1)

Notes 55(9)

Second Bianchi Identity, Schwarzschild Metric

In form notation this is:

$$D \wedge R^a{}_b = 0 \quad -(1)$$

i.e.

$$d \wedge R^a{}_b = j^a{}_b = R^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge R^c{}_b$$
$$-(2)$$

In tensor notation:

$$\partial_\mu \tilde{R}^a{}_b{}^{\mu\nu} = \tilde{j}^a{}_b{}^\nu \quad -(3)$$

where Hodge dual is

$$\partial_\mu R^a{}_b{}^{\mu\nu} = J^a{}_b{}^\nu \quad -(4)$$

The operators ∂_μ are cartesian operators and so refer to a cartesian unit vector system $\underline{i}, \underline{j}, \underline{k}$. Thus in vector notation:

$$\underline{R}^a{}_b \underset{\text{(orbital)}}{)} = R^a{}_{b01}\,\underline{i} + R^a{}_{b02}\,\underline{j} + R^a{}_{b03}\,\underline{k}$$
$$-(5)$$

$$\underline{R}^a{}_b(\text{spin}) = R^2{}_{323}\,\underline{i} + R^1{}_{331}\,\underline{j} + R^1{}_{212}\,\underline{k}$$
$$-(6)$$

It is also possible to define:

$$\underline{R}^a{}_b(\text{orbital}) = R^0{}_1{}^{01}\,\underline{i} + R^0{}_2{}^{02}\,\underline{j} + R^0{}_3{}^{03}\,\underline{k}$$
$$-(7)$$

$$\underline{R}^a{}_b(\text{spin}) = R^2{}_3{}^{23}\,\underline{i} + R^1{}_3{}^{31}\,\underline{j} + R^1{}_2{}^{12}\,\underline{k}$$

In the Schwarzschild metric:

$$\underline{R}^{0}{}_{1} = R^{0}{}_{1}{}^{01}\,\underline{i}\;;\; \underline{R}^{0}{}_{2} = R^{0}{}_{2}{}^{02}\,\underline{j}\;;\; \underline{R}^{0}{}_{3} = R^{0}{}_{3}{}^{03}\,\underline{k} \quad -(9)$$

so we define:

$$\underline{R} = \underline{R}^{0}{}_{1} + \underline{R}^{0}{}_{2} + \underline{R}^{0}{}_{3} \quad -(10)$$

Thus:

$$\underline{R}\,(\text{orbital}) = R^{0}{}_{1}{}^{01}\,\underline{i} + R^{0}{}_{2}{}^{02}\,\underline{j} + R^{0}{}_{3}{}^{03}\,\underline{k} \quad -(11)$$

$$\underline{R}\,(\text{spin}) = R^{2}{}_{3}{}^{23}\,\underline{i} + R^{1}{}_{3}{}^{31}\,\underline{j} + R^{1}{}_{2}{}^{12}\,\underline{k} \quad -(12)$$

The equations of motion are therefore:

$$\partial_\mu \tilde{R}^{\mu\nu} = \tilde{j}^{\nu}\;;\; \partial_\mu R^{\mu\nu} = J^{\nu} \quad -(13)$$

where:

$$R^{\mu\nu} = \begin{bmatrix} 0 & R^{0}{}_{1}{}^{01} & R^{0}{}_{2}{}^{02} & R^{0}{}_{3}{}^{03} \\ R^{0}{}_{1}{}^{10} & 0 & R^{1}{}_{2}{}^{12} & R^{1}{}_{3}{}^{13} \\ R^{0}{}_{2}{}^{20} & R^{1}{}_{2}{}^{21} & 0 & R^{2}{}_{3}{}^{23} \\ R^{0}{}_{3}{}^{30} & R^{1}{}_{3}{}^{31} & R^{2}{}_{3}{}^{32} & 0 \end{bmatrix} \quad -(14)$$

3) In vector notation:

$$\underline{\nabla} \cdot \underline{R}\,(\text{spin}) = \tilde{j}^{\,0} \quad -(15)$$

$$\underline{\nabla} \times \underline{R}\,(\text{orbital}) + \frac{1}{c}\frac{\partial}{\partial t}\underline{R}\,(\text{spin}) = \underline{\tilde{j}} \quad -(16)$$

$$\underline{\nabla} \cdot \underline{R}\,(\text{orbital}) = J^{\,0} \quad -(17)$$

$$\underline{\nabla} \times \underline{R}\,(\text{spin}) - \frac{1}{c}\frac{\partial \underline{R}}{\partial t}(\text{orbital}) = \underline{J} \quad -(18)$$

Eq (15) is analogous to the Gauss Law of magnetism, eq (16) to the Faraday law of induction, eq. (17) to the Coulomb Law and eq (18) to the Ampère Maxwell law.

The whole of Newtonian dynamics emerge from eq. (17).