1) 130(1): Lorentz Transform of the Dirac Spinor

The Dirac spinor in ECE theory is a tetrad, so transforms as a tetrad under the general coordinate transform in general relativity. In the Minkowski limit it transforms as a Lorentz transform. In SU(2) representation space:

$$\phi(\underline{p}) = \begin{bmatrix} \phi^R(\underline{p}) \\ \phi^L(\underline{p}) \end{bmatrix} = \Lambda \begin{bmatrix} \phi^R(0) \\ \phi^L(0) \end{bmatrix} \quad -(1)$$

where:

$$\Lambda = \begin{bmatrix} \exp\left(\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right) & 0 \\ 0 & \exp\left(-\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right) \end{bmatrix} \quad -(2)$$

Here:

$$\exp\left(\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right) = \frac{1}{E^{(0)}}\left(E_0 + mc^2 + \underline{\sigma}\cdot\underline{p}\right) \quad -(3)$$

$$\exp\left(-\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right) = \frac{1}{E^{(0)}}\left(E_0 + mc^2 - \underline{\sigma}\cdot\underline{p}\right) \quad -(4)$$

with

$$E^{(0)} = \left(2mc^2\left(E + mc^2\right)\right)^{1/2} \quad -(5)$$

When

$$\underline{p} = \underline{0} \quad -(6)$$

then

$$E = E_0 = mc^2 \quad -(7)$$

and

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(8)$$

1) 130(2). Quantum F.

The Dirac equation is:

$$\left(i \gamma^\mu \partial_\mu - \frac{mc}{\hbar} \right) \psi = 0 \qquad - (1)$$

and is obtained from the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} \right) = 0 \qquad - (2)$$

with lagrangian:

$$\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{mc}{\hbar} \bar{\psi} \psi . \qquad - (3)$$

Strictly speaking this has the units of inverse distance, but this notation is used in quantum field theory.

Eq. (3) is:

$$\mathcal{L} = i \bar{\psi} \left(\gamma^0 \partial_0 + \gamma^i \partial_i \right) \psi - \frac{mc}{\hbar} \bar{\psi} \psi \qquad - (4)$$

$$= i \bar{\psi} \gamma^0 \dot{\psi} + i \bar{\psi} \gamma^i \partial_i \psi - \frac{mc}{\hbar} \bar{\psi} \psi$$

so the canonical momentum is:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \bar{\psi} \gamma^0 = i \psi^+ \qquad - (5)$$

The Hamiltonian is:

$$H = \pi \dot{\psi} - \mathcal{L} \qquad - (6)$$

2) Therefore:

$$H = i\psi^{+}\dot{\psi} - i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + \frac{mc}{\hbar}\bar{\psi}\psi$$

$$= i\psi^{+}\dot{\psi} - i\psi^{+}\gamma^{0}\gamma^{\mu}\partial_{\mu}\psi + \frac{mc}{\hbar}\psi^{+}\gamma^{0}\psi$$

$$= \psi^{+}\gamma^{0}\frac{mc}{\hbar}\psi + i\psi^{+}\left(\dot{\psi} - \gamma^{0}\gamma^{\mu}\partial_{\mu}\psi\right)$$

$$= \psi^{+}\gamma^{0}\frac{mc}{\hbar}\psi + i\psi^{+}\left(\dot{\psi} - \gamma^{0}\left(\gamma^{0}\partial_{0} + \gamma^{i}\partial_{i}\right)\psi\right) \quad -(7)$$

Now we: $\gamma^{0}\gamma^{0} = 1$ — (8)

$\partial_{0}\psi = \dot{\psi}$. — (9)

So: $H = \psi^{+}\gamma^{0}\frac{mc}{\hbar}\psi - i\psi^{+}\gamma^{i}\partial_{i}\psi$ — (10)

From eq. (1):

$$\left(i\left(\gamma^{0}\partial_{0} + \gamma^{i}\partial_{i}\right) - \frac{mc}{\hbar}\right)\psi = 0 \quad -(11)$$

Therefore $\boxed{H = i\psi^{+}\partial_{0}\psi}$ — (12)

This is the expectation value of the energy, using

$p_{0} = i\hbar\partial_{0}$ — (13)

$E_{n} = i\hbar\frac{\partial}{\partial t}$ — (14)

3) In quantum field theory the units are reduced units ($\hbar = c = 1$). It is easiest to introduce the correct S.I. units at the end of the calculation, so

$$\boxed{H = i\hbar\phi^{+}\frac{\partial\phi}{\partial t}} \quad -(15)$$

In the usual standard development H is regarded as being not positive definite, because of the standard model's use of negative energy plane wave solutions of the Dirac equation. For example, in a rest particle, positive energy solutions are defined as

$$\phi = u(0)\exp\left(-\frac{imc^2}{\hbar}t\right) \quad -(16)$$

and negative energy solutions are defined as:

$$\phi = v(0)\exp\left(\frac{imc^2}{\hbar}t\right). \quad -(17)$$

This interpretation is however rejected in ECE theory because the interpretation is based on the assumption that there is negative energy in the classical level. In the standard interpretation there are two spinor components of type (16), and two spinor components of type (17). In the ECE interpretation there are four spinor components of positive energy, all with negative exponent.

4) The particle is described by:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar} \right) \psi = 0 \quad -(18)$$

and the anti-particle is described by eq. (18) of opposite parity. Therefore:

Particle $$\left(i\gamma^0 \partial_0 + \gamma^i \partial_i - \frac{mc}{\hbar} \right) \psi_p = 0 \quad -(19)$$

Anti-Particle $$\left(i\gamma^0 \partial_0 + \gamma^i \partial_i - \frac{mc}{\hbar} \right) \psi_a = 0 \quad -(20)$$

Eq. (20) is generated from eq. (19) by:

$$\boxed{\gamma^5 \rightarrow -\gamma^5} \quad -(21)$$

where:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad -(22)$$

For example, if the particle is travelling in the Z axis, eq. (19) is:

$$\left(i\gamma^0 \partial_0 + \gamma^3 \partial_3 - mc/\hbar \right) \psi_p = 0 \quad -(23)$$

and eq. (20) is:

$$\left(i\gamma^0 \partial_0 - \gamma^3 \partial_3 - mc/\hbar \right) \psi_a = 0 \quad -(24)$$

The quantum field theory of eqs. (23) and (24) is developed in the next note.

2) Therefore:

$$\phi^R(\underline{p}) = \frac{1}{E^{(0)}}\left(E_0 + mc^2 + \underline{\sigma}\cdot\underline{p}\right)\phi^R(0) \quad -(9)$$

and
$$\phi^L(\underline{p}) = \frac{1}{E^{(0)}}\left(E_0 + mc^2 - \underline{\sigma}\cdot\underline{p}\right)\phi^L(0) \quad -(10)$$

with
$$\phi^R(0) = \phi^L(0). \quad -(11)$$

Eqs. (9) to (11) give the Dirac equation:

$$\begin{bmatrix} -mc^2 & E_0 + c\underline{\sigma}\cdot\underline{p} \\ E_0 - c\underline{\sigma}\cdot\underline{p} & -mc^2 \end{bmatrix}\begin{bmatrix} \phi^R(\underline{p}) \\ \phi^L(\underline{p}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad -(12)$$

which is the same as:

$$\left(\gamma^\mu p_\mu - mc\right)\phi = 0 \quad -(13)$$

i.e.
$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\phi = 0 \quad -(14)$$

and
$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0. \quad -(15)$$

The Dirac equation is the general coordinate transform of the Cartan tetrad in the Poincaré limit, the free fermion limit.

