introduction

It is well known that the Bianchi identity as developed by Cartan [1] relates the torsion form and curvature form. It has been shown during the development [2–10] of ECE theory that this Bianchi identity is a rigorous identity that states that the cyclically symmetric sum of three curvature tensors is identically equal to the same cyclically symmetric sum of fundamental definitions of the same curvature tensors. The definitions of the curvature and torsion tensors arise [1] from the action of the commutator of covariant derivatives on the four vector. The curvature and torsion tensors are defined by the same commutator, and therefore the one tensor cannot exist without the other tensor. The Bianchi identity is derivable from the commutator given the tetrad postulate [2–10], and the Bianchi identity must always link the torsion to the curvature. In Section 2 a well defined Hodge dual of the same

Bianchi identity is derived rigorously by considering the Hodge dual of the commutator of covariant derivatives acting on the four vector. The resulting Hodge dual identity, when developed in tensor notation in the base manifold, shows that there is a fundamental self inconsistency in the Einstein Hilbert (EH) theory of general relativity and in the Einstein Hilbert field equation. The Christoffel connection of the EH theory is fundamentally incompatible with the Bianchi identity as developed by Cartan. In previous work [2–10] a development of the EH equation has been suggested, a development based on the Bianchi identity. Furthermore, the Einstein Cartan Evans (ECE) field equations of dynamics and electrodynamics are based on the Bianchi identity of Cartan and its Hodge dual, proven rigorously in Section 11.2.

## 11.2  Proof of the Hodge Dual Bianchi Identity

Consider the commutator of covariant derivatives acting on the four vector $V^\rho$ in a four-dimensional space-time:

$$[D_\mu, \ D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda_{\mu\nu}D_\lambda V^\rho \qquad (11.1)$$

Here $R^\rho{}_{\sigma\mu\nu}$ is the curvature tensor [1], $T^\lambda_{\mu\nu}$ is the torsion tensor, and $D_\lambda$ is the covariant derivative. It has been shown [2–10] that Eq. (11.1) is the basis for the Bianchi identity of Cartan [1–10]:

$$D \wedge T := R \wedge q \qquad (11.2)$$

where a shorthand notation has been adopted suppressing indices for structural clarity. In this notation T is the torsion form of Cartan's differential geometry , R is the Riemann form, q is the tetrad form, $\wedge$ is the wedge product and $D\wedge$ is the covariant exterior derivative of Cartan's differential geometry. In order for Eq. (11.2) to be true, the basic definitions generated by Eq. (11.1) must be used in Eq. (11.2), and the tetrad postulate [1–10] must also be used. The Bianchi identity (11.2) is the basic structure used for the homogeneous field equation of ECE theory, both in dynamics and electrodynamics.

It is proven that there exists the following identity:

$$D \wedge \widetilde{T} := \widetilde{R} \wedge q \qquad (11.3)$$

which is the basis for the inhomogeneous field equation of ECE theory, both in dynamics and electrodynamics. The proof is as follows.

Consider the Hodge dual transformations [1–10]:

$$[D^\mu, \ D^\nu]_{HD} = \frac{1}{2} \parallel g \parallel^{\frac{1}{2}} \epsilon^{\mu\nu\alpha\beta}[D_\alpha, \ D_\beta], \qquad (11.4)$$

$$\widetilde{R}^{\rho}{}_{\sigma}{}^{\mu\nu} = \frac{1}{2} \parallel g \parallel^{\frac{1}{2}} \epsilon^{\mu\nu\alpha\beta} R^{\rho}{}_{\sigma\alpha\beta}, \tag{11.5}$$

$$\widetilde{T}^{\lambda\mu\nu} = \frac{1}{2} \parallel g \parallel^{\frac{1}{2}} \epsilon^{\mu\nu\alpha\beta} T^{\lambda}_{\alpha\beta}. \tag{11.6}$$

Here $\epsilon^{\mu\nu\alpha\beta}$ is the four dimensional totally anti-symmetric unit tensor of Minkowski space-time [1], and the Hodge dual transformations are weighted by definition [1] by the square root of the modulus of the determinant of the metric, denoted by $\|g\|^{1/2}$. The Hodge dual is denoted by $\nu$. In four dimensions, the Hodge dual of an anti-symmetric tensor (i.e. differential two-form [1–10]) is another anti-symmetric tensor. Using Eqs. (11.4) to (11.6) in Eq. (11.1):

$$[D^{\mu}, D^{\nu}]_{HD} V^{\rho} = \widetilde{R}^{\rho}{}_{\sigma}{}^{\mu\nu} V^{\sigma} - \widetilde{T}^{\lambda\mu\nu} D_{\lambda} V^{\rho} \tag{11.7}$$

