Some Additional Details of the Euler Method

Start with:

$$a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = f(x) \quad -(1)$$

and use:

$$x = e^t \quad -(2)$$

then:

$$t = \log_e x. \quad -(3)$$

and

$$\frac{dt}{dx} = \frac{1}{x}. \quad -(4)$$

then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}. \quad -(5)$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) \quad -(6)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) \quad -(7)$$

Now use eq. (5) with $y \to dy/dt$:

$$\frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{1}{x}\frac{d^2y}{dt^2}. \quad -(8)$$

So in eq. (7):

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2} \quad -(9)$$

From eq. (5):

$$x\frac{dy}{dx} = \frac{dy}{dt}. \quad -(10)$$

From eq. (9):

$$x^2\frac{d^2y}{dx^2} = -\frac{dy}{dt} + \frac{d^2y}{dt^2} \quad -(11)$$

2) Eqs. (10) and (11) are given by Stephenson, "Mathematical Methods for Science Students". It the does not give the intermediate steps (6) to (9). Using eqs. (10) and (11) in eq. (1) gives the following partial differential equation with constant coefficients:

$$a_0 \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + a_1 \frac{dy}{dt} + a_2 y = f(e^t) \quad - (12)$$

i.e.

$$\frac{d^2 y}{dt^2} + \left(\frac{a_1 - a_0}{a_0} \right) \frac{dy}{dt} + a_2 y = f(e^t) \quad - (13)$$

We now apply this method to the resonance equation:

$$\frac{d^2 \phi}{dZ^2} + \frac{1}{Z} \frac{d\phi}{dZ} - \frac{\phi}{Z^2} = -\frac{\rho_0}{\epsilon_0} \cos(\kappa Z) - (14)$$

The change of variable in eq. (2) is replaced by:

$$Z = Z_0 e^{i\kappa x} \quad - (15)$$

where Z_0 is a constant, κ is a constant, and x is a variable. In order to reduce the number of variables we assume that:

$$\kappa = 1 / Z_0 \quad - (16)$$

From eq. (15):

$$i\kappa x = \log_e \left(\frac{Z}{Z_0} \right) \quad - (17)$$

3)

So
$$x = -\frac{i}{k}\left(\log_e Z - \log_e Z_0\right) \quad -(18)$$

$$\frac{dx}{dZ} = -\frac{i}{kZ} \quad -(19)$$

Now use:
$$\frac{d\phi}{dZ} = \frac{d\phi}{dx}\frac{dx}{dZ} \quad -(20)$$

i.e
$$\frac{d\phi}{dZ} = -\frac{i}{kZ}\frac{d\phi}{dx} \quad -(21)$$

So:
$$Z\frac{d\phi}{dZ} = -\frac{i}{k}\frac{d\phi}{dx} \quad -(22)$$

From eq. (21):
$$\frac{d^2\phi}{dZ^2} = -\frac{i}{k}\frac{d}{dZ}\left(\frac{1}{Z}\frac{d\phi}{dx}\right)$$
$$= \frac{i}{kZ^2}\frac{d\phi}{dx} - \frac{i}{kZ}\frac{d}{dZ}\left(\frac{d\phi}{dx}\right) \quad -(23)$$

Now use eq. (21) wth $\phi \rightarrow d\phi / dx$:
$$\frac{d}{dZ}\left(\frac{d\phi}{dx}\right) = -\frac{i}{kZ}\frac{d^2\phi}{dx^2} \quad -(24)$$

So:
$$\frac{d^2\phi}{dZ^2} = \frac{i}{kZ^2}\frac{d\phi}{dx} - \frac{1}{k^2Z^2}\frac{d^2\phi}{dx^2} \quad -(25)$$

$$Z^2\frac{d^2\phi}{dZ^2} = \frac{i}{k}\frac{d\phi}{dx} - \frac{1}{k^2}\frac{d^2\phi}{dx^2} \quad -(26)$$

4) Eq. (14) is:

$$z^2 \frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} - \phi = -\frac{\rho_0}{\epsilon_0} z^2 \cos(kz) \quad -(27)$$

Using eq. (22) and eq. (26) in eq. (27):

$$\frac{i}{k}\frac{d\phi}{dx} - \frac{1}{k^2}\frac{d^2\phi}{dx^2} - \frac{i}{k}\frac{d\phi}{dx} - \phi = -\frac{\rho_0}{\epsilon_0} z^2 \cos(kz) \quad -(28)$$

i.e.

$$\boxed{\frac{d^2\phi}{dx^2} + k^2\phi = \frac{\rho_0}{\epsilon_0} k^2 z^2 \cos(kz)} \quad -(29)$$

Using eqs (15) and (16):

$$\frac{d^2\phi}{dx^2} + k^2\phi = \frac{\rho_0}{\epsilon_0} e^{2ikx} \cos(e^{ikx}) \quad -(30)$$

$$:= \frac{\rho_0}{\epsilon_0} \cos(k' x)$$

Real Part of RHS of Eq. (30)

By de Moivre's Theorem:

$$e^{2ikx} = \cos(2kx) + i\sin(2kx) \quad -(31)$$

$$e^{ikx} = \cos(kx) + i\sin(kx) \quad -(32)$$

So we expand the exponentials:

$$e^{2ikx}\cos(e^{ikx}) = (\cos(2kx) + i\sin(2kx))(\cos(\cos(kx) + i\sin(kx)))$$

5) Using:

$$\cos(A+B) = \cos A\cos B + \sin A \sin B \quad -(33)$$

and

$$\cos(ix) = \cosh x \quad -(34)$$

$$\sin(ix) = i\sinh x \quad -(35)$$

Then:

$$\left.\begin{aligned}&\cos\left(\cos(kx) + i\sin(kx)\right)\\ &= \cos(\cos(kx))\cosh(\sin(kx))\\ &\quad + i\sin(\cos(kx))\sinh(\sin(kx))\end{aligned}\right\} -(36)$$

So:

$$\left.\begin{aligned}&\mathrm{Real}\left(e^{2ikx}\cos\left(e^{ikx}\right)\right)\\ &= \cos(2kx)\cos(\cos(kx))\cosh(\sin(kx))\\ &\quad - \sin(2kx)\sin(\cos(kx))\sinh(\sin(kx))\\ &\quad := \cos(k'x)\end{aligned}\right\} -(37)$$

So:

$$\boxed{k' = \frac{1}{x}\cos^{-1}\left(\cos(2kx)\cos(\cos(kx))\cosh(\sin(kx)) - \sin(2kx)\sin(\cos(kx))\sinh(\sin(kx))\right)} \quad -(38)$$

and

$$\boxed{\frac{d^2\phi}{dx^2} + k^2\phi = \frac{\rho_0}{\epsilon_0}\cos(k'x)} \quad -(39)$$

6) Since κ' is a function of x given by eq. (38) eq. (39) has to be solved numerically in general, but has the form of an undamped oscillator equation.

If κ' can be regarded as only a slowly varying function of x:

$$\kappa' \sim \kappa_0' \quad - (40)$$

where κ_0' is a constant. Then:

$$\phi \doteq \frac{\rho_0}{\epsilon_0} \frac{\cos(\kappa_0' x)}{\kappa^2 - \kappa_0'^2} \quad - (41)$$

This is only an approximate solution.