The solutions of the Dirac equation are given by eqns. (9) and (10) with:

$$\phi^R(0) = \phi^L(0) = \exp\left(-\frac{imc^2}{\hbar}\right)t \quad -(16)$$

Note carefully that in the ECE interpretation the rest energy is rigorously positive on the classical level:

$$E_0 = mc^2 \quad -(17)$$

<u>The Anti-particle</u> (<u>Anti-fermion</u>)

<u>The anti-fermion</u> is <u>generated</u> from <u>the fermion</u> <u>by using</u> <u>the coordinate</u> <u>system of opposite</u> <u>chirality</u>.

<u>Proof</u> Let

$$\sigma^3 \rightarrow -\sigma^3 \quad -(18)$$

then

$$\gamma^3 = \begin{bmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{bmatrix} \rightarrow -\gamma^3 \quad -(19)$$

and

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \rightarrow -\gamma^5} \quad -(20)$$

The Dirac γ^5 matrix is <u>the operator</u> of <u>chirality</u> in the chiral representation:

$$\phi(\underline{p}) = \begin{bmatrix} \phi^R(\underline{p}) \\ \phi^L(\underline{p}) \end{bmatrix} \quad -(21)$$

where $\phi^R(\underline{p})$ and $\phi^L(\underline{p})$ are eigenstates of chirality. Therefore:

$$\bar{\phi}\gamma^5\phi = \phi^{L*}\phi^R - \phi^{R*}\phi^L \qquad (22)$$

is negative under parity, and is a pseudo-sclar. In order to conserve CPT, the charge conjugation operator must be negative if P is negative and T is positive.

The anti-fermion is generated by reversing the Dirac γ^5 matrix, its electric charge is opposite to that of the fermion.

The Dirac γ^5 matrix is:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad (23)$$

Fourier Expansion of the Dirac Spinor

The methods used in the quantum field theory of the Dirac equation are based on a Fourier expansion. This is claimed to produce a multi particle theory and to produce the Pauli exclusion principle. However the standard argument is, mathematically, a Fourier analysis. In its simplest terms the wave function is expanded as:

$$\phi = \phi(0)\left(b e^{-i(\omega t - \kappa Z)} + d^{+} e^{i(\omega t - \kappa Z)}\right) \quad -(1)$$

so

$$\phi^{+} = \phi(0)\left(b^{+} e^{i(\omega t - \kappa Z)} + d e^{-i(\omega t - \kappa Z)}\right) \quad -(2)$$

The next step is to work out the hamiltonian:

$$H = i\phi^{+} \frac{\partial \phi}{\partial t} \quad -(3)$$

where

$$\frac{\partial \phi}{\partial t} = -i\omega\phi(0)\left(b e^{-i(\omega t - \kappa Z)} - d^{+} e^{i(\omega t - \kappa Z)}\right) \quad -(4)$$

so

$$H = \omega\phi^{2}(0)\left(b^{+}b - dd^{+} + db e^{-2i\phi} - b^{+}d^{+} e^{2i\phi}\right) \quad -(5)$$

where

$$\phi = \omega t - \kappa Z \quad -(6)$$

In S.I. units the mean hamiltonian is:

$$\boxed{\langle H \rangle = \hbar\omega\phi^{2}(0)\left(b^{+}b - dd^{+}\right)} \quad -(7)$$

because

$$\langle e^{-2i\phi} \rangle = \langle e^{2i\phi} \rangle = 0. \quad -(8)$$

The actual result of the standard interpretation

2) is essentially eq. (7). In second quantization $b^+b - dd^+$.
! ϕ is itself regarded as an operator, so
is an operator. In standard second quantization b
is the annihilation operator and d^+ is the creation
operator. It is claimed that b^+ creates particles
and d^+ creates antiparticles. The wave function ϕ
in second quantization is a hermitian operator:

$$\langle m | \Omega | n \rangle = \langle n | \Omega | m \rangle^* \quad -(9)$$

which has real eigenvalues.

It is also claimed in the standard approach
that

$$dd^+ = - d^+ d \quad -(10)$$

so:

$$\langle H \rangle = \hbar\omega \left(b^+ b + d^+ d \right) \quad -(11)$$

The technique of normal ordering is used, in which
all annihilation operators b are written to the right of
creation operators d. Using the assumption (10) the
probability density is

$$\phi^+ \phi = b^+ b - d^+ d \quad -(12)$$

and it is claimed that if b^+ creates particles
then d^+ creates antiparticles.

In order to derive eq. (11) from eq. (7),
the Jordan Wigner anti-commutators are used:

3)

$$[A, B] := AB + BA, \quad -(13)$$

i.e.

$$[b_\alpha, b^+_{\alpha'}] = \{d_\alpha, d^+_{\alpha'}\} = \delta^3(\underline{k}' - \underline{k}')\delta_{\alpha\alpha'} \quad -(14)$$

$$\{b_\alpha, b_{\alpha'}\} = [b^+_\alpha, b^+_{\alpha'}] = 0 \quad -(15)$$

$$\{d_\alpha, d_{\alpha'}\} = \{d^+_\alpha, d^+_{\alpha'}\} = 0 \quad -(16)$$

The change of sign, eq. (10), is empirical. There is no fundamental justification for it. Also, eqns. (14) to (16) are empirical.

In the standard model the procedure is therefore greatly simplified. It can be observed from the outset by adapting the rule used in ECE theory, that the antiparticle is generated from the particle by

$$\gamma^5 \rightarrow -\gamma^5. \quad -(17)$$

The natural anti-commutator in geometry is:

$$2g_{\mu\nu} = \{\gamma_\mu, \gamma_\nu\} \quad -(18)$$

and in a geometrical theory such as ECE, this is the only fundamental anti-commutator.

1) 130(4): Dirac Algebra and γ^5

The fundamental ECE hypothesis is that the antiparticle is generated from the particle by:

$$\gamma^5 \to -\gamma^5 \quad -(1)$$

where:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad -(2)$$

is the operator of chirality or handedness. This hypothesis gets rid of the unobservable Dirac sea and is a geometrical explanation of the existence of antiparticles. The antiparticle must have opposite electric charge to the particle because:

$$CPT = (-C)(-P)T \quad -(3)$$

and CPT is conserved. The γ^5 matrix is pure geometry. The Minkowski metric is defined by the anti-commutator of the Dirac matrices:

$$2g_{\mu\nu} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu, \quad -(4)$$

and

$$2g^{\mu\nu} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \{\gamma^\mu, \gamma^\nu\}. \quad -(5)$$

where

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(6)$$

and

$$x_\mu = g_{\mu\nu} x^\nu. \quad -(7)$$

The Dirac matrices are:

$$\gamma^\mu = (\gamma^0, \gamma^i) \quad -(8)$$

2) where $\gamma^0 = \begin{bmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{bmatrix}$, $\gamma^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix}$. — (9)

The four Pauli matrices are:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad — (10)$$

so

$$\left[\frac{\sigma^1}{2}, \frac{\sigma^2}{2} \right] = i \frac{\sigma^3}{2} \quad — (11)$$

in cyclic permutation of 1, 2 and 3. — (12)

Therefore:

$$\gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \gamma^2 = i \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \gamma^3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The γ^5 matrix is therefore:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad — (13)$$

Some Examples

1) $$\gamma^0\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \gamma^{00} I \quad — (14)$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad — (15)$$

is the 4×4 unit matrix.

