1) 112(3): Some Detailed Proofs Concerning the Bianchi Identity

Proof One

Consider the operator:

$$[D_\mu, D_\nu]\nabla\rho = D_\mu(D_\nu\nabla\rho) - D_\nu(D_\mu\nabla\rho), \quad -(1)$$

multiply both sides by $\frac{1}{2}\|g\|^{1/2}\epsilon^{\mu\nu\alpha\beta}$, where $\|g\|^{1/2}$ is the square root of the determinant of the metric and $\epsilon^{\mu\nu\alpha\beta}$ is the antisymmetric tensor in four dimensions. Therefore:

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}[D_\mu, D_\nu]\nabla\rho = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\left(D_\mu(D_\nu\nabla\rho) - D_\nu(D_\mu\nabla\rho)\right), \quad -(2)$$

i.e. $$[D^\alpha, D^\beta]\nabla\rho = D^\alpha(D^\beta\nabla\rho) - D^\beta(D^\alpha\nabla\rho). \quad -(3)$$

Now lower indices with the metrics:

$$g_{\mu\alpha}g_{\nu\beta}[D^\mu, D^\nu]\nabla\rho = g_{\mu\alpha}g_{\nu\beta}\left(D^\mu(D^\nu\nabla\rho) - D^\nu(D^\mu\nabla\rho)\right) \quad -(4)$$

i.e $$[D_\alpha, D_\beta]\nabla\rho = D_\alpha(D_\beta\nabla\rho) - D_\beta(D_\alpha\nabla\rho). \quad -(5)$$

The Hodge dual is defined as:

$$[D^\alpha, D^\beta] = \frac{1}{2}\|g\|^{1/2}\epsilon^{\alpha\beta\mu\nu}[D_\mu, D_\nu] \quad -(6)$$

For example, if $\mu = 0$ and $\nu = 1$ in eq. (1) then:

2)

$$[D_0, D_1] V^\rho = D_0 (D_1 V^\rho) - D_1 (D_0 V^\rho) \quad - (7)$$

and eq. (5) is:

$$[D_2, D_3] V^\rho = D_2 (D_3 V^\rho) - D_3 (D_2 V^\rho) \quad - (8)$$

Eqs (7) and (8) are both examples of eq (1). However, eq (8) is by definition the Hodge dual of eq (7). It follows that the Hodge dual of eq. (1) is defined in the same way as eq. (1).

Proof Two

Eq. (1) may be written as:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad - (9)$$

where the curvature tensor is:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \quad - (10)$$

and the torsion tensor is:

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \quad - (11)$$

Full details of this proof are given in paper 99.

From proof one it follows that:

$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad - (12)$$

3)

Proof Three

Given the curvature and torsion tensors (9) and (10) respectively, and given the tetrad postulate:

$$D_\mu q^a_{\ \nu} = 0 \quad -(13)$$

it follows that:

$$D \wedge T^a := R^a_{\ b} \wedge q^b \quad -(14)$$

This proof is given in all detail in paper 102. Eq (14) is equivalent to eq. (9), so it follows from proof one that:

$$D \wedge \tilde{T}^a := \tilde{R}^a_{\ b} \wedge q^b . \quad -(15)$$

Eqs (14) and (15) may be considered to be expressions of the Bianchi identity. They may be written respectively as:

$$D_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^{a\ \ \mu\nu}_{\ \mu} \quad -(16)$$

and:

$$D_\mu T^{a\mu\nu} = R^{a\ \ \mu\nu}_{\ \mu} . \quad -(17)$$

Multiply both sides of eqs (16) and (17) by $q^\mu_{\ a}$ and use the tetrad postulate to find that a particular solution is:

$$\boxed{D_\mu \tilde{T}^{\kappa\mu\nu} = \tilde{R}^{\kappa\ \ \mu\nu}_{\ \mu} \quad -(18)}$$

and

$$\boxed{D_\mu T^{\kappa\mu\nu} = R^{\kappa\ \ \mu\nu}_{\ \mu} \quad -(19)}$$

4) The structures of eqs. (18) and (19) are similar to the Maxwell Heaviside equations:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad -(20)$$

$$\partial_\mu F^{\mu\nu} = J^\nu / \epsilon_0 \qquad -(21)$$

However, eqns. (18) and (19) are duality invariant, and are the field equations of dynamics. Eqs. (18) and (19) are the same as:

$$D_\rho T^\kappa{}_{\mu\nu} + D_\nu T^\kappa{}_{\rho\mu} + D_\mu T^\kappa{}_{\nu\rho} = R^\kappa{}_{\rho\mu\nu} + R^\kappa{}_{\nu\rho\mu} + R^\kappa{}_{\mu\nu\rho} \qquad -(22)$$

and

$$D_\rho \tilde{T}^\kappa{}_{\mu\nu} + D_\nu \tilde{T}^\kappa{}_{\rho\mu} + D_\mu \tilde{T}^\kappa{}_{\nu\rho} = \tilde{R}^\kappa{}_{\rho\mu\nu} + \tilde{R}^\kappa{}_{\nu\rho\mu} + \tilde{R}^\kappa{}_{\mu\nu\rho} \qquad -(23)$$

Eqs (18) and (19), or eqs. (22) and (23), were tested by computer algebra for exact solutions of the Einstein field equation as in paper 93. The Einstein field equation failed this test because:

$$\tilde{T}^{\kappa\mu\nu} = \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} = T^{\kappa\mu} = 0 \qquad -(24)$$

but:

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq 0. \qquad -(25)$$