1. 115(3): Integral Form of the ECE Newton Law

The ECE Coulomb law is:

$$\underline{\nabla} \cdot \underline{E} = \rho_e / \epsilon_0 \quad -(1)$$

and the ECE Newton law is:

$$\boxed{\underline{\nabla} \cdot \underline{g} = 4\pi G \rho} \quad -(2)$$

Therefore as in Jackson, third edition, p. 28, there exists the dynamical law:

$$\boxed{\oint_S \underline{g} \cdot \underline{n} \, da = 4\pi G \int_V \rho(\underline{x}) \, d^3x} \quad -(3)$$

for a distribution of "point masses" inside a volume V.

Consider a point mass m and a closed surface S. Let r be the distance from the mass to a point on the surface, $\underline{n}$ be the outwardly directed unit normal to the surface, da be an element of surface area. If the acceleration due to gravity $\underline{g}$ at the point on the surface due to m makes an angle θ with the unit normal, then:

$$\underline{g} \cdot \underline{n} \, da = G m \frac{\cos\theta}{r^2} da \quad -(4)$$

The vector $\underline{g}$ is directed along the line from the surface element to m, so:

$$\cos\theta \, da = r^2 d\Omega \quad -(5)$$

where $d\Omega$ is the element of solid angle. So:

$$\underline{g} \cdot \underline{n} \, da = G m \, d\Omega \quad -(6)$$

Now integrate the normal component of $\underline{g}$ over the whole surface:

$$\oint_S \underline{g} \cdot \underline{n} \, da = \begin{cases} 4\pi m G & \text{if } m \text{ is inside } S \\ 0 & \text{if } m \text{ is outside } S \end{cases} \quad -(7)$$

The integral form of the Newton / ECE law for one mass m is:

$$\boxed{\oint_S \underline{g} \cdot \underline{n} \, da = 4\pi G m} \quad -(8)$$

For many masses:

$$\oint_S \underline{g} \cdot \underline{n} \, da = 4\pi G \sum_i m_i \quad -(9)$$

For continuous mass density:

$$\oint_S \underline{g} \cdot \underline{n} \, da = 4\pi g \rho \quad -(10)$$

Note that this result comes from the inverse square law and weak equivalence principle:

$$\underline{F} = \cancel{M} \underline{g} = - \frac{mMG}{r^2} \underline{r}, \quad -(11)$$

$$\underline{g} = - \frac{mG}{r^2} \underline{r} \quad -(12)$$

The acceleration $\underline{g}$ is directed towards the mass m and by convention is given a negative sign. The electric field strength $\underline{E}$ is by convention directed outward (away) from the charge e and is given a positive sign. In eq. (4), all the quantities on the right hand side are by convention positive, except for $\cos\theta$, which is negative for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. So the situation is as follows:

The force must also be central and there must be a linear superposition of the effects of different masses. As Jackson mentions on p. 28 of the third ed., Gauss law holds for Newtonian gravitational force fields. This conclusion is generalized to relativistic force fields in eq. (3).

The divergence theorem states that:

$$\oint_S \underline{g} \cdot \underline{n} \, da = \int_V \underline{\nabla} \cdot \underline{g} \, d^3x \qquad -(13)$$

Therefore from eqs. (3) and (13):

4)

$$\int_V \underline{\nabla} \cdot \underline{g} \, d^3x = 4\pi G \int_V \rho(x) \, d^3x \quad -(14)$$

so:

$$\underline{\nabla} \cdot \underline{g} = 4\pi G \rho(x) \quad -(15)$$

which is eq. (2), QED.

Conclusion

1) In corollary, the inverse square law of Newton is given by eq. (2), in the weak field limit.

2) The generally covariant form of eq. (2) is

$$\underline{D} \cdot \underline{g} = 4\pi G \rho \quad -(16)$$

which is written in the form (2) when the spin connection is written in to the definition of mass density ρ.

3) Eq. (2) is the $\kappa = 0$ part of:

$$D_\mu T^{\kappa\mu\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(17)$$