~~3)~~

2)
$$\gamma^1\gamma^1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
$$= -1\,I \quad -(16)$$

and so on.

Therefore eq. (5) means:
$$2g^{\mu\nu} I = \{\gamma^\mu, \gamma^\nu\} \quad -(17)$$
i.e.
$$2g^{00} I = \{\gamma^0, \gamma^0\} \quad -(18)$$
and so on. Here:
$$g^{00} = 1 \quad -(19)$$

3)
$$\gamma^0\gamma^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad -(20)$$

$$\gamma^1\gamma^0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad -(21)$$

so
$$\gamma^0\gamma^1 + \gamma^1\gamma^0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -(22)$$

and so on. Therefore:
$$\gamma^5 = i^2 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
$$-(23)$$

4) Interpretation of γ^5

The Dirac spinor in the chiral representation is:

$$\psi = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix}, \quad -(24)$$

where:

$$\phi^R = \begin{bmatrix} \phi_1^R \\ \phi_2^R \end{bmatrix}, \quad \phi^L = \begin{bmatrix} \phi_1^L \\ \phi_2^L \end{bmatrix} \quad -(25)$$

As shown in previous notes, the adjoint Dirac spinor is:

$$\bar{\psi} = \left[\psi_3^* \; \psi_4^* \; \psi_1^* \; \psi_2^* \right] = \left[\phi_1^{L*} \; \phi_2^{L*} \; \phi_1^{R*} \; \phi_2^{R*} \right] \quad -(26)$$

Therefore:

$$\bar{\psi}\gamma^5\psi = \left[\phi_1^{L*} \; \phi_2^{L*} \; \phi_1^{R*} \; \phi_2^{R*} \right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_1^R \\ \phi_2^R \\ \phi_1^L \\ \phi_2^L \end{bmatrix} \quad -(27)$$

$$\boxed{\bar{\psi}\gamma^5\psi = \phi_1^{L*}\phi_1^R + \phi_2^{L*}\phi_2^R - \phi_1^{R*}\phi_1^L - \phi_2^{R*}\phi_2^L} \quad -(28)$$

In concise notation:

$$\bar{\psi}\gamma^5\psi = \phi^{L+}\phi^R - \phi^{R+}\phi^L \quad -(29)$$

Under parity:

$$\hat{P}\left(\bar{\psi}\gamma^5\psi\right) = -\bar{\psi}\gamma^5\psi \quad -(30)$$

so $\bar{\psi}\gamma^5\psi$ is a pseudoscalar. Thus γ^5 is the operator of chirality and ϕ^R and ϕ^L are eigenstates of chirality.

130(5) : Pauli Matrices as Wavefunctions

Consider the wave equation:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2 \right) \phi = 0 \quad -(1)$$

where ϕ is a time dependent wave function. Define the following four solutions:

$$\phi_1 = \phi_1^R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^{-i\phi} \quad -(2)$$

$$\phi_2 = \phi_2^R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e^{-i\phi} \quad -(3)$$

$$\phi_3 = \phi_1^L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} e^{-i\phi} \quad -(4)$$

$$\phi_4 = \phi_2^L = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} e^{-i\phi} \quad -(5)$$

where

$$\phi = \frac{mc^2}{\hbar} t. \quad -(6)$$

We have:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\left(\sigma^0 + \sigma^3\right) \quad -(7)$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\left(\sigma^1 + i\sigma^2\right) \quad -(8)$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}\left(\sigma^1 - i\sigma^2\right) \quad -(9)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\left(\sigma^0 - \sigma^3\right) \quad -(10)$$

where σ^0, σ^1, σ^2 and σ^3 are the Pauli matrices. The latter are tetrad components:

$$\sigma^0 = q_0^0, \; \sigma^1 = q_X^1, \; \sigma^2 = q_Y^2, \; \sigma^3 = q_Z^3 \quad -(11)$$

as shown in note 129(1).

2)

Eq (1) may be factorized into:

$$\left(i\gamma^{\mu}\partial_{\mu} - mc/\hbar \right)\phi = 0 \quad -(12)$$

where γ^{μ} is the 4x4 Dirac matrix.

The important mathematical result is obtained in this note that this factorization can be made with 2x2 Pauli matrices.

For example:

$$\boxed{\begin{aligned}\begin{bmatrix}1&0\\0&0\end{bmatrix} &= \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix} = \frac{1}{4}\left(\sigma^1+i\sigma^2\right)\left(\sigma^1-i\sigma^2\right) \quad -(13)\\ \begin{bmatrix}0&0\\1&0\end{bmatrix} &= \begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} = \frac{1}{4}\left(\sigma^1-i\sigma^2\right)\left(\sigma^1+i\sigma^2\right) \quad -(14)\end{aligned}}$$

and so on.

Therefore:

$$\boxed{\begin{aligned}\phi^R_1 &= \frac{1}{4}\left(\sigma^1+i\sigma^2\right)\left(\sigma^1-i\sigma^2\right)e^{-i\phi} \quad -(15)\\ \phi^L_1 &= \frac{1}{4}\left(\sigma^1-i\sigma^2\right)\left(\sigma^1+i\sigma^2\right)e^{-i\phi} \quad -(16)\end{aligned}}$$

and:

$$i\frac{\partial\phi^R_1}{\partial t} = \frac{mc^2}{\hbar}\phi^L_1 \quad -(17)$$

$$\frac{1}{c^2}\frac{\partial^2\phi^R_1}{\partial t^2} = -\frac{m^2c^2}{\hbar^2}\phi^R_1 \quad -(18)$$

Eq (17) is the Dirac equation for a rest particle's component ϕ^R_1. Eq (18) is the

3) wave form of the Dirac equation for ϕ^R_1.

It is seen that chirality or handedness is the result of the non-commutative property of the 2×2 matrices in eqs. (13) and (14). Therefore, for the first time, the Dirac equation has been written in terms of the 2×2 Pauli matrices. There is no need for the 4×4 Dirac matrices, and the Pauli matrices are tetrad components.

<u>These are</u> major advances in mathematics and physics.

Another example is:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad — (19)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad — (20)$$

So

$$\boxed{\begin{aligned} \phi^R_2 &= \frac{1}{4}\left(\sigma^0 + \sigma^3\right)\left(\sigma^1 + i\sigma^2\right)e^{-i\phi} \quad — (21) \\ \phi^L_2 &= \frac{1}{4}\left(\sigma^1 + i\sigma^2\right)\left(\sigma^0 - \sigma^3\right)e^{-i\phi} \quad — (22) \end{aligned}}$$

and

$$i\frac{\partial \phi^R_2}{\partial t} = \frac{mc^2}{\hbar}\phi^L_2 \quad — (23)$$

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi^R_2 = 0 \quad — (24)$$

1) Note 135(6): Origin of Anti-Commutator

In note 135(5) it was shown that

$$\phi_1^R = \frac{1}{4}(\sigma^1 + i\sigma^2)(\sigma^1 - i\sigma^2)e^{-i\phi} \quad -(1)$$

$$\phi_1^L = \frac{1}{4}(\sigma^1 - i\sigma^2)(\sigma^1 + i\sigma^2)e^{-i\phi} \quad -(2)$$

