88(7): A New Relation between Curvature and Torsion.

In note 88(6) it was shown that the conventionally named second Bianchi identity is true if and only if there is no torsion present. In the presence of torsion, Cartan geometry shows that

$$R^a{}_b \wedge q^b := D \wedge T^a \quad — (1)$$

and this is a true identity. In this note, a new fundamental relation between curvature and torsion is proven by taking the $D\wedge$ derivative of both sides of eq. (1):

$$D \wedge (R^a{}_b \wedge q^b) := D \wedge (D \wedge T^a). \quad — (2)$$

The general rule for the exterior derivative of an n-form is:

$$(d \wedge A)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]} \quad — (3)$$

For a 3-form for example:

$$(d \wedge A)_{\mu_1 \mu_2 \mu_3} = 3\, \partial_{[\mu_1} A_{\mu_2 \mu_3]}. \quad — (4)$$

The cyclic permutations are:

$$\mu_1 \mu_2 \mu_3, \quad \mu_3 \mu_1 \mu_2, \quad \mu_2 \mu_3 \mu_1. \quad — (5)$$

For a 4-form:

$$(d \wedge A)_{\mu_1 \mu_2 \mu_3 \mu_4} = 4\, \partial_{[\mu_1} A_{\mu_2 \mu_3 \mu_4]}. \quad — (6)$$

The cyclic permutations are:

$$\mu_1 \mu_2 \mu_3 \mu_4, \; \mu_4 \mu_1 \mu_2 \mu_3, \; \mu_3 \mu_4 \mu_1 \mu_2, \; \mu_2 \mu_3 \mu_4 \mu_1 \quad — (7)$$

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In tensor notation:

$$R^a{}_b \wedge q^b = R^a{}_{\mu\nu\rho} + R^a{}_{\nu\rho\mu} + R^a{}_{\rho\mu\nu}. \quad -(8)$$

Therefore, with:

$$\mu_1 = \mu,\ \mu_2 = \nu,\ \mu_3 = \rho,\ \mu_4 = \sigma \quad -(9)$$

we have:

$$D \wedge (R^a{}_b \wedge q^b)$$

$$= D \wedge (R^a{}_{\mu\nu\rho} + R^a{}_{\nu\rho\mu} + R^a{}_{\rho\mu\nu}) \quad -(10)$$

$$= D \wedge R^a{}_{\mu\nu\rho} + D \wedge R^a{}_{\nu\rho\mu} + D \wedge R^a{}_{\rho\mu\nu}$$

Using eqns. (6), (7) and (9):

$$\boxed{\begin{aligned} D \wedge (R^a{}_b \wedge q^b) &= D_\sigma R^a{}_{\mu\nu\rho} + D_\mu R^a{}_{\nu\rho\sigma} + D_\rho R^a{}_{\sigma\mu\nu} + D_\nu R^a{}_{\rho\sigma\mu} \\ &+ D_\sigma R^a{}_{\nu\rho\mu} + D_\mu R^a{}_{\sigma\nu\rho} + D_\rho R^a{}_{\mu\sigma\nu} + D_\nu R^a{}_{\rho\mu\sigma} \\ &+ D_\sigma R^a{}_{\rho\mu\nu} + D_\nu R^a{}_{\sigma\rho\mu} + D_\mu R^a{}_{\nu\sigma\rho} + D_\rho R^a{}_{\mu\nu\sigma} \end{aligned}}$$

$$-(11)$$

We now use the antisymmetry in the last two indices of the Riemann forms:

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$$D_\nu R^a{}_{\rho\sigma\mu} = - D_\nu R^a{}_{\rho\mu\sigma} \quad - (12)$$

$$D_\mu R^a{}_{\nu\rho\sigma} = - D_\mu R^a{}_{\nu\sigma\rho} \quad - (13)$$

$$D_\rho R^a{}_{\mu\sigma\nu} = - D_\rho R^a{}_{\mu\nu\sigma}. \quad - (14)$$

So:

$$\boxed{\begin{aligned} D \wedge \left(R^a{}_b \wedge q^b \right) = & \; D_\sigma \left(R^a{}_{\mu\nu\rho} + R^a{}_{\nu\rho\mu} + R^a{}_{\rho\mu\nu} \right) \\ & + D_\rho R^a{}_{\sigma\mu\nu} + D_\mu R^a{}_{\sigma\nu\rho} + D_\nu R^a{}_{\sigma\rho\mu} \end{aligned}}.$$

$$- (15)$$

Now use:

