199(3): Check on Note 99(2)

We have:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \quad -(1)$$

i.e.

$$[D_\mu, D_\nu] V^0 = R^0{}_{0\mu\nu} V^0 + R^0{}_{1\mu\nu} V^1 + R^0{}_{2\mu\nu} V^2 + R^0{}_{3\mu\nu} V^3$$

$$[D_\mu, D_\nu] V^1 = R^1{}_{0\mu\nu} V^0 + R^1{}_{1\mu\nu} V^1 + R^1{}_{2\mu\nu} V^2 + R^1{}_{3\mu\nu} V^3$$

$$[D_\mu, D_\nu] V^2 = R^2{}_{0\mu\nu} V^0 + R^2{}_{1\mu\nu} V^1 + R^2{}_{2\mu\nu} V^2 + R^2{}_{3\mu\nu} V^3$$

$$[D_\mu, D_\nu] V^3 = R^3{}_{0\mu\nu} V^0 + R^3{}_{1\mu\nu} V^1 + R^3{}_{2\mu\nu} V^2 + R^3{}_{3\mu\nu} V^3 \quad -(2)$$

For the Riemann tensor:

$$R^0{}_{0\mu\nu} = R^1{}_{1\mu\nu} = R^2{}_{2\mu\nu} = R^3{}_{3\mu\nu} = 0 \quad -(3)$$

because:

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \quad -(4)$$

so in this case:

$$[D_\mu, D_\nu] V^\rho = \begin{bmatrix} 0 & R^0{}_1 & R^0{}_2 & R^0{}_3 \\ -R^0{}_1 & 0 & R^1{}_2 & R^1{}_3 \\ -R^0{}_2 & -R^1{}_2 & 0 & R^2{}_3 \\ -R^0{}_3 & -R^1{}_3 & -R^2{}_3 & 0 \end{bmatrix}_{\mu\nu} \begin{bmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} \quad -(5)$$

2)

$$= [D_\mu, D_\nu] \begin{bmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} \quad — (6)$$

So we can write the <u>operator</u> relation:

$$\boxed{[D_\mu, D_\nu] = R_{\mu\nu}} \quad — (7)$$

where $R_{\mu\nu}$ is the matrix in eq. (5):

$$R_{\mu\nu} := \begin{bmatrix} 0 & R^0{}_{1\mu\nu} & R^0{}_{2\mu\nu} & R^0{}_{3\mu\nu} \\ -R^0{}_{1\mu\nu} & 0 & R^1{}_{2\mu\nu} & R^1{}_{3\mu\nu} \\ -R^0{}_{2\mu\nu} & -R^1{}_{2\mu\nu} & 0 & R^2{}_{3\mu\nu} \\ -R^0{}_{3\mu\nu} & -R^1{}_{3\mu\nu} & -R^2{}_{3\mu\nu} & 0 \end{bmatrix} \quad — (8)$$

So:

$$[D_\sigma, [D_\mu, D_\nu]] + [D_\nu, [D_\sigma, D_\mu]] + [D_\mu, [D_\nu, D_\sigma]] := 0 \quad — (9)$$

i.e. $$\boxed{D \wedge R_{\mu\nu} = 0} \quad — (10)$$

$$R_{\mu\nu} = - R_{\nu\mu} \quad — (11)$$

3) If the formal operator definition is made:

$$[D^a_\mu, D^b_\nu] = R^a{}_{b\mu\nu} \quad -(12)$$

i.e. $$D^a \wedge D^b = R^a{}_b \quad -(13)$$

then the Jacobi identity (9) leads to the conventionally named second Bianchi identity:

$$D \wedge R^a{}_b = 0. \quad -(14)$$

However we know from paper 88 that eq. (14) is incomplete. The reason is that torsion has been neglected.