If we write these equations as:

$$\phi_1^R = ABe^{-i\phi} \; ; \; \phi_1^L = BAe^{-i\phi} \quad -(3)$$

then:

$$i\frac{\partial}{\partial t}\left(ABe^{-i\phi}\right) = \frac{mc^2}{\hbar}ABe^{-i\phi} \quad -(4)$$

$$i\frac{\partial}{\partial t}\left(BAe^{-i\phi}\right) = \frac{mc^2}{\hbar}BAe^{-i\phi} \quad -(5)$$

Add:

$$\boxed{i\frac{\partial\phi_1^R}{\partial t} + i\frac{\partial\phi_1^L}{\partial t} = \frac{mc^2}{\hbar}\left(\phi_1^R + \phi_1^L\right)} \quad -(6)$$

i.e.

$$i\frac{\partial}{\partial t}\left(\{A, B\}e^{-i\phi}\right) = \frac{mc^2}{\hbar}\{A, B\}e^{-i\phi} \quad -(7)$$

where the anticommutator is:

$$\{A, B\} = AB + BA \quad -(8)$$

Eqn. (6) is:

$$i\frac{\partial}{\partial t}\left(\phi_1^R + \phi_1^L\right) = \frac{mc^2}{\hbar}\left(\phi_1^R + \phi_1^L\right) \quad -(9)$$

2)

Also:

$$i\frac{\partial}{\partial t}\left([A,B]e^{-i\phi}\right) = \frac{nc^2}{\hbar}[A,B]e^{-i\phi} \quad -(10)$$

where:

$$\boxed{i\frac{\partial}{\partial t}\left(\phi^R_1 - \phi^L_1\right) = \frac{nc^2}{\hbar}\left(\phi^R_1 - \phi^L_1\right)} \quad -(11)$$

$$[A,B] = AB - BA \quad -(12)$$

and

$$i\frac{\partial}{\partial t}\left([A,B]e^{-i\phi}\right) = \frac{nc^2}{\hbar}[A,B]e^{-i\phi} \quad -(13)$$

so

Here:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad -(14)$$

so:

$$\{A,B\} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad -(15)$$

and

$$[A,B] = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \quad -(16)$$

Remarks

It is possible to represent the real fermion with 2×2 matrices. If:

$$\sigma^2 \rightarrow -\sigma^2 \quad -(17)$$

in eqns. (1) and (2), then:

$$\phi^R_1 \rightarrow \phi^L_1 \quad -(18)$$

3) Eqs. (6) and (11) are more symmetrically written as

$$i\frac{\partial}{\partial t}\left(\phi_1^L+\phi_1^R\right)=\frac{mc^2}{\hbar}\left(\phi_1^R+\phi_1^L\right) \quad -(19)$$

and

$$i\frac{\partial}{\partial t}\left(\phi_1^L-\phi_1^R\right)=-\frac{mc^2}{\hbar}\left(\phi_1^R-\phi_1^L\right) \quad -(20)$$

Therefore combinations of states appear when using 2×2 matrices.

1) 130(7) : Complete Matrix Factorization

This is as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad - (1)$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad - (2)$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad - (3)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad - (4)$$

Therefore:

$$\phi_1^R = \frac{1}{4}(\sigma^0+\sigma^3)(\sigma^0+\sigma^3)e^{-i\phi} \quad - (5)$$

$$= \frac{1}{4}(\sigma^1+i\sigma^2)(\sigma^1-i\sigma^2)e^{-i\phi}$$

$$\phi_1^L = \frac{1}{4}(\sigma^1-i\sigma^2)(\sigma^0+\sigma^3)e^{-i\phi} \quad - (6)$$

$$= \frac{1}{4}(\sigma^0-\sigma^3)(\sigma^1-i\sigma^2)e^{-i\phi}$$

$$\phi_2^R = \frac{1}{4}(\sigma^0+\sigma^3)(\sigma^1+i\sigma^2)e^{-i\phi} \quad - (7)$$

$$= \frac{1}{4}(\sigma^1+i\sigma^2)(\sigma^0-\sigma^3)e^{-i\phi}$$

$$\phi_2^L = \frac{1}{4}(\sigma^0-\sigma^3)(\sigma^0-\sigma^3)e^{-i\phi} \quad - (8)$$

$$= \frac{1}{4}(\sigma^1-i\sigma^2)(\sigma^1+i\sigma^2)e^{-i\phi}$$

Eq. (4) corrects an error in Eq. (20) of note 130(5)

2) Under $\sigma^3 \rightarrow -\sigma^3$ — (9)

then $\phi^R_1 \rightarrow \phi^L_2$ — (10)

and under $\sigma^1 + i\sigma^2 \rightarrow \sigma^1 - i\sigma^2$ — (11)

$\phi^L_1 \rightarrow \phi^R_2$ — (12)

These are helicity transformations. In eq. (9) the helicity is reversed along Z, and in eq. (11) it is reversed along Y. Therefore the existence of R and L states is a consequence of the matrix properties of Pauli matrices. The states ϕ^R_1, ϕ^L_1, ϕ^R_2 and ϕ^L_2 are combinations of tetrad elements. For example:

$$\phi^R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^{-i\phi} = \frac{1}{2}\left(\sigma^0 + \sigma^3\right) e^{-i\phi}, \quad (13)$$

also:

$$\phi = mc^2 t / \hbar. \quad (14)$$

Here $\sigma^0 = q^0_0$, $\sigma^3 = q^3_Z$. — (15)

We have:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) \phi^R_1 = 0 \quad (16)$$

and

$$i \frac{\partial}{\partial t}\left(\phi^R_1 + \phi^L_2\right) e^{-i\phi} = \left(\phi^L_2 + \phi^R_1\right) e^{-i\phi}. \quad (17)$$

There are patterns such as:

$$\boxed{\begin{aligned} \phi^R_1 &= AB e^{-i\phi} = C^2 e^{-i\phi} \quad (18) \\ \phi^L_2 &= BA e^{-i\phi} = D^2 e^{-i\phi} \quad (19) \end{aligned}}$$

3)

$$\boxed{\begin{aligned}\phi^L_1 &= BC e^{-i\phi} = DB e^{-i\phi} \quad &-(20)\\ \phi^R_2 &= CB e^{-i\phi} = BD e^{-i\phi} \quad &-(21)\end{aligned}}$$

We have:

$$\left(\Box + (mc/\hbar)^2\right)\phi^R_1 = 0 \quad -(22)$$

$$\left(\Box + (mc/\hbar)^2\right)\phi^R_2 = 0 \quad -(23)$$

$$\left(\Box + (mc/\hbar)^2\right)\phi^L_1 = 0 \quad -(24)$$

$$\left(\Box + (mc/\hbar)^2\right)\phi^L_2 = 0 \quad -(25)$$

Therefore:

$$\left(\Box + (mc/\hbar)^2\right)\begin{bmatrix}\phi^R_1 & \phi^R_2\\ \phi^L_1 & \phi^L_2\end{bmatrix} = 0 \quad -(26)$$

where:

$$\begin{bmatrix}V^R\\ V^L\end{bmatrix} = \begin{bmatrix}\phi^R_1 & \phi^R_2\\ \phi^L_1 & \phi^L_2\end{bmatrix}\begin{bmatrix}V^1\\ V^2\end{bmatrix}. \quad -(27)$$

