1) Calculation to be Checked by Computer

1) $$R^0{}_{101} = -2(\partial_1\alpha)^2 - \partial_1(\partial_1\alpha) \quad -(1)$$

where: $$\partial_1\alpha = \frac{1}{2r}\left(e^{-2\alpha} - 1\right) \quad -(2)$$

$$R^0{}_{101} = -\frac{1}{2r^2}\left(e^{-2\alpha}-1\right)^2 - \frac{\partial}{\partial r}\left(\frac{1}{2r}\left(e^{-2\alpha}-1\right)\right)$$

$$= -\frac{1}{2r^2}\left(e^{-2\alpha}-1\right)^2 + \frac{1}{2r^2}\left(e^{-2\alpha}-1\right) - \frac{1}{2r}\frac{\partial}{\partial r}\left(e^{-2\alpha}-1\right)$$

$$= -\frac{1}{2r^2}\left(e^{-2\alpha}-1\right)^2 + \frac{1}{2r^2}\left(e^{-2\alpha}-1\right) + \frac{1}{r}(\partial_1\alpha)\,e^{-2\alpha}$$

$$= -\frac{1}{2r^2}\left(e^{-2\alpha}-1\right)^2 + \frac{1}{2r^2}\left(e^{-2\alpha}-1\right) + \frac{1}{2r^2}\left(e^{-2\alpha}-1\right)e^{-2\alpha}$$

$$= \frac{1}{2r^2}\left(e^{-2\alpha}-1\right)\left(1 - e^{-2\alpha} + 1 + e^{-2\alpha}\right)$$

$$R^0{}_{101} = \frac{1}{r^2}\left(e^{-2\alpha} - 1\right) \quad -(2)$$

where $$e^{-2\alpha} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \quad -(3)$$

$$R^0{}_{101} = \frac{1}{r^2}\left(\frac{1}{1-\frac{2GM}{rc^2}} - 1\right)$$

$$R^0{}_{101} = \frac{1}{r^2}\cdot\frac{2GM}{rc^2}\left(1 - \frac{2GM}{rc^2}\right)^{-1} \quad -(4)$$

2)

Finally use:

$$R^0{}_{110} = -R^0{}_{101} \quad -(5)$$

and: $$R^0{}_1{}^{10} = -R^0{}_1{}^{01} = -g^{11} g^{00} R^0{}_{101} \quad -(6)$$

where: $$g^{11} = \left(1 - \frac{2mG}{rc^2}\right), \quad g^{00} = -\left(1 - \frac{2mG}{rc^2}\right)^{-1}$$

So: $$\boxed{R^0{}_1{}^{10} = R^0{}_{101} = \frac{1}{r^2}\left(\frac{2Gm}{rc^2}\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1}} \quad -(7)$$

2) $$R^0{}_{202} = -\frac{1}{2} e^{-2\beta}\left(e^{-2\alpha} - 1\right) \quad -(8)$$

$$= -r e^{-2\beta} \partial_1 \alpha$$

$$R^0{}_2{}^{20} = \frac{1}{2} g^{22} g^{00} e^{-2\beta}\left(e^{-2\alpha} - 1\right)$$

where $$g^{22} = \frac{1}{r^2}, \quad g^{00} = -\left(1 - \frac{2mG}{rc^2}\right)^{-1}$$

So:

$$R^0{}_2{}^{20} = -\frac{1}{2r^2}\left(1 - \frac{2mG}{rc^2}\right)^{-1} e^{-2\beta}\left(e^{-2\alpha} - 1\right) \quad -(9)$$

where $$e^{-2\beta} = e^{2\alpha} \quad -(10)$$

So:

3)

$$R^{0}{}_{2}{}^{20} = \frac{-1}{2r^{2}}\left(1-\frac{2mG}{rc^{2}}\right)^{-1}\left(1-e^{2\alpha}\right)$$

where:
$$e^{2\alpha} = -\left(1-\frac{2mG}{rc^{2}}\right),$$

so:
$$R^{0}{}_{2}{}^{20} = -\frac{1}{r^{2}}\left(1-\frac{2mG}{rc^{2}}\right)^{-1}\left(2+\frac{2mG}{rc^{2}}\right)$$

$$\boxed{R^{0}{}_{2}{}^{20} = -\frac{2}{r^{2}}\left(1+\frac{2mG}{rc^{2}}\right)\left(1-\frac{2mG}{rc^{2}}\right)^{-1}} \quad -(11)$$

3) $$R^{0}{}_{303} = R^{0}{}_{202}\sin^{2}\theta \quad -(12)$$

$$R^{0}{}_{3}{}^{30} = -g^{33}g^{00}R^{0}{}_{303} \quad -(13)$$

where:
$$g^{00} = -\left(1-\frac{2mG}{rc^{2}}\right)^{-1},\quad g^{33} = \frac{1}{r^{2}\sin^{2}\theta} \quad -(14)$$

So:
$$R^{0}{}_{3}{}^{30} = \frac{-\sin^{2}\theta}{2r^{2}\sin^{2}\theta}\left(1-\frac{2mG}{rc^{2}}\right)^{-1}e^{-2\beta}\left(e^{-2\alpha}-1\right)$$

$$= R^{0}{}_{2}{}^{20} \quad -(15)$$

4)

So:

$$R^{0}{}_{2}{}^{20} = R^{0}{}_{3}{}^{30} = -\frac{2}{r^2}\left(1+\frac{2mG}{rc^2}\right)\left(1-\frac{2mG}{rc^2}\right)^{-1}$$

$$R^{0}{}_{1}{}^{10} = \frac{1}{r^2}\left(\frac{2Gm}{rc^2}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1}$$

So:

$$R^{0}{}_{1}{}^{10} + R^{0}{}_{2}{}^{20} + R^{0}{}_{3}{}^{30}$$

$$= \left(-\frac{4}{r^2}\left(1+\frac{2mG}{rc^2}\right) + \frac{1}{r^2}\left(\frac{2mG}{rc^2}\right)\right)\left(1-\frac{2Gm}{rc^2}\right)$$

$$= \left(-\frac{4}{r^2} - \frac{3}{r^2}\left(\frac{2mG}{rc^2}\right)\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1}$$

$$= -\frac{1}{r^2}\left(4+3\left(\frac{2mG}{rc^2}\right)\right)\left(1-\frac{2mG}{rc^2}\right)^{-1}$$

$$\boxed{\underline{\nabla}\cdot\underline{E} = \frac{\phi}{r^2}\left(4+3\left(\frac{2mG}{rc^2}\right)\right)\left(1-\frac{2mG}{rc^2}\right)^{-1}}$$

$$\longrightarrow \frac{4\phi}{r^2} \quad \text{if } mG \ll rc^2$$