1) 64(5): Demonstration of Resonances with a Fourier Series

Consider the basic resonance equation:

$$\frac{d^2\phi}{dR^2} + \kappa^2\phi = \frac{\rho(0)}{\epsilon_0} f(\kappa R) \quad -(1)$$

where:

$$f(\kappa R) = e^{2i\kappa R}\cos\left(e^{i\kappa R}\right). \quad -(2)$$

If $f(\kappa R)$ is single valued and continuous in an interval such as $\pi < f(\kappa R) \leq \pi$ it can be expanded in a Fourier series:

$$f(\kappa R) = \frac{a_0}{2} + \sum_{\alpha=1}^{\infty}\left(a_\alpha \cos(\alpha\kappa R) + b_\alpha \sin(\alpha\kappa R)\right) \quad -(3)$$

where:

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\kappa R)\, d(\kappa R) \\ a_\alpha &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\kappa R)\cos(\alpha\kappa R)\, d(\kappa R) \\ b_\alpha &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\kappa R)\sin(\alpha\kappa R)\, d(\kappa R) \end{aligned}\right] \quad -(3a)$$

These integrals can easily be evaluated by computer in any interval, not just $\pi < f(\kappa R) \leq \pi$.

Therefore eq (1) becomes: $\quad (4)$

$$\frac{d^2\phi}{dR^2} + \kappa^2\phi = \frac{\rho(0)}{\epsilon_0}\left(\frac{a_0}{2} + a_1\cos(\kappa R) + a_2\cos(2\kappa R) + \ldots + b_1\sin(\kappa R) + b_2\sin(2\kappa R) + \ldots\right)$$

2) Assume a solution to eq. (4) of the form:

$$\phi = \frac{\rho(0)}{\epsilon_0}\left(A_0 \frac{a_0}{2} + A_1 a_1 \cos(\kappa R) + A_2 a_2 \cos(2\kappa R) + \cdots + B_1 b_1 \sin(\kappa R) + B_2 b_2 \sin(2\kappa R) + \cdots\right) \quad -(5)$$

Then substituting eq. (5) into eq. (4) and comparing term by term:

$$\left.\begin{aligned} \kappa^2 A_0 \frac{a_0}{2} &= \frac{a_0}{2} \\ A_1 \kappa^2 (1-a_1)\cos(\kappa R) &= \cos(\kappa R) \\ A_2 \kappa^2 (1-4a_2)\cos(2\kappa R) &= \cos(2\kappa R) \\ &\vdots \\ B_1 \kappa^2 (1-b_1)\sin(\kappa R) &= \sin(\kappa R) \\ B_2 \kappa^2 (1-4b_2)\sin(2\kappa R) &= \sin(2\kappa R) \end{aligned}\right] \quad -(6)$$

$-(7)$

Thus:

$$\boxed{\phi = \frac{\rho(0)}{\epsilon_0 \kappa^2}\left(\frac{a_0}{2} + \frac{\cos(\kappa R)}{(1-a_1)} + \frac{\cos(2\kappa R)}{(1-4a_2)} + \cdots + \frac{\sin(\kappa R)}{(1-b_1)} + \frac{\sin(2\kappa R)}{(1-4b_2)} + \cdots\right)}$$

<u>Resonances</u> occur at:

$$\left.\begin{aligned} a_1 &= 1 \\ a_2 &= 1/4 \\ a_3 &= 1/9 \\ &\vdots \end{aligned}\right\} \quad -(8)$$

3) From eq (8) & eq. (3a), resonances occur at:

$$\mathrm{Real}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\cos(\kappa R)\,d(\kappa R)\right) = 1$$

— (9)

$$\mathrm{Real}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\cos(2\kappa R)\,d(\kappa R)\right) = \frac{1}{4} \quad \text{— (10)}$$

$$\mathrm{Real}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\cos(3\kappa R)\,d(\kappa R)\right) = \frac{1}{9} \quad \text{— (11)}$$

etc.

In general:

$$\boxed{\mathrm{Real}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\cos(n\kappa R)\,d(\kappa R)\right) = \frac{1}{n^2}}$$

— (12)

The general solution of eq (1) is eq (7).

— (13)

$$\mathrm{Real}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\sin(n\kappa R)\,d(\kappa R)\right) = \frac{1}{\cdots}$$

4) This method can be used for any function:

$$f(\kappa R) = e^{2i\kappa R} f_1\left(e^{i\kappa R}\right) \quad — (14)$$

Resonances occur at:

$$\mathrm{Real}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2i\kappa R} f_1\left(e^{i\kappa R}\right)\cos(n\kappa R)\, d(\kappa R)\right) = \frac{1}{n^2} \quad — (15)$$

$$\mathrm{Real}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2i\kappa R} f_1\left(e^{i\kappa R}\right)\sin(n\kappa R)\, d(\kappa R)\right) = \frac{1}{n^2} \quad — (16)$$

Any interval can be chosen, not just the interval $-\pi < \kappa R \leq \pi$. The Dirichlet condition means that $f(\kappa R)$ must be single valued and continuous in the interval.