This is an example of: $V^a = q^a_\mu V^\mu$ $-(28)$

All the information needed for the existence of spin in a particle has already been obtained, without the use of 4x4 matrices. The conventional Dirac formalism is obtained by rewriting eqn. (26) as:

$$\left(\Box + (mc/\hbar)^2\right)\begin{bmatrix}\phi^R_1\\ \phi^R_2\\ \phi^L_1\\ \phi^L_2\end{bmatrix} = 0 \quad -(29)$$

i.e. as

$$\left(\Box + (mc/\hbar)^2\right)\psi = 0 \quad -(30)$$

which factorizes into:

$$\left(\gamma^\mu \partial_\mu - imc/\hbar\right)\psi = 0. \quad -(31)$$

This does not give any more information

130(8): 2 × 2 Matrix Dirac Equation for the Rest Fermion

Recall that the 4 × 4 matrix Dirac equation for the rest fermion is:

$$i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{\partial \psi}{\partial t} = \frac{mc^2}{\hbar} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \psi \quad -(1)$$

where

$$\psi = \begin{bmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{bmatrix} = \begin{bmatrix} \phi^R_1 \\ \phi^R_2 \\ \phi^L_1 \\ \phi^L_2 \end{bmatrix} \quad -(2)$$

In ECE theory the rest particle spinors are all considered to be positive energy spinors with the same sign of phase. The solutions are:

$$\begin{bmatrix} \psi^1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \phi^R_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \exp\left(-i\frac{mc^2}{\hbar}t\right) \quad -(3)$$

$$\begin{bmatrix} 0 \\ \psi^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi^R_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \exp\left(-i\frac{mc^2}{\hbar}t\right) \quad -(4)$$

$$\begin{bmatrix} 0 \\ 0 \\ \psi^3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi^L_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \exp\left(-i\frac{mc^2}{\hbar}t\right) \quad -(5)$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \psi^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \phi^L_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \exp\left(-i\frac{mc^2}{\hbar}t\right) \quad -(6)$$

From these equations, the following relations are obtained between the scalar components:

2)

$$i\frac{\partial \phi_1^R}{\partial t} = \frac{mc^2}{\hbar}\phi_1^L \quad -(7)$$

$$i\frac{\partial \phi_2^R}{\partial t} = \frac{mc^2}{\hbar}\phi_2^L \quad -(8)$$

$$i\frac{\partial \phi_1^L}{\partial t} = \frac{mc^2}{\hbar}\phi_1^R \quad -(9)$$

$$i\frac{\partial \phi_2^L}{\partial t} = \frac{mc^2}{\hbar}\phi_2^R \quad -(10)$$

These same relations can be observed in ECE theory with 2×2 matrices as follows. The ECE wave equation in the free fermion limit is:

$$(\square + \kappa^2)\phi = 0 \quad -(11)$$

where

$$\kappa = \frac{mc}{\hbar}, \quad -(12)$$

and

$$\phi = \begin{bmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{bmatrix} = \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} \quad -(13)$$

is a <u>tetrad</u>. Thus:

$$e^{-i\phi}, \quad \phi = \frac{mc^2}{\hbar}t \quad -(14)$$

$$\begin{bmatrix} \phi^1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \phi_1^R & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^{-i\phi} \quad -(15)$$

$$\begin{bmatrix} 0 & \phi^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \phi_2^R \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e^{-i\phi} \quad -(16)$$

$$\begin{bmatrix} 0 & 0 \\ \phi^3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \phi_1^L & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} e^{-i\phi} \quad -(17)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & \phi^4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \phi_2^L \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} e^{-i\phi}$$

3) Now use:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \sigma^1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad -(18)$$

This means:

$$\begin{bmatrix} \phi_1^R & 0 \\ 0 & 0 \end{bmatrix} = \sigma^1 \begin{bmatrix} 0 & 0 \\ \phi_1^L & 0 \end{bmatrix}. \quad -(19)$$

Similarly:

$$\begin{bmatrix} 0 & \phi_2^R \\ 0 & 0 \end{bmatrix} = \sigma^1 \begin{bmatrix} 0 & 0 \\ 0 & \phi_2^L \end{bmatrix} \quad -(20)$$

because: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad -(21)$

Also $\begin{bmatrix} 0 & 0 \\ \phi_1^L & 0 \end{bmatrix} = \sigma^1 \begin{bmatrix} \phi_1^R & 0 \\ 0 & 0 \end{bmatrix} \quad -(22)$

because $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad -(23)$

Finally: $\begin{bmatrix} 0 & 0 \\ 0 & \phi_2^L \end{bmatrix} = \sigma^1 \begin{bmatrix} 0 & \phi_2^R \\ 0 & 0 \end{bmatrix} \quad -(24)$

because: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad -(25)$

It follows that:

$$\boxed{\left(i\sigma^1 \partial_0 - \frac{mc}{\hbar} \right) \phi = 0} \quad -(26)$$

where

$$\phi = \begin{bmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{bmatrix} \quad -(27)$$

Eq. (26) is the ECE equation of the rest fermion. It may be written as:

$$\boxed{i\frac{\partial\phi}{\partial t} = \sigma^1\frac{mc^2}{\hbar}\phi} \quad -(28)$$

It contains only 2 x 2 matrices. For example:

$$i\frac{\partial}{\partial t}\begin{bmatrix}\phi^1 & 0\\ 0 & 0\end{bmatrix} = \frac{mc^2}{\hbar}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ \phi^3 & 0\end{bmatrix}, \quad -(29)$$

i.e.

$$i\frac{\partial\phi_1^R}{\partial t} = \frac{mc^2}{\hbar}\phi_1^L \quad -(30)$$

which is eq. (7).

Finally eq. (26) is derived from the ECE wave equation of the rest particle:

$$(\partial^0\partial_0 + \kappa^2)\phi = 0 \quad -(31)$$

using:

$$\partial^0\partial_0 = (-i\sigma^1\partial_0)(i\sigma^1\partial^0) \quad -(32)$$

and

$$\sigma^1\sigma^1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \quad -(33)$$

Thus

$$\left(i\sigma^1\partial_0 - \frac{mc}{\hbar}\right)\left(i\sigma^1\partial^0 + \frac{mc}{\hbar}\right)\phi = 0 \quad -(34)$$

or

$$\left(i\sigma^1\partial^0 + \frac{mc}{\hbar}\right)\left(\left(i\sigma^1\partial_0 - \frac{mc}{\hbar}\right)\phi\right) = 0$$

giving eq. (26), QED.

130(9): The Algebra of the SU(2) Group.

