98(5): Proof of the Ryder Relation

Ryder defines:

$$x^\mu = \epsilon^\mu{}_\nu x^\nu = X^\mu{}_{\rho\sigma} \epsilon^{\rho\sigma} \quad -(1)$$

and derives:

$$X^\mu{}_{\rho\sigma} = \frac{1}{2}\left(\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho\right) \quad -(2)$$

Proof

The Kronecker delta is:

$$\delta^\mu{}_\sigma = g^{\mu\nu} g_{\nu\sigma}. \quad -(3)$$

By definition:

$$x^\nu = g^{\nu\sigma} x_\sigma \quad -(4)$$

$$\epsilon^{\rho\sigma} = g^{\rho\nu} \epsilon^\sigma{}_\nu \quad -(5)$$

so:

$$\epsilon^\mu{}_\nu g^{\nu\sigma} x_\sigma = X^\mu{}_{\rho\sigma} g^{\rho\nu} \epsilon^\sigma{}_\nu \quad -(6)$$

Multiply both sides of eq. (6) by $g_{\rho\nu}$:

$$\left(g_{\rho\nu} g^{\nu\sigma}\right) \epsilon^\mu{}_\nu x_\sigma = X^\mu{}_{\rho\sigma}\left(g^{\rho\nu} g_{\rho\nu}\right) \epsilon^\sigma{}_\nu \quad -(7)$$

i.e.

$$\delta^\sigma{}_\rho \epsilon^\mu{}_\nu x_\sigma = 4 X^\mu{}_{\rho\sigma} \epsilon^\sigma{}_\nu \quad -(8)$$

Change $\sigma \rightarrow \mu \quad -(9)$

$$\delta^\mu{}_\rho \epsilon^\mu{}_\nu x_\sigma = 4 X^\mu{}_{\rho\sigma} \epsilon^\mu{}_\nu \quad -(9)$$

i.e.

$$X^\mu{}_{\rho\sigma} = \frac{1}{4} \delta^\mu{}_\rho x_\sigma. \quad -(10)$$

2) Use: $$\delta^{\mu}_{\rho} x_{\sigma} = -\delta^{\mu}{}_{\rho} x_{\sigma} \quad -(11)$$

$$\Rightarrow \quad X^{\mu}{}_{\rho\sigma} = \frac{1}{2}\left(\delta^{\mu}{}_{\rho} x_{\sigma} - \delta^{\mu}{}_{\sigma} x_{\rho}\right) \quad -(12)$$

Q.E.D.

Technical Note

The left and right hand sides of eq. (6) are multiplied by $g_{\rho\nu}$. On the left hand side of eq. (7) summation is no longer implied over the repeated σ index. This is also true in eq. (8) so the upper σ index in eq. (8) may be replaced by a μ index. In eq. (7) there are three ν indices on the left hand side and summation takes place on the ν indices inside the brackets. On the right hand side of eq. (7) there are also three ν indices and summation takes place on the indices inside the bracket. Similarly with the ρ indices on the RHS of eq. (7). In eq. (8) therefore, summation is no longer implied over the σ indices.

Ryder's eqn. (12) does therefore hold in any spacetime, but the proof is not straightforward.

3)

! Therefore Ryder appears to restrict the gener

equation (6) <u>to a special case</u>. The most generally valid relation is:

$$\boxed{X^{\mu}{}_{\rho\sigma}\, g^{\rho\nu}\, \epsilon^{\sigma}{}_{\nu} = \epsilon^{\mu}{}_{\nu}\, g^{\nu\sigma}\, x_{\sigma}} \quad - ($$

In general μ, ρ and σ can take any va

in eq. (6), but in eq. (12):

$$\mu = \rho \quad - (13)$$

$$\mu = \sigma \quad - (14)$$

This must mean that if $\mu = \rho$ then $\rho \neq \sigma$,

and if $\mu = \sigma$ then $\sigma \neq \rho$. For example:

$$X^{1}{}_{10} = \frac{1}{2}\left(\delta^{1}{}_{1}\, x_{0} - \delta^{1}{}_{0}\, x_{1}\right)$$

$$= \frac{1}{2} x_{0} \quad - (15)$$

This is therefore a dubious exercise by Ryder

nd it is therefore best to proceed as in not

98(4):

$$\boxed{x^{\mu} = X^{\mu}{}_{\rho\sigma}\, \epsilon^{\rho\sigma}} \quad - (16)$$