98(6): Field Equation in the Base Manifold

The field equation in form notation is:

$$D \wedge F^a = R^a{}_b \wedge A^b = -A^b \wedge R^a{}_b \qquad -(1)$$

where:

$$F^a_{\mu\nu} = q^a{}_\kappa F^\kappa_{\mu\nu}. \qquad -(2)$$

The left hand side of eq. (1) is:

$$D \wedge F^a = D_\mu F^a{}_{\nu\sigma} + D_\sigma F^a_{\mu\nu} + D_\nu F^a{}_{\sigma\mu} \qquad -(3)$$

and the tetrad postulate is:

$$D_\mu q^a{}_\kappa = 0. \qquad -(4)$$

So:

$$D_\mu \left(q^a{}_\kappa F^\kappa{}_{\nu\sigma} \right) + D_\sigma \left(q^a{}_\kappa F^\kappa_{\mu\nu} \right) + D_\nu \left(q^a{}_\kappa F^\kappa{}_{\sigma\mu} \right)$$

$$= q^a{}_\kappa \left(D_\mu F^\kappa{}_{\nu\sigma} + D_\sigma F^\kappa_{\mu\nu} + D_\nu F^\kappa{}_{\sigma\mu} \right) \qquad -(5)$$

Similarly the right hand side of eq. (1) is:

$$A^{(0)} q^a{}_\kappa \left(R^\kappa{}_{\sigma\mu\nu} + R^\kappa{}_{\nu\sigma\mu} + R^\kappa{}_{\mu\nu\sigma} \right) \qquad -(6)$$

So:

$$q^a{}_\kappa \left(D_\mu F^\kappa{}_{\nu\sigma} + D_\sigma F^\kappa_{\mu\nu} + D_\nu F^\kappa{}_{\sigma\mu} \right)$$

$$= q^a{}_\kappa A^{(0)} \left(R^\kappa{}_{\sigma\mu\nu} + R^\kappa{}_{\nu\sigma\mu} + R^\kappa{}_{\mu\nu\sigma} \right)$$

$$-(7)$$

2)

A particular solution of eq. (7) is:

$$D_\mu F^{\kappa}{}_{\nu\sigma} + D_\sigma F^{\kappa}{}_{\mu\nu} + D_\nu F^{\kappa}{}_{\sigma\mu}$$
$$= A^{(0)} \left(R^{\kappa}{}_{\sigma\mu\nu} + R^{\kappa}{}_{\nu\sigma\mu} + R^{\kappa}{}_{\mu\nu\sigma} \right) \quad -(8)$$

By definition:

$$D_\mu F^{\kappa}{}_{\nu\sigma} = \partial_\mu F^{\kappa}{}_{\nu\sigma} + \omega^{\kappa}{}_{\mu b} F^{b}{}_{\nu\sigma} \quad -(9)$$

The structure of eq. (8) is therefore

$$\partial_\mu F^{\kappa}{}_{\nu\sigma} + \partial_\sigma F^{\kappa}{}_{\mu\nu} + \partial_\nu F^{\kappa}{}_{\sigma\mu}$$
$$= \frac{A^{(0)}}{\mu_0} \left(j^{\kappa}{}_{\mu\nu\sigma} + j^{\kappa}{}_{\sigma\mu\nu} + j^{\kappa}{}_{\nu\sigma\mu} \right) \quad -(10)$$

For all practical purposes this is:

$$\boxed{\partial_\mu \tilde{F}^{\kappa\mu\nu} = 0} \quad -(11)$$

whose Hodge dual is

$$\boxed{\partial_\mu F^{\kappa\mu\nu} = -\frac{A^{(0)}}{\mu_0} R^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad -(12)$$

i)

It is seen that rank three tensors enter into the field equation (11) and (12). These are now known from paper 98 to be dual to a four vector as follows:

$$V^{\mu} = V^{\mu}{}_{\rho\sigma}\,\epsilon^{\rho\sigma} \quad -(13)$$

where in general:

$$\epsilon^{\rho}{}_{\sigma} = \frac{\partial x^{\rho}}{\partial x^{\sigma}}. \quad -(14)$$

This leads to an interpretation of the electromagnetic field tensor as:

$$F^{\mu}{}_{\rho\sigma} = -F^{\mu}{}_{\sigma\rho} \quad -(15)$$

and the Cartan torsion tensor as:

$$T^{\mu}{}_{\rho\sigma} = -T^{\mu}{}_{\sigma\rho}. \quad -(16)$$

The canonical angular momentum density tensor is then:

$$J^{\mu}{}_{\rho\sigma} = -J^{\mu}{}_{\sigma\rho}. \quad -(17)$$

Eq. (12) therefore has a clear interpretation in terms of a rank three tensor $F^{\kappa\mu\nu}$ whose elements are electric and magnetic field vector components, and in terms of a Ricci type tensor

4) $R^{\kappa}{}_{\mu}{}^{\mu\nu}$. The technical correctness of eq. (1
has been evaluated by computer in papers 93 onwards.
It has been checked by computer that in Ricci
flat spacetimes:

$$R^{\kappa}{}_{;\mu}{}^{\mu\nu} = 0 \qquad -(18)$$

meaning that:

$$\partial_\mu F^{\kappa\mu\nu} = 0 \qquad -(19)$$

self-consistently. Eq. (19) can be translated into vector notation as follows:

$$\underline{\nabla} \cdot \underline{E} = 0 \qquad -(20)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \underline{0}. \qquad -(21)$$

The charge-current density in classical e/m is recognized therefore as being $-A^{(0)} R^{\kappa}{}_{\mu}{}^{\mu\nu} / \mu_0$ and proportional directly to the Ricci tensor. The vacuum in classical electrodynamics is therefore defined as being Ricci flat in a generally covariant unified field theory. This is, self-consistently, the vacuum solution of the Einstein Hilbert field equation.