In the O(3) group there are operator relations such as:

$$\exp(iJ_Z\theta) = 1 + iJ_Z\theta - J_Z^2\frac{\theta^2}{2!} - iJ_Z^3\frac{\theta^3}{3!} + \dots \quad (1)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \theta\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\theta^2}{2!}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\theta^3}{3!}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dots \quad (2)$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is a rotation about the Z axis. A finite rotation about an axis $\underline{n}$ through an angle θ is denoted:

$$R_n(\theta) = \exp(i\underline{J}\cdot\underline{\theta}) = \exp(i\underline{J}\cdot\underline{n}\theta) \quad - (3)$$

The SU(2) group represents this rotation in a <u>2×2 complex space</u>. The space is represented by the spinor:

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \quad - (4)$$

<u>This concept</u> was also <u>introduced by Cartan</u>, in 1913. The SU(2) group is defined by 2×2 unitary matrices with unit determinant:

$$UU^+ = 1, \ \det U = 1 \quad - (5)$$

The superscript + denotes the complex conjugate of the transposed matrix. Unitary matrices are therefore defined by

$$U^+ = U^{-1} \quad - (6)$$

where U^{-1} is the inverse of U. Therefore we have:

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ U^+ = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}, \ U^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$- (7)$$

2.

$$\det U = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 1 \quad -(8)$$

and

$$UU^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad -(9)$$

So: $a^* = d, \; b^* = -c, \; aa^* + bb^* = 1. \quad -(10)$

The matrix U transforms the spinor:

$$\xi' = U\xi \quad -(11)$$

i.e.

$$\begin{bmatrix} \xi_1' \\ \xi_2' \end{bmatrix} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \quad -(12)$$

Similarly:

$$\xi^+ = \begin{bmatrix} \xi_1^* & \xi_2^* \end{bmatrix} \quad -(13)$$

and

$$\xi^{+\prime} = \xi^+ U^+ \quad -(14)$$

i.e.

$$\begin{bmatrix} \xi_1^{*\prime} & \xi_2^{*\prime} \end{bmatrix} = \begin{bmatrix} \xi_1^* & \xi_2^* \end{bmatrix} \begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix} \quad -(15)$$

Thus:

$$\xi^+\xi = \begin{bmatrix} \xi_1^* & \xi_2^* \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \xi_1^*\xi_1 + \xi_2^*\xi_2 \quad -(16)$$

and

$$\xi^{+\prime}\xi' = \begin{bmatrix} \xi_1^* & \xi_2^* \end{bmatrix} U^+ U \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad -(17)$$

$$= \begin{bmatrix} \xi_1^* & \xi_2^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

$$\boxed{\xi^{+\prime}\xi' = \xi^+\xi} \quad -(18)$$

3) Eq. (18) denotes invariance under a SU(2) transformation. Similarly $X^2+Y^2+Z^2$ is invariant under an O(3) transformation.

Now consider the spinor $\begin{bmatrix} -\xi_2^* \\ \xi_1^* \end{bmatrix}$, which transforms in the same way as $\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$:

$$\left.\begin{aligned} \xi_1' &= a\xi_1 + b\xi_2 \\ \xi_2' &= -b^*\xi_1 + a^*\xi_2 \end{aligned}\right\} - (19)$$

$$\left.\begin{aligned} -\xi_2^{*\prime} &= a(-\xi_2^*) + b\xi_1^* \\ \xi_1^{*\prime} &= -b^*(-\xi_2^*) + a^*\xi_1^* \end{aligned}\right\} - (20)$$

Note that:

$$\begin{bmatrix} -\xi_2^* \\ \xi_1^* \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix} - (21)$$

$$\xi = \zeta\xi^* - (22)$$

i.e. The ξ spinor in eqs. (21) and (22) is redefined as:

$$\xi = \begin{bmatrix} -\xi_2^* \\ \xi_1^* \end{bmatrix} - (23)$$

and transforms under SU(2) in the same way as $\zeta\xi^*$.

This is denoted:

$$\boxed{\xi \sim \zeta\xi^*} - (24)$$

Also:

$$\xi^T \sim (\zeta\xi^*)^T - (25)$$

$$= [-\xi_2 \ \xi_1]$$

Therefore:

$$\xi\,\xi^{+}\,(\text{original}) \sim \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \begin{bmatrix} -\xi_2 & \xi_1 \end{bmatrix} \quad -(26)$$

This equation means that the originally defined $\xi\,\xi^{+}$ (eqs. (4) and (13)) transform under $SU(2)$ in the same way as the matrix on the right hand side of eq. (26):

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \begin{bmatrix} -\xi_2 & \xi_1 \end{bmatrix} = \begin{bmatrix} -\xi_1 \xi_2 & \xi_1^2 \\ -\xi_2^2 & \xi_1 \xi_2 \end{bmatrix} \quad -(27)$$

The matrix H is defined as:

$$\boxed{H = \begin{bmatrix} \xi_1 \xi_2 & -\xi_1^2 \\ \xi_2^2 & -\xi_1 \xi_2 \end{bmatrix}} \quad -(28)$$

and transforms as:

$$H' = U H U^{+}. \quad -(29)$$

It is <u>hermitian</u>:

$$H^{+} = H \quad -(30)$$

and <u>traceless</u>. <u>The Pauli matrices</u> <u>are examples of H matrices.</u>

$$\sigma^1 = \sigma^{1+} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad -(31)$$

$$\sigma^2 = \sigma^{2+} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad -(32)$$

$$\sigma^3 = \sigma^{3+} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad -(33)$$

and

$$\underline{\sigma} \cdot \underline{r} = \begin{bmatrix} Z & X - iY \\ X + iY & -Z \end{bmatrix} \quad -(34)$$

130(10): SU(2) Transformation Matrix and Origin of Half Integral Spin.

Consider:

$$\underline{\sigma} \cdot \underline{r} = \begin{bmatrix} Z & X - iY \\ X + iY & -Z \end{bmatrix} = \begin{bmatrix} \xi_1 \xi_2 & -\xi_1^2 \\ \xi_2^2 & -\xi_1 \xi_2 \end{bmatrix} \quad -(1)$$

then:

$$X = \frac{1}{2}\left(\xi_2^2 - \xi_1^2\right) \quad -(2)$$

$$Y = -\frac{i}{2}\left(\xi_1^2 + \xi_2^2\right) \quad -(3)$$

$$Z = \xi_1 \xi_2 \quad -(4)$$

Now consider the properties of X, Y and Z under the SU(2) transform:

$$\xi_1' = a\,\xi_1 + b\,\xi_2 \quad -(5)$$

$$\xi_2' = -b^*\xi_1 + a^*\xi_2 \quad -(6)$$

Thus:

$$X' = \frac{1}{2}\left(\xi_2'^2 - \xi_1'^2\right) \quad -(7)$$

$$= \frac{1}{2}\left(\left(-b^*\xi_1 + a^*\xi_2\right)^2 - \left(a\xi_1 + b\xi_2\right)^2\right)$$

$$= \frac{1}{2}\left(\left(a^{*2} - b^2\right)\xi_2^2 + \left(b^{*2} - a^2\right)\xi_1^2\right) - \xi_1\xi_2\left(ab + a^*b^*\right)$$

Now use:

$$a^{*2}\xi_2^2 - a^2\xi_1^2 = \frac{1}{2}\left(a^{*2} + a^2\right)\left(\xi_2^2 - \xi_1^2\right) + \frac{1}{2}\left(a^{*2} - a^2\right)\left(\xi_1^2 + \xi_2^2\right)$$

$$= \left(a^{*2} + a^2\right)X - i\left(a^2 - a^{*2}\right)Y \quad -(8)$$

$$b^{*2}\xi_1^2 - b^2\xi_2^2 = -\frac{1}{2}\left(\xi_2^2 - \xi_1^2\right)\left(b^2 + b^{*2}\right) + \frac{1}{2}\left(\xi_1^2 + \xi_2^2\right)\left(b^{*2} - b^2\right)$$

$$= -\left(b^{*2} + b^2\right)X - i\left(b^2 - b^{*2}\right)Y \quad -(9)$$

2) So:

$$X' = \frac{1}{2}\left(a^2 + a^{*2} - b^2 - b^{*2}\right)X - \frac{i}{2}\left(a^2 - a^{*2} + b^2 - b^{*2}\right)Y - \left(a^*b^* + ab\right)Z \quad -(10)$$

$$Y' = \frac{i}{2}\left(a^2 - a^{*2} - b^2 + b^{*2}\right)X + \frac{1}{2}\left(a^2 + a^{*2} + b^2 + b^{*2}\right)Y - i\left(ab - a^*b^*\right)Z \quad -(11)$$

$$Z' = \left(ab^* + ba^*\right)X + i\left(ba^* - ab^*\right)Y + \left(aa^* - bb^*\right)Z \quad -(12)$$

Now choose:

$$a = \exp\left(i\frac{\alpha}{2}\right), \; b = 0, \quad -(13)$$

so that:

$$aa^* + bb^* = \underline{1} \quad -(14)$$

$$\det U = 1 \quad -(15)$$

i.e.

