Notes 55(7)

Schwarzschild Metric (SM)

In notes 55(6) the second Bianchi identity of Cartan geometry was used to show that:

$$\underline{R}^a{}_b = \frac{kR}{c^2}\underline{F}^a{}_b . \quad -(1)$$

In these notes the force $\underline{F}^a{}_b$ is worked out for the SM. In spherical polar coordinates the SM is:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)(c\,dt)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$

$$- r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \quad -(2)$$

This gives six elements of the Riemann tensor:

$$R^0{}_{101} = e^{2(\beta-\alpha)}\left(\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\,\partial_0\beta\right) + \partial_1\alpha\,\partial_1\beta - \partial_1^2\alpha - (\partial_1\alpha)^2 \quad -(3)$$

$$R^0{}_{202} = -r\,e^{-2\beta}\partial_1\alpha / r^2 \quad -(4)$$

$$R^0{}_{303} = \sin^2\theta\, R^0{}_{202} \quad -(5)$$

$$R^1{}_{212} = r\,e^{-2\beta}\partial_1\beta / r^2 \quad -(6)$$

$$R^1{}_{313} = R^2{}_{323} = \left(1 - e^{-2\beta}\right)\sin^2\theta / r^2 \quad -(7)$$

where:

$$e^{2\alpha} = e^{-2\beta} = 1 - \frac{2GM}{rc^2} . \quad -(8)$$

2)

Thus:

$$R^{0}{}_{202} + R^{0}{}_{303} = -\frac{GM}{c^2 r^3}\left(1+\sin^2\theta\right) \quad -(9)$$

and $$R^{0}{}_{101} = e^{-4\alpha}\left(-(\partial_1\alpha)^2 - \partial_1^2\alpha - (\partial_1\alpha)^2\right) \quad -(10)$$

The Newtonian limit is given by:

$$\underline{\nabla}\cdot\underline{g}^{a}{}_{b} = -c^2\left(R^{0}{}_{202}+R^{0}{}_{303}\right)$$
$$= \frac{2GM}{r^3}, \quad -(11)$$

so: $$\underline{g}^{a}{}_{b} = \frac{1}{m}\underline{F}^{a}{}_{b} = -\frac{GM}{r^2}\underline{k}^{a}{}_{b} \quad -(12)$$

and $$\boxed{\underline{F}^{a}{}_{b} = -\frac{GmM}{r^2}\underline{k}^{a}{}_{b}} \quad -(13)$$

In the flat space-time of the Newtonian limit:

$$\underline{F}^{a}{}_{b} = \underline{F}, \quad \underline{k}^{a}{}_{b} = \underline{k}, \quad -(14)$$

and we obtain the inverse square law of Newton:

$$\underline{F} = -\frac{GmM}{r^2}\underline{k}. \quad -(15)$$

3)

Orbital and Spin Elements of the Riemann Tensor.

The Riemann tensor is defined as:

$$R^{\rho}{}_{\sigma\mu\nu} = q^{b}{}_{\sigma} q^{\rho}{}_{a} R^{a}{}_{b\mu\nu} \quad -(16)$$

Therefore:

$$\underline{R}^{a}{}_{b}\,(\text{orbital}) = R^{0}{}_{101}\, \underline{i}^{a}{}_{b} + R^{0}{}_{202}\, \underline{j}^{a}{}_{b} + R^{0}{}_{303}\, \underline{k}^{a}{}_{b} \quad -(17)$$

$$\underline{R}^{a}{}_{b}\,(\text{spin}) = R^{2}{}_{323}\, \underline{i}^{a}{}_{b} + R^{1}{}_{331}\, \underline{j}^{a}{}_{b} + R^{1}{}_{212}\, \underline{k}^{a}{}_{b} \quad -(18)$$

The Riemann form in the SM is:

$$R^{a}{}_{b\mu\nu} = \begin{bmatrix} 0 & -R^{0}{}_{101} & -R^{0}{}_{202} & -R^{0}{}_{303} \\ R^{0}{}_{101} & 0 & -R^{1}{}_{212} & R^{1}{}_{313} \\ R^{0}{}_{202} & R^{1}{}_{212} & 0 & -R^{2}{}_{323} \\ R^{0}{}_{303} & -R^{1}{}_{313} & R^{2}{}_{323} & 0 \end{bmatrix} \quad -(19)$$

It is seen that the six non-vanishing elements of the SM are precisely the three elements of the orbital Riemann vector (17) and spin Riemann vector (18).

4)

Since a and b always refer to a Minkowski spacetime then eqn. (14) is always true, so

$$\boxed{\begin{aligned} \underline{R}\,(\text{orbital}) &= R^0{}_{101}\,\underline{i} + R^0{}_{202}\,\underline{j} + R^0{}_{303}\,\underline{k} \qquad -(20) \\ \underline{R}\,(\text{spin}) &= R^2{}_{323}\,\underline{i} + R^1{}_{331}\,\underline{j} + R^1{}_{212}\,\underline{k}. \end{aligned}}$$
$$-(21)$$

The vectors in eqns (20) and (21) obey the same type of equations as in the ECE theory of electrodynamics. The Newton equation is only one of these. There are three more equations.