where the subscript HD on the left hand side denotes Hodge dual of the commutator, which is anti-symmetric in $\mu$ and $\nu$. Eq. (11.7) proves Eq. (11.3) with raised indices $\mu$ and $\nu$. In order to obtain the final form of the ECE field equations it is necessary to prove Eq. (11.7) with lowered indices $\mu$ and $\nu$. To lower indices requires the use of the metric by definition [1–10]. Therefore the three anti-symmetric tensors in Eq. (11.7) are expressed in terms of their equivalents with lowered indices as follows:

$$[D^{\mu}, D^{\nu}]_{HD} = g^{\mu\alpha} g^{\nu\beta} [D_{\alpha}, D_{\beta}]_{HD}, \tag{11.8}$$

$$\widetilde{R}^{\rho}{}_{\sigma}{}^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \widetilde{R}^{\rho}{}_{\sigma\alpha\beta}, \tag{11.9}$$

$$\widetilde{T}^{\lambda\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \widetilde{T}^{\lambda}_{\alpha\beta}. \tag{11.10}$$

It follows by use of Eqs. (11.8) to (11.10) in Eq. (11.7) that:

$$g^{\mu\alpha} g^{\nu\beta} [D_{\alpha}, D_{\beta}]_{HD} V^{\rho} = g^{\mu\alpha} g^{\nu\beta} (\widetilde{R}^{\rho}{}_{\sigma\alpha\beta} V^{\sigma} - \widetilde{T}^{\lambda}_{\alpha\beta} D_{\lambda} V^{\rho}) \tag{11.11}$$

a particular solution of which is:

$$[D_{\alpha}, D_{\beta}]_{HD} V^{\rho} := \widetilde{R}^{\rho}{}_{\sigma\alpha\beta} V^{\sigma} - \widetilde{T}^{\lambda}_{\alpha\beta} D_{\lambda} V^{\rho}. \tag{11.12}$$

This is the required identity with lowered indices. Eq. (11.12) means that:

$$(D \wedge \widetilde{T}^{a})_{\mu\nu} := (\widetilde{R}^{a}{}_{b} \wedge q^{b})_{\mu\nu} \tag{11.13}$$

with lowered $\mu$ and $\nu$ indices, and it is seen that the metric has been eliminated from consideration provided that the particular solution (11.12) is used. In tensor notation Eq. (11.13) is:

$$D_\mu \widetilde{T}^a_{\nu\rho} + D_\rho \widetilde{T}^a_{\mu\nu} + D_\nu \widetilde{T}^a_{\rho\mu} := \widetilde{R}^a_{\mu\nu\rho} + \widetilde{R}^a_{\rho\mu\nu} + \widetilde{R}^a_{\nu\rho\mu} \tag{11.14}$$

where the rules for wedge product and covariant exterior derivative [1–10] have been used. Eq. (11.14) is the same as:

$$D_\mu T^{a\mu\nu} := R^a{}_\mu{}^{\mu\nu} \tag{11.15}$$

because the four dimensional totally anti-symmetric unit tensor is the same by definition [1–10] as that used in Minkowski or flat space-time. The factor $\|g\|^{1/2}$ has been cancelled out on both sides of Eq. (11.15). A particular solution of Eq. (11.15) is the base manifold equation:

$$D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu} \tag{11.16}$$

which is the inhomogeneous field equation of ECE theory, Q.E.D.

By compute algebra [2–10] it has been shown that for the Christoffel connection:

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \tag{11.17}$$

the right hand side of Eq. (11.16) is not zero in general. The computer algebra shows that it is zero only in a Ricci flat space-time. For the same Christoffel connection the left hand side of Eq. (11.16) is always zero because:

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \tag{11.18}$$

So the use of the Christoffel connection is incompatible with fundamental geometry.

This conclusion signals the collapse of the EH theory of general relativity and conclusions based thereon, because all so called "exact solutions" of EH are based on the Christoffel symbol and line elements deduced therefrom. Other severe limitations and internal inconsistencies of EH are given by Crothers on www.aias.us and Santilli on www.telesio-galilei.com.

Similarly, the Bianchi identity (11.2) translates into:

$$D_\mu T^a_{\nu\rho} + D_\rho T^a_{\mu\nu} + D_\nu T^a_{\rho\mu} := R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu} \tag{11.19}$$

which in form notation is:

$$(D \wedge T^a)_{\mu\nu} := (R^a{}_b \wedge q^b)_{\mu\nu}. \tag{11.20}$$

This equation is the same as:

$$D_\mu \widetilde{T}^{a\mu\nu} := \widetilde{R}^a{}_\mu{}^{\mu\nu}. \tag{11.21}$$