Eqs. (10) to (12) become:

$$X' = X\cos\alpha + Y\sin\alpha \quad -(16)$$

$$Y' = -X\sin\alpha + Y\cos\alpha \quad -(17)$$

$$Z' = Z \quad -(18)$$

This is a rotation about the Z axis through an angle α.

This rotation is produced by:

$$\boxed{U = \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}} \quad -(19)$$

The famous 1/2 factor has appeared.

3) It is seen that:

$$U^{+} = \begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{bmatrix} \quad -(20)$$

and:

$$\boxed{U \begin{bmatrix} Z & X - iY \\ X + iY & -Z \end{bmatrix} U^{+} = \begin{bmatrix} Z & e^{i\alpha}(X - iY) \\ e^{-i\alpha}(X + iY) & -Z \end{bmatrix}} \quad -(21)$$

The transformed matrix is hermitian and traceless and has the same determinant as the original matrix. Thus determinant is

$$r^2 = X^2 + Y^2 + Z^2. \quad -(22)$$

In general:

$$\boxed{U = \exp\left(i \sigma_2 \frac{\alpha}{2}\right)} \quad -(23)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \sigma_2 \frac{\alpha}{2} - \sigma_2^2 \frac{\alpha^2}{4} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\alpha}{2} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\alpha^2}{4} + \cdots$$

$$= \begin{bmatrix} 1 + i\alpha/2 - \alpha^2/4 + \cdots & 0 \\ 0 & 1 - i\alpha/2 - \alpha^2/4 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix} \quad -(24)$$

For rotation about any axis:

4)

$$\boxed{\begin{aligned} U &= \exp\left(i \underline{\sigma} \cdot \underline{\theta} / 2 \right) \\ &= \cos\frac{\theta}{2} + i \underline{\sigma} \cdot \underline{n} \sin\frac{\theta}{2} \end{aligned}} \quad - (25)$$

using the de Moivre Theorem.

The same rotation in O(3) is given by the rotation operator:

$$\boxed{R = \exp\left(i \underline{J} \cdot \underline{\theta} \right)} \quad - (26)$$

In O(3) the angle rotated through is θ, but in SU(2) the angle rotated through is $\theta / 2$.

Now use:

$$e^{i\alpha/2} = \cos\frac{\alpha}{2} + i \sin\frac{\alpha}{2} \quad - (27)$$

$$\left. \begin{aligned} \text{If} \quad \alpha \to \alpha + 2\pi, \quad \cos\frac{\alpha}{2} &\to -\cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} &\to -\sin\frac{\alpha}{2} \end{aligned} \right\} - (28)$$

$$\left. \begin{aligned} \text{and} \quad e^{i\alpha/2} &\to -e^{i\alpha/2} \\ U &\to -U \quad (\alpha \to \alpha + 2\pi) \end{aligned} \right\} - (29)$$

so

$$R \to R \quad (\alpha \to \alpha + 2\pi)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to$$

TWO TO ONE MAPPING

$- (30)$

130(11): ECE Equation of the Fermion

The structure of this equation is:

$$(\Box + \kappa^2)\begin{bmatrix}\phi^1 & \phi^2 \\ \phi^3 & \phi^4\end{bmatrix} = 0 \quad -(1)$$

which may be written as:

$$(i\gamma^\mu \partial_\mu - \kappa)\begin{bmatrix}\phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4\end{bmatrix} = 0 \quad -(2)$$

Here:

$$p^\mu = i\hbar\partial^\mu \quad -(3)$$

and

$$\kappa = \frac{mc}{\hbar}. \quad -(4)$$

Eq. (2) is:

$$(\sigma^0 E + c\underline{\sigma}\cdot\underline{p})\begin{bmatrix}0 & 0 \\ \phi^3 & \phi^4\end{bmatrix} = mc^2\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\phi^1 & \phi^2 \\ 0 & 0\end{bmatrix} \quad -(5)$$

and its parity inverted format

$$(\sigma^0 E - c\underline{\sigma}\cdot\underline{p})\begin{bmatrix}\phi^1 & \phi^2 \\ 0 & 0\end{bmatrix} = mc^2\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 \\ \phi^3 & \phi^4\end{bmatrix} \quad -(6)$$

Eq. (5) is

$$\boxed{i\sigma^\mu\partial_\mu\begin{bmatrix}0 & 0 \\ \phi^3 & \phi^4\end{bmatrix} = \frac{mc}{\hbar}\sigma^1\begin{bmatrix}\phi^1 & \phi^2 \\ 0 & 0\end{bmatrix}} \quad -(7)$$

2) The basic property of the $SL(2, C)$ group is:

$$\phi^R(\underline{p}) = \exp\left(\frac{1}{2}\underline{\sigma} \cdot \underline{\phi}\right)\phi^R(\underline{0}) \quad -(8)$$

and
$$\phi^L(\underline{p}) = \exp\left(-\frac{1}{2}\underline{\sigma} \cdot \underline{\phi}\right)\phi^L(\underline{0}) \quad -(9)$$

and this leads to the Dirac equation of the fermion. The ECE equation of the fermion, eq. (7), is obtained directly from geometry.

If motion is considered along the Z axis in eq. (7) then:

$$(E_n + cp_Z)\phi^3 = mc^2\phi^1 \quad -(8)$$
$$(E_n - cp_Z)\phi^4 = mc^2\phi^2 \quad -(9)$$

Parity inversion of eqns. (8) and (9) gives:

$$(E_n - cp_Z)\phi^1 = mc^2\phi^3 \quad -(10)$$
$$(E_n + cp_Z)\phi^2 = mc^2\phi^4 \quad -(11)$$

Thus:
$$E_n\phi^3 = mc^2\phi^1 \quad -(12)$$
$$E_n\phi^4 = mc^2\phi^2 \quad -(13)$$
$$E_n\phi^1 = mc^2\phi^3 \quad -(14)$$
$$E_n\phi^2 = mc^2\phi^4 \quad -(15)$$

for the rest fermion, as in paper 129.

