1)

Revised Method for Paper 85

The radiative correction to the Schrödinger equation of the H atom is, to first order in α:

$$-\frac{\hbar^2}{2m}\nabla^2\left(1+\frac{\alpha}{2\pi}\right)\phi - \frac{e^2}{4\pi\epsilon_0 r}\phi = E\phi. \quad -(1)$$

This is equivalent to:

$$-\frac{\hbar^2}{2m}\nabla^2\phi - \frac{e^2}{4\pi\epsilon_0(r+r(\mathrm{vac}))}\phi = E\phi. \quad -(2)$$

Assume that $r(\mathrm{vac})$ is small, so:

$$\phi = \phi_0 \quad -(3)$$

where ϕ_0 is the H atom wave function in the absence of radiative corrections. Now subtract eq. (2) from eq. (1):

$$-\frac{\hbar^2\alpha}{4\pi m}\nabla^2\phi_0 = \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r+r(\mathrm{vac})}\right)\phi_0 \quad -(4)$$

i.e.

$$\boxed{\nabla^2\phi_0 = -\frac{4\pi mc}{\hbar}\left(\frac{1}{r} - \frac{1}{r+r(\mathrm{vac})}\right)\phi_0} \quad -(5)$$

Evaluating this equation by computer algebra gives:

2)

$$\frac{1}{r + r_{2p}(\text{vac})} - \frac{1}{r + r_{2s}(\text{vac})} = \frac{1}{4\pi^{3/2}} \frac{\hbar}{mc} \frac{1}{r^2} \qquad -(6)$$

The change in potential energy is:

$$\Delta V = \alpha \left(\frac{1}{r + r_{2p}(\text{vac})} - \frac{1}{r + r_{2s}(\text{vac})} \right) \text{cm}^{-1} \qquad -(7)$$

$$= \frac{\alpha}{4\pi^{3/2}} \frac{\hbar}{mc} \frac{1}{r^2} \qquad -(8)$$

The change in total energy is:

$$\Delta E = \frac{r}{2n^2 a} \Delta V = \frac{\alpha}{32\pi^{3/2}} \frac{\hbar}{mc} \frac{1}{a} \cdot \frac{1}{r}$$

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$$\boxed{\Delta E = \left(\frac{1}{32\pi^{3/2}} \frac{\alpha}{a} \frac{\hbar}{mc} \right) \frac{1}{r} \ \text{cm}^{-1} = 0.0353 \ \text{cm}^{-1}} \qquad -(9)$$

This gives:

$$r = 8.46 \times 10^{-4} \text{m}. \qquad -(10)$$

From eq. (6):

$$\frac{r_{2s}(\text{vac}) - r_{2p}(\text{vac})}{(r + r_{2p}(\text{vac}))(r + r_{2s}(\text{vac}))} = \frac{1}{4\pi^{3/2}} \frac{\hbar}{mc} \frac{1}{r^2} \qquad -(11)$$

Eq (3) is consistent with the assumption:

3)

$$r > r_{2S} \sim r_{2P}, \quad -(12)$$

$$r \gg \left(r_{2S}(\text{vac}) - r_{2P}(\text{vac})\right) \quad -(13)$$

So from eq. (11):

$$\boxed{r_{2S}(\text{vac}) - r_{2P}(\text{vac}) \sim \frac{1}{4\pi^{3/2}} \frac{\hbar}{mc}} \quad -(14)$$

i.e. $$r_{2S}(\text{vac}) - r_{2P}(\text{vac}) \sim \frac{1}{8\pi^{5/2}} \frac{h}{mc} \quad -(15)$$

where $$\lambda_c = \text{Compton wavelength} = \frac{h}{mc} = 2.426 \times 10^{-12} \text{ m}$$

so $$\boxed{r_{2S}(\text{vac}) - r_{2P}(\text{vac}) = 1.74 \times 10^{-14} \text{ m}} \quad -(16)$$

and indeed this is much less than r (eq. (10))

This result can be compared with the classical radius of the electron:

$$r(\text{classical}) = \frac{1}{4\pi\epsilon_0} \frac{e^2}{mc^2} = 2.818 \times 10^{-15} \text{ m} \quad -(17)$$

and the Bohr radius:

$$a = 5.292 \times 10^{-11} \text{ m} \quad -(18)$$

From these results it is seen that the vacuum perturbs the electron as expected, i.e. much less than the Bohr radius and a little greater than its classical radius