A particular solution of Eq. (11.21) is:

$$D_\mu \widetilde{T}^{\kappa\mu\nu} = \widetilde{R}^\kappa{}_\mu{}^{\mu\nu} \tag{11.22}$$

which is the homogeneous ECE field equation's structure. Eqs. (11.16) and (11.22) are the ECE field equations for dynamics, and using the fundamental hypothesis:

$$F^a_{\mu\nu} = A^{(0)} T^a_{\mu\nu} \tag{11.23}$$

become the ECE field equations for electrodynamics. The homogeneous electro-dynamical equation is:

$$D_\mu \widetilde{F}^{\kappa\mu\nu} = A^{(0)} \widetilde{R}^\kappa{}_\mu{}^{\mu\nu} \tag{11.24}$$

and the inhomogeneous electro-dynamical equation is:

$$D_\mu F^{\kappa\mu\nu} = A^{(0)} R^\kappa{}_\mu{}^{\mu\nu}. \tag{11.25}$$

The covariant derivative may be expanded in terms of the spin connection to give:

$$\partial_\mu \widetilde{F}^{\kappa\mu\nu} = A^{(0)} (\widetilde{R}^\kappa{}_\mu{}^{\mu\nu} - \omega^\kappa_{\mu\lambda} \widetilde{T}^{\lambda\mu\nu}), \tag{11.26}$$

and

$$\partial_\mu F^{\kappa\mu\nu} = A^{(0)} (R^\kappa{}_\mu{}^{\mu\nu} - \omega^\kappa_{\mu\lambda} T^{\lambda\mu\nu}), \tag{11.27}$$

Experimental data show [2–10] that for all practical purposes (F. A. P. P.) in the laboratory:

$$\widetilde{R}^\kappa{}_\mu{}^{\mu\nu} = \omega^\kappa_{\mu\lambda} \widetilde{T}^{\lambda\mu\nu} \tag{11.28}$$

but:

$$R^\kappa{}_\mu{}^{\mu\nu} \neq \omega^\kappa_{\mu\lambda} T^{\lambda\mu\nu}. \tag{11.29}$$

Using these data, Eqs. (11.26) and (11.27) reduce to:

$$\partial_\mu \widetilde{F}^{\kappa\mu\nu} = 0, \tag{11.30}$$

$$\partial_\mu F^{\kappa\mu\nu} = J^{\kappa\nu}/\epsilon_0. \tag{11.31}$$

It has been shown [2–10] that in vector form these become the same as the Maxwell Heaviside field equations as follows:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{11.32}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}, \tag{11.33}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0, \tag{11.34}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{J}. \tag{11.35}$$

but written in a more general space-time with curvature and torsion. The relation between the electric and magnetic fields and the potentials are developed as follows in terms of the spin connection vector $\boldsymbol{\omega}$, and scalar:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} + \phi\boldsymbol{\omega} - \omega\boldsymbol{A}, \tag{11.36}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} - \boldsymbol{\omega} \times \boldsymbol{A}. \tag{11.37}$$

In standard S.I. units $\mathbf{B}$ is the magnetic flux density, $\mathbf{E}$ is the electric field strength, $\epsilon_0$ is the vacuum permittivity, $\rho$ is the electric charge density, $\mu_0$ is the vacuum permeability, and $\mathbf{J}$ is the electric current density. In Eqs. (11.36) and (11.37) is the scalar potential, $\mathbf{A}$ is the vector potential, $\boldsymbol{\omega}$ is the spin connection vector and $\omega$ is the spin connection scalar. It is seen that all the equations are derived from the Bianchi identity.

## Acknowledgments

# References

[1] S. P. Carroll, "Space-time and Geometry: an Introduction to General Relativity" (Addison Wesley, New York, 2004, and lecture notes of 1997 downloadable from the web).

[2] M. W. Evans, "Generally Covariant Unified Field Theory : the Geometrization of Physics" (Abramis, 2005 onwards), in four volumes to date, volume five in prep. (Papers 71 to 93 of www.aias.us).

[3] L. Felker, "The Evans Equations of Unified Field Theory" (Abramis, 2007).

[4] K. Pendergast, "Crystal Spheres" (Abramis in preps. Preprint on www.aias.us).

[5] M. W. Evans and H. Eckardt, Physica B, **400**, 175 (2007).

[6] M. W. Evans, Acta Phys. Polonica B, **38**, 2211 (2007).

[7] M. W. Evans, Physica B, **403**, 517 (2008).

[8] M. W. Evans et al., Omnia Opera section of www.aias.us, 1992 to present.

[9] M. W. Evans and J.-P. Vigier, "The Enigmatic Photon" (Kluwer Dordrecht, 1994 to 2002, hardback and softback), in five volumes.

[10] M. W. Evans, "Modern Non-Linear Optics", in "Advances in Chemical Physics" (Wiley Interscience, 1992 to 2001, first and second editions, hardback and softback), vols. 85 and 119; M. W. Evans and L. B. Crowell, "Classical and Quantum Electrodynamics and the B(3) Field" (World scientific, 2001).