1)

93(1): Calculation of Light Bending from the S Tensor of Paper 91, Background

The traditional method is described by Carroll in chap... It starts by choosing u-coordinates such that the metric on the entire manifold is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{IJ}(v)\, dv^I dv^J + f(v)\, \gamma_{ij}(u)\, du^i du^j \quad -(1)$$

This is taken as a basic theorem. It implies in four dimensions that:

$$ds^2 = g_{aa}(a,b)\, da^2 + g_{ab}(a,b)(da\,db + db\,da) + g_{bb}(a,b)\, db^2 + r^2(a,b)\, d\Omega^2 \quad -(2)$$

$$= g_{aa}(a,r)\, da^2 + g_{ar}(a,r)(da\,dr + dr\,da) + g_{rr}(a,r)\, dr^2 + r^2 d\Omega^2 \quad -(3)$$

The chain rule is then used:

$$dt = \frac{\partial t}{\partial a} da + \frac{\partial t}{\partial r} dr \quad -(4)$$

$$dt^2 = \left(\frac{\partial t}{\partial a}\right)^2 da^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(da\,dr + dr\,da) + \left(\frac{\partial t}{\partial r}\right)^2 dr^2 \quad -(5)$$

The first three terms of eq. (3) are replaced by:

$$m\, dt^2 + n\, dr^2 \quad -(6)$$

i.e.:

$$m\left(\frac{\partial t}{\partial a}\right)^2 = g_{aa}, \quad n + m\left(\frac{\partial t}{\partial r}\right)^2 = g_{rr}; \quad -(7)$$

$$m\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right) = g_{ar}. \quad -(8)$$

Therefore:

$$ds^2 = m(t,r)\, dt^2 + n(t,r)\, dr^2 + r^2 d\Omega^2 \quad -(9)$$

This is then compared with the Minkowski metric:

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$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad -(10)$$

leading to the general ~~metric~~ line-element for a spherically symmetric space-time:

$$\boxed{ds^2 = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + r^2 d\Omega^2} \quad -(11)$$

This is a purely geometrical result.

The EH equation:

$$G_{\mu\nu} = k T_{\mu\nu} \quad -(12)$$

is then used to find α and β. This is done by considering:

$$G_{\mu\nu} \rightarrow 0 \quad -(13)$$

and this is called a vacuum solution. From eq. (13) it is found that:

$$\alpha = -\beta \quad -(14)$$

$$e^{2\alpha} = 1 + \frac{\mu}{r} \quad -(15)$$

so:

$$ds^2 = -\left(1 + \frac{\mu}{r}\right) dt^2 + \left(1 + \frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad -(16)$$

Finally the weak field limit is considered:

$$g_{00}(r \rightarrow \infty) = -\left(1 + \frac{\mu}{r}\right) \quad -(17)$$

$$g_{rr}(r \rightarrow \infty) = 1 - \frac{\mu}{r}. \quad -(18)$$

In the weak field limit.

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$$g_{00} = -\left(1 + 2\Phi\right) \quad \text{--- (19)}$$

$$g_{rr} = 1 - 2\Phi \quad \text{--- (20)}$$

where:

$$\Phi = -\frac{GM}{r} \quad \text{--- (21)}$$

is the Newtonian potential. Therefore: — (22)

$$ds^2 = -\left(1 - 2\frac{GM}{c^2 r}\right) dt^2 + \left(1 - 2\frac{GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Notes

1) Carroll leaves out the c^2 because he uses the normalized system of units rather than S.I.
2) He refers to eq. (22) as the Schwarzchild metric, but Crothers shows that it was derived by Hilbert.
3) Carroll does not discuss the problem of singularities.

