1)

Notes for Paper 55, Part 4

Evaluation of the Orbital Torque

The complete orbital torque is:

$$\underline{N}_L^{\ a} = -\frac{\partial \underline{J}^a}{\partial t} - c\underline{\nabla} J^{0a} - c\omega^{0a}{}_b \underline{J}^b + c\underline{\omega}^a{}_b J^{0b} \quad -(1)$$

This orbital torque is a consequence of the Cartan structure equation itself. If angular momentum is considered to be a space-like property the scalars J^{0a} and J^{0b} vanish because the 0 index is time-like and the a and b indices are space-like.

So eq. (1) reduces to:

$$\underline{N}_L^{\ a} = -\frac{\partial \underline{J}^a}{\partial t} - c\omega^{0a}{}_b \underline{J}^b . \quad -(2)$$

The components of eq. (2) are:

$$\underline{N}_L^{\ 1} = -\frac{\partial \underline{J}^1}{\partial t} - c\omega^{01}{}_b \underline{J}^b \quad -(3)$$

$$\underline{N}_L^{\ 2} = -\frac{\partial \underline{J}^2}{\partial t} - c\omega^{02}{}_b \underline{J}^b \quad -(4)$$

$$\underline{N}_L^{\ 3} = -\frac{\partial \underline{J}^3}{\partial t} - c\omega^{03}{}_b \underline{J}^b . \quad -(5)$$

For rotational motion not affected by central

2)

gravitation:

$$\omega^a{}_b = -\frac{\kappa}{2}\epsilon^a{}_{bc}\, q^c \quad - (6)$$

$$\epsilon^a{}_{bc} = \eta^{ad}\epsilon_{dbc} \quad - (7)$$

so:

$$\omega^1{}_{\mu 2} = -\frac{\kappa}{2} q^3_\mu \quad - (8)$$

$$\omega^3{}_{\mu 1} = -\frac{\kappa}{2} q^2_\mu \quad - (9)$$

$$\omega^3{}_{\mu 2} = -\frac{\kappa}{2} q^1_\mu . \quad - (10)$$

It follows ~~th~~ that:

$$\omega^1{}_{02} = -\frac{\kappa}{2} q^3_0 = 0 \quad - (11)$$

$$\omega^3{}_{01} = -\frac{\kappa}{2} q^2_0 = 0 \quad - (12)$$

$$\omega^3{}_{02} = -\frac{\kappa}{2} q^1_0 = 0, \quad - (13)$$

and

$$\boxed{\underline{N}^a_L = -\frac{\partial \underline{J}^a}{\partial t}} . \quad - (14)$$

The Newtonian definition of orbital torque is:

$$\underline{N} = \frac{d\underline{J}}{dt} . \quad - (15)$$

3)

The sign change in eq. (14) is a convention, and the indices a in eq. (14) are space-like. So eqns (14) and (15) are mathematically the same but philosophically very different. If central gravitation affects rotational motion then eqs. (6) to (13) are no longer true and there is an additional torque term as in eq. (2). This leads to additional physical effects, e.g. on a rotating object in a gravitational field. If for some reason J^{0a} and J^{0b} are not zero then all four terms on the right hand side of eq. (1) contribute in general, and in this case there may be additional effects which are not observed in Newtonian dynamics.

These notes are for orbital torque only. The next set of notes deal with spin torque:

$$\underline{N}_S^{\,a} = c\left(\underline{\nabla} \times \underline{J}^{\,a} - \underline{\omega}^a{}_b \times \underline{J}^{\,b}\right) \tag{16}$$