$$\begin{aligned} D \wedge R^a{}_b &= D_\rho R^a{}_{\sigma\mu\nu} + D_\mu R^a{}_{\sigma\nu\rho} + D_\nu R^a{}_{\sigma\rho\mu} \\ &= D_\rho R^a{}_{b\sigma\mu\nu} q^b{}_\sigma + D_\mu R^a{}_{b\nu\rho} q^b{}_\sigma + D_\nu R^a{}_{b\rho\mu} q^b{}_\sigma \\ &= q^b{}_\sigma \left(D_\rho R^a{}_{b\mu\nu} + D_\mu R^a{}_{b\nu\rho} + D_\nu R^a{}_{b\rho\mu} \right) \quad - (16) \end{aligned}$$

$$R^a{}_b \wedge q^b = R^a{}_{\mu\nu\rho} + R^a{}_{\nu\rho\mu} + R^a{}_{\rho\mu\nu}. \quad - (17)$$

Thus:

$$\boxed{D \wedge \left(R^a{}_b \wedge q^b \right) = D \wedge R^a + D \left(R^a{}_b \wedge q^b \right)}$$

$$- (18)$$

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where:

$$D \wedge R^a := q^b_\sigma \left(D_\rho R^a{}_{b\mu\nu} + D_\mu R^a{}_{b\nu\rho} + D_\nu R^a{}_{b\rho\mu} \right) \quad -(1$$

$$D\left(R^a{}_b \wedge q^b\right) := D_\sigma \left(R^a{}_{\mu\nu\rho} + R^a{}_{\nu\rho\mu} + R^a{}_{\rho\mu\nu} \right) \quad -(20$$

Now we:

$$R^a{}_b \wedge q^b := D \wedge T^a \quad -(21)$$

So the true second Bianchi identity is:

$$\boxed{D \wedge R^a := D \wedge (D \wedge T^a) - D(D \wedge T^a)} \quad -(22)$$

However, we know that:

$$D \wedge (D \wedge T^a) =$$

$$= D \wedge \left(D_\mu T^a{}_{\nu\rho} + D_\nu T^a{}_{\rho\mu} + D_\rho T^a{}_{\mu\nu} \right)$$

$$= D_\sigma \left(D_\mu T^a{}_{\nu\rho} + D_\nu T^a{}_{\rho\mu} + D_\rho T^a{}_{\mu\nu} \right)$$
$$+ D_\rho D_\sigma T^a{}_{\mu\nu} + D_\mu D_\sigma T^a{}_{\nu\rho} + D_\nu D_\sigma T^a{}_{\rho\mu}$$

$$= 2 D_\sigma \left(D_\mu T^a{}_{\nu\rho} + D_\nu T^a{}_{\rho\mu} + D_\rho T^a{}_{\mu\nu} \right)$$

$$:= 2 D (D \wedge T^a) \quad -(23)$$

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So:

$$\boxed{D \wedge R^a := D(D \wedge T^a)} \quad -(24)$$

This is the true second Bianchi identity, and a new result in Cartan geometry. In eq. (24):

$$D \wedge R^a = q^b_\sigma \left(D_\rho R^a{}_{b\mu\nu} + D_\mu R^a{}_{b\nu\rho} + D_\nu R^a{}_{b\rho\mu} \right)$$

$$:= q^b_\sigma \, D \wedge R^a{}_b \quad -(25)$$

So:

$$\boxed{D \wedge R^a{}_b := q^\sigma_b D_\sigma (D \wedge T^a)} \quad -(26)$$

The conventional "second Bianchi identity" is recovered if and only if:

$$D \wedge T^a = 0 \quad -(27)$$

i.e.

$$R^a{}_b \wedge q^b = 0 \quad -(28)$$

which is the conventional "first Bianchi identity",

Q.E.D.

This has basic consequence for general relativity because the Einstein Hilbert field equation is conventionally:

$$D \wedge R^a{}_b = k D \wedge N^a{}_b \quad -(29)$$

where k is the Einstein constant and where $N^a{}_b$ the Noether form:

$$N^a{}_b := N^a{}_{b\mu\nu}. \quad -(30)$$

Thus:

$$\boxed{R^a{}_b = k N^a{}_b} \quad -(31)$$

is the EH field equation.

From eq. (24) we can see that eq. (31) can be written as:

$$\boxed{T^a_{\mu\nu} = k N^a_{\mu\nu}}. \quad -(32)$$

Here:

$$R^a{}_{b\mu\nu} = -R^a{}_{b\nu\mu} \quad -(33)$$

$$T^a{}_{\mu\nu} = -T^a{}_{\nu\mu} \quad -(34)$$

So:

$$N^a{}_{b\mu\nu} = -N^a{}_{b\nu\mu} \quad -(35)$$

$$N^a_{\mu\nu} = -N^a_{\nu\mu} \quad -(36)$$

The whole of cosmology can be developed as torsion.