Suggested More General Solution

It is not necessary to consider the vacuum solution. It is possible to consider the weak field limit of eq. (12) directly. This means that $T_{\mu\nu}$ is non-zero as it should be. In this weak field limit:

$$\boxed{\begin{aligned} g_{00} &= -e^{2\alpha(t,r)} \rightarrow -\left(1 + 2\frac{\Phi}{c^2}\right) \\ g_{rr} &= e^{2\beta(t,r)} \rightarrow 1 - 2\frac{\Phi}{c^2} \end{aligned}}$$

— (23)

— (24)

4)

Now use:

$$e^{2\alpha} = 1 + 2\alpha + \dots \quad -(25)$$

$$e^{2\beta} = 1 + 2\beta + \dots \quad -(26)$$

to find that in the weak field limit:

$$\alpha \rightarrow \Phi / c^2 \quad -(27)$$

$$\beta \rightarrow -\Phi / c^2 \quad -(28)$$

Note that the factor c^2 is again left out by Carroll because he uses $c = 1$ units. In S.I. units it is needed.

So it is found that:

$$\boxed{ds^2 \rightarrow -\exp\left(\frac{2\Phi}{c^2}\right)dt^2 + \exp\left(-\frac{2\Phi}{c^2}\right)dr^2 + r^2 d\Omega^2} \quad -(29)$$

in the weak field limit of eq. (12).

5 Tensor and Christoffel Symbol

Carroll shows that associated with eq. (11) there are non-vanishing Christoffel symbols, Riemann and Ricci tensor elements. Unfortunately there are dimensional inconsistencies in this part of Carroll's book, but I corrected them all in volume two, putting them in S.I. units.

5) Carroll uses the notation:

$$(t, r, \theta, \phi) = (0, 1, 2, 3) \quad -(30)$$

The Christoffel symbol relevant to the S tensor of paper 91 is:

$$\Gamma^{r}_{00} := \Gamma^{1}_{00} = e^{2(\alpha-\beta)} \frac{\partial \alpha}{\partial r} \quad -(31)$$

$$\boxed{\Gamma^{r}_{00} = \frac{1}{c^2} \exp\left(\frac{4\Phi}{c^2}\right) \frac{\partial \Phi}{\partial r}} \quad -(32)$$

where

$$\Phi = -\frac{GM}{r} \quad -(33)$$

The S tensor is:

$$\boxed{S^{r}_{00} = \Gamma^{r}_{00}}. \quad -(34)$$

Thus:

$$\frac{\partial \Phi}{\partial r} = \frac{GM}{r^2}, \quad -(35)$$

and:

$$\exp\left(\frac{4\Phi}{c^2}\right) \rightarrow 1 + \frac{4\Phi}{c^2} \quad -(36)$$

So:

$$S^{r}_{00} = \Gamma^{r}_{00} \rightarrow \left(1 + 4GM \right) GM \quad (37)$$

6)

From paper 91:

$$S^r_{00} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial r}, \quad -(30)$$

so

$$g = c^2 S^r_{00} = -\frac{GM}{r^2}. \quad -(32)$$

Traditionally, g is defined as positive valued, so we redefine:

$$g := -c^2 S^r_{00} = \frac{GM}{r^2} \quad -(33)$$

From eq. (32):

$$\boxed{g = \frac{GM}{r^2}\left(1 + \frac{4GM}{c^2 r}\right) : c^2 S^r_{00}} \quad -(34)$$

Thus:

$$\boxed{\begin{aligned} g(\text{Newton}) &= \frac{GM}{r^2} \\ g(S\ \text{tensor}) &= g(\text{Newton})\left(1 + \frac{4GM}{c^2 r}\right) \end{aligned}} \quad -(35)$$

This change in g leads to relativistic changes in orbits, and to the result that the light deflection is twice the Newtonian value, precession of ellipticals, Shapiro delay, Lense Thirring etc.

It is interesting to note that the deflection of light due to gravity in g.r. is:

$$\delta = \frac{4MG}{c^2 r_0} \quad \text{— (36)}$$

where r_0 is the distance of closest approach. By coincidence this is the correction factor to g in eq. (35). So:

$$\boxed{\frac{\Delta g}{g} = \delta(r = r_0)} \quad \text{— (37)}$$

The actual calculation of δ in ECE theory is given in paper 58, which has been distributed in typeset form. This calculation can now be repeated for the more general result (29).