1) Note 115(12): Rules for Coordinate Transformation

For the general tensor:

$$T^{\mu_1'\ldots\mu_k'}_{\nu_1'\ldots\nu_l'} = \left(\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}}\right)\left(\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\cdots\frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}}\right)T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} \quad -(1)$$

For a mixed index tensor:

$$T^{a'\mu'}_{b'\nu'} = \Lambda^{a'}_{a}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\Lambda^{b}_{b'}\frac{\partial x^{\nu}}{\partial x^{\nu'}}T^{a\mu}_{b\nu}. \quad -(2)$$

For a partial derivative:

$$\partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial_{\mu}. \quad -(3)$$

For a covariant derivative:

$$D_{\mu'}V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}D_{\mu}V^{\nu} \quad -(4)$$

Transformation of the Maxwell Heaviside Equations

a) Homogeneous

$$\boxed{\left(\partial_\mu\tilde{F}^{\mu\nu}\right)' = \Lambda^{\nu'}_{\nu}\partial_\mu\tilde{F}^{\mu\nu} + \Lambda^{\mu}_{\mu'}\tilde{F}^{\mu\nu}\partial_\mu\left(\Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}\right)} \quad -(5)$$

in general, where $\Lambda$ denotes Lorentz transform

2) because the MH equation are written in Minkowski spacetime. It is seen that there is an extra term on the RHS of eq. (5) from the Leibnitz theorem. This is because the partial derivative $\partial_\mu$ does not transform covariantly (see Carroll). However, the only parameter in the Lorentz transform is a constant velocity $v$, which does not change with time or distance. So:

$$\partial_\mu \left( \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} \right) = 0 \quad -(6)$$

and:

$$\boxed{\left( \partial_\mu \tilde{F}^{\mu\nu} \right)' = \Lambda^{\nu'}_{\nu} \partial_\mu \tilde{F}^{\mu\nu} = 0} \quad -(7)$$

as derived previously.

The homogeneous MH equation is invariant under the Lorentz transform. This means:

$$\left( \underline{\nabla} \cdot \underline{B} \right)' = \underline{\nabla} \cdot \underline{B} = \underline{0} \quad -(8)$$

$$\left( \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} \right)' = \left( \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} \right) = \underline{0} \quad -(9)$$

The Gauss law of magnetism and Faraday's law of induction are the same in a frame K' moving at v with respect to a frame K. They are Lorentz invariant.

b) [3] Free Space Inhomogeneous Equation

Similarly:

$$\boxed{\left(\partial_\mu F^{\mu\nu}\right)' = \Lambda^{\nu'}_{\nu}\,\partial_\mu F^{\mu\nu}} \quad -(10)$$

$$= 0$$

So this law is also invariant.

c) Inhomogeneous Equation in the Presence of Matter

Similarly:

$$\left(\partial_\mu F^{\mu\nu} = \frac{J^\nu}{\epsilon_0}\right)' = \Lambda^{\nu'}_{\nu}\left(\partial_\mu F^{\mu\nu} = \frac{J^\nu}{\epsilon_0}\right). \quad -(11)$$

i.e. in frame K':

$$\left(\underline{\nabla}\cdot\underline{D} = \rho\right)' \quad -(12)$$

$$\left(\underline{\nabla}\times\underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{J}\right)' \quad -(13)$$

and in frame K:

$$\underline{\nabla}\cdot\underline{D} = \rho \quad -(14)$$

$$\underline{\nabla}\times\underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{J}. \quad -(15)$$

The laws are Lorentz covariant.

4)

## Discussion

From eqs. (9) it is seen that the Faraday law of induction is the same precisely in frame K′ and K. Therefore it is indistinguishable and has no physical effect in one frame that is not present in the other frame. In the Faraday disk generator:

$$\left(\frac{\partial \underline{B}}{\partial t} = \underline{0}, \ \underline{\nabla} \times \underline{E} = \underline{0}\right)' \quad \text{and also}$$

$$\left(\frac{\partial \underline{B}}{\partial t} = \underline{0}, \ \underline{\nabla} \times \underline{E} = \underline{0}\right).$$

In neither frame is the electric field induced by the Faraday law is $\partial \underline{B} / \partial t = \underline{0}$, because the Faraday law is the same in both frames.

The fields themselves are the Lorentz transform:

$$\tilde{F}^{\mu\nu\,\prime} = \frac{\partial x^{\mu\prime}}{\partial x^{\mu}} \frac{\partial x^{\nu\prime}}{\partial x^{\nu}} \tilde{F}^{\mu\nu}$$

$$\boxed{\tilde{F}^{\mu\nu\,\prime} = \Lambda^{\mu\prime}_{\mu} \Lambda^{\nu\prime}_{\nu} \tilde{F}^{\mu\nu}} \quad -(16)$$

i.e.

$$\underline{E}' = \underline{E} + \underline{v} \times \underline{B} \quad -(17)$$

$$\underline{B}' = \underline{B} - \frac{1}{c^2} \underline{v} \times \underline{E} \quad -(18)$$

but the relation between them (Faraday law) is unchanged. The Faraday law has no explanation for the disk generator.

5.)

Technical Details

$$\partial'_{\mu} \tilde{F}^{\mu\nu'} = \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) \partial_{\mu} \tilde{F}^{\mu\nu'}$$

$$= \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) \partial_{\mu} \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \tilde{F}^{\mu\nu}\right)$$

$$= \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right)\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right)\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right) \partial_{\mu} \tilde{F}^{\mu\nu}$$

$$+ \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) \tilde{F}^{\mu\nu} \partial_{\mu} \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}}\right)$$

$$= \left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right) \partial_{\mu} \tilde{F}^{\mu\nu} + \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) \tilde{F}^{\mu\nu} \partial_{\mu} \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}}\right)$$

In the Lorentz transform of special relativity:

$$\boxed{\partial'_{\mu} \tilde{F}^{\mu\nu'} = \Lambda^{\nu'}_{\nu} \partial_{\mu} \tilde{F}^{\mu\nu} + \Lambda^{\mu}_{\mu'} \tilde{F}^{\mu\nu} \partial_{\mu} \left(\Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu}\right)}$$

QED