3)

If: $\phi^R := \begin{bmatrix}\psi^1 & \psi^2\end{bmatrix}$ — (16)

$\phi^L := \begin{bmatrix}\psi^3 & \psi^4\end{bmatrix}$ — (17)

$0 := \begin{bmatrix}0, & 0\end{bmatrix}$ — (18)

then eq. (5) is:

$$(\sigma^0 E + c\underline{\sigma}\cdot\underline{p})\,\phi^L = mc^2\,\phi^R \quad -(19)$$

and eq. (6) is:

$$(\sigma^0 E - c\underline{\sigma}\cdot\underline{p})\,\phi^R = mc^2\,\phi^L \quad -(20)$$

Eqs. (19) and (20) are mathematically the same as the Dirac equation, but in the Dirac equation:

$$\phi^R = \begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix},\quad \phi^L = \begin{bmatrix}\psi^3\\ \psi^4\end{bmatrix} \quad -(21)$$

Adding eqns (5) and (6):

$$\sigma^0 E\begin{bmatrix}\psi^1 & \psi^2\\ \psi^3 & \psi^4\end{bmatrix} + c\underline{\sigma}\cdot\underline{p}\left(\begin{bmatrix}0 & 0\\ \psi^3 & \psi^4\end{bmatrix} - \begin{bmatrix}\psi^1 & \psi^2\\ 0 & 0\end{bmatrix}\right)$$

$$= mc^2\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi^1 & \psi^2\\ \psi^3 & \psi^4\end{bmatrix} \quad -(22)$$

$$= mc^2\begin{bmatrix}\psi^3 & \psi^4\\ \psi^1 & \psi^2\end{bmatrix}$$

4)

Therefore the ECE fermion equation is:

$$\boxed{\sigma^0 E \begin{bmatrix} \psi^1 & \psi^2 \\ \psi^3 & \psi^4 \end{bmatrix} + c\underline{\sigma}\cdot\underline{p}\begin{bmatrix} -\psi^1 & -\psi^2 \\ \psi^3 & \psi^4 \end{bmatrix} = mc^2 \begin{bmatrix} \psi^3 & \psi^4 \\ \psi^1 & \psi^2 \end{bmatrix}} \quad -(23)$$

with:

$$E = i\hbar\frac{\partial}{\partial t}, \quad \underline{p} = -i\hbar\underline{\nabla} \quad -(24)$$

Finally use:

$$\begin{bmatrix} -\psi^1 & -\psi^2 \\ \psi^3 & \psi^4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \psi^1 & \psi^2 \\ \psi^3 & \psi^4 \end{bmatrix} \quad -(25)$$

to find:

$$\boxed{\sigma^0 E\psi - \sigma^3 c\underline{\sigma}\cdot\underline{p}\,\psi = mc^2\sigma^1\psi} \quad -(26)$$

where

$$\psi = \begin{bmatrix} \psi^1 & \psi^2 \\ \psi^3 & \psi^4 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad -(27)$$

1) 130(12): Comparison of ECE and Dirac Equation

ECE Equation

$$\boxed{\left(\sigma^0 E - \sigma^3 c\,\underline{\sigma}\cdot\underline{p}\right)\phi = mc^2\sigma^1\phi.} \quad -(1)$$

This is:

$$E\begin{bmatrix}\phi^1 & \phi^2\\ \phi^3 & \phi^4\end{bmatrix} + c\,\underline{\sigma}\cdot\underline{p}\begin{bmatrix}-\phi^1 & -\phi^2\\ \phi^3 & \phi^4\end{bmatrix} = mc^2\begin{bmatrix}\phi^3 & \phi^4\\ \phi^1 & \phi^2\end{bmatrix}$$

$$-(2)$$

These are the four equations:

$$\left.\begin{array}{l}(E - c\,\underline{\sigma}\cdot\underline{p})\phi^1 = mc^2\phi^3\\ (E - c\,\underline{\sigma}\cdot\underline{p})\phi^2 = mc^2\phi^4\end{array}\right\} \quad (E - c\,\underline{\sigma}\cdot\underline{p})\phi^R = mc^2\phi^L$$

$$\left.\begin{array}{l}(E + c\,\underline{\sigma}\cdot\underline{p})\phi^3 = mc^2\phi^1\\ (E + c\,\underline{\sigma}\cdot\underline{p})\phi^4 = mc^2\phi^2\end{array}\right\} \quad (E + c\,\underline{\sigma}\cdot\underline{p})\phi^L = mc^2\phi^R$$

$$-(3)$$

Here:

$$\phi^R = \left[\phi^1 \ \ \phi^2\right] \quad -(4)$$

$$\phi^L = \left[\phi^3 \ \ \phi^4\right]$$

and

$$\phi = \begin{bmatrix}\phi_1^R & \phi_2^R\\ \phi_1^L & \phi_2^L\end{bmatrix} = \begin{bmatrix}\phi^1 & \phi^2\\ \phi^3 & \phi^4\end{bmatrix} \quad -(5)$$

In general ϕ is a tetrad defined by:

$$\begin{bmatrix}\phi^R\\ \phi^L\end{bmatrix} = \phi\begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix} \quad -(6)$$

i.e.

$$V^a = q^a_{\ \mu} V^\mu \quad -(7)$$

2)

Dirac Equation

$$(\gamma^\mu p_\mu - mc)\phi = 0 \quad -(8)$$

where: $\gamma^0 = \begin{bmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{bmatrix}$, $\gamma^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} \quad -(9)$

Therefore 4×4 matrices are used:

$$E\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & \underline{\sigma} & \\ 0 & 0 & & \\ -\underline{\sigma} & & 0 & 0 \\ & & 0 & 0 \end{bmatrix} \cdot \underline{p}\begin{bmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{bmatrix} = mc^2\begin{bmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{bmatrix}$$

$-(10)$

i.e.

$$(E + c\underline{\sigma}\cdot\underline{p})\begin{bmatrix} \psi^3 \\ \psi^4 \end{bmatrix} = mc^2\begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} \quad -(11)$$

$$(E - c\underline{\sigma}\cdot\underline{p})\begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} = mc^2\begin{bmatrix} \psi^3 \\ \psi^4 \end{bmatrix} \quad -(12)$$

These are the same as eqns. (5) Q.E.D. Here

$$\phi^R = \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix}, \quad \phi^L = \begin{bmatrix} \psi^3 \\ \psi^4 \end{bmatrix} \quad -(13)$$

If the two parts of eq. (3) are multiplied together:

$$(E - c\underline{\sigma}\cdot\underline{p})(E + c\underline{\sigma}\cdot\underline{p})\phi^R\phi^L = m^2c^4\phi^L\phi^R$$

$-(14)$

i.e. $$E^2 = c^2p^2 + m^2c^4, \quad -(15)$$

which is the Einstein energy equation.

Using the de Broglie wave particle dualism:

$$p^\mu = i\hbar \partial^\mu \quad — (16)$$

eq. (15) is:

$$(\Box + \kappa^2) \psi = 0 \quad — (17)$$

where κ is the Compton wavenumber:

$$\kappa = \frac{mc}{\hbar} \quad — (18)$$

and

$$\psi = \begin{bmatrix} \psi^1 & \psi^2 \\ \psi^3 & \psi^4 \end{bmatrix} \quad — (19)$$

Eq. (17) is the free fermion limit of the ECE wave equation:

$$(\Box + kT) q^a_\mu = 0 \quad — (20)$$

or ECE Lemma

$$\Box q^a_\mu = R q^a_\mu, \quad — (21)$$

which is the tetrad postulate:

$$D_\mu q^a_\nu = 0 \quad — (22)$$

with the basic hypothesis of general relativity:

$$R = -kT \quad — (23)$$

The ECE equation is one of generally covariant unified field theory. The Dirac equation is restricted to special relativity and is not unified with other fields.