1)

109(3): New Theorem of Differential Geometry

Theorem

If:

$$[D_\mu, D_\nu] V^\kappa = A^\kappa{}_{\sigma\mu\nu} V^\sigma - B^\lambda{}_{\mu\nu} D_\lambda V^\kappa \quad -(1)$$

then:

$$D_\mu B^{\kappa\mu\nu} = A^{\kappa}{}_{\mu}{}^{\mu\nu}. \quad -(2)$$

Proof

It follows from eq. (1) that the structure of A and B must be:

$$A^\kappa{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\kappa_{\nu\sigma} - \partial_\nu \Gamma^\kappa_{\mu\sigma} + \Gamma^\kappa_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\kappa_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \quad -(3)$$

and

$$B^\kappa{}_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu}. \quad -(4)$$

This structure implies:

$$D_\mu B^\kappa{}_{\nu\sigma} + D_\sigma B^\kappa{}_{\mu\nu} + D_\nu B^\kappa{}_{\sigma\mu} = A^\kappa{}_{\mu\nu\sigma} + A^\kappa{}_{\sigma\mu\nu} + A^\kappa{}_{\nu\sigma\mu} \quad -(5)$$

i.e.

$$D_\mu \tilde{B}^{\kappa\mu\nu} = \tilde{A}^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(6)$$

Now from the Hodge duals:

2)

$$\tilde{A}^{\kappa}{}_{\sigma}{}^{\alpha\beta} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} A^{\kappa}{}_{\sigma\mu\nu} \quad -(7)$$

$$\tilde{B}^{\kappa\alpha\beta} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} B^{\kappa}{}_{\mu\nu} \quad -(8)$$

in four dimensions.

The particular solution:

$$[D_\mu, D_\nu]_{HD} V^{\kappa} = \tilde{A}^{\kappa}{}_{\sigma\mu\nu} V^{\sigma} - \tilde{B}^{\lambda}{}_{\mu\nu} D_\lambda V^{\kappa} \quad -(9)$$

follows, where:

$$[D^\mu, D^\nu]_{HD} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} [D_\alpha, D_\beta]. \quad -(10)$$

For example:

$$\tilde{A}^{\kappa}{}_{\sigma}{}^{01} = \|g\|^{1/2} A^{\kappa}{}_{\sigma 23} \quad -(11)$$

$$\tilde{B}^{\kappa 01} = \|g\|^{1/2} B^{\kappa}{}_{23}. \quad -(12)$$

The tensors $\tilde{A}$ and $\tilde{B}$ therefore have the same structure as A and B, and use the same connections:

$$\tilde{A}^{\kappa}{}_{\sigma}{}^{01} = \|g\|^{1/2} \left(\partial_2 \Gamma^{\kappa}_{3\sigma} - \partial_3 \Gamma^{\kappa}_{2\sigma} + \Gamma^{\kappa}_{2\lambda} \Gamma^{\lambda}_{3\sigma} - \Gamma^{\kappa}_{3\lambda} \Gamma^{\lambda}_{2\sigma} \right) \quad -(13)$$

$$\tilde{B}^{\kappa 01} = \|g\|^{1/2} \left(\Gamma^{\kappa}_{23} - \Gamma^{\kappa}_{32} \right). \quad -(14)$$

In eq. (5) fix $\mu = 2, \nu = 3$:

3)

$$D_2 B^{\kappa}_{\ 3\sigma} + D_\sigma B^{\kappa}_{\ 23} + D_3 B^{\kappa}_{\ \sigma 2} = A^{\kappa}_{\ 23\sigma} + A^{\kappa}_{\ \sigma 23} + A^{\kappa}_{\ 3\sigma 2} \quad -(15)$$

Now multiply through by $\|g\|^{1/2}$ and:

$$\epsilon^{0123} = 1 \quad -(16)$$

to obtain:

$$D_0 \tilde{B}^{\kappa}_{\ 1\sigma} + D_\sigma \tilde{B}^{\kappa}_{\ 01} + D_0 \tilde{B}^{\kappa}_{\ \sigma 1} = \tilde{A}^{\kappa}_{\ 01\sigma} + \tilde{A}^{\kappa}_{\ \sigma 01} + \tilde{A}^{\kappa}_{\ 0\sigma 1} \quad -(17)$$

as a particular solution after lowering indices..
Finally fix σ at 2 in eq. (17) to obtain:

$$D_0 \tilde{B}^{\kappa}_{\ 12} + D_2 \tilde{B}^{\kappa}_{\ 01} + D_0 \tilde{B}^{\kappa}_{\ 21} = \tilde{A}^{\kappa}_{\ 012} + \tilde{A}^{\kappa}_{\ 201} + \tilde{A}^{\kappa}_{\ 021} \quad -(18)$$

This is an example of:

$$D_\mu \tilde{B}^{\kappa}_{\ \nu\sigma} + D_\sigma \tilde{B}^{\kappa}_{\ \mu\nu} + D_\nu \tilde{B}_{\sigma\mu} = \tilde{A}^{\kappa}_{\ \mu\nu\sigma} + \tilde{A}^{\kappa}_{\ \sigma\mu\nu} + \tilde{A}^{\kappa}_{\ \nu\sigma\mu} \quad -(19)$$

which is

$$D_\mu B^{\kappa\mu\nu} = A^{\kappa\ \ \mu\nu}_{\ \mu} . \quad -(20)$$

Q.E.D.

The tensor A is the curvature tensor R
and B is the torsion tensor T.

4)

Therefore it has been proven from fundamental first principles that:

$$D_\mu T^{\kappa\mu\nu} = R^{\kappa\ \ \mu\nu}_{\ \ \mu} \quad -(21)$$

where:

$$T^{\kappa}_{\ \mu\nu} = \Gamma^{\kappa}_{\ \mu\nu} - \Gamma^{\kappa}_{\ \nu\mu}, \quad -(22)$$

$$R^{\kappa}_{\ \sigma\mu\nu} = \partial_\mu \Gamma^{\kappa}_{\ \nu\sigma} - \partial_\nu \Gamma^{\kappa}_{\ \mu\sigma} + \Gamma^{\kappa}_{\ \mu\lambda}\Gamma^{\lambda}_{\ \nu\sigma} - \Gamma^{\kappa}_{\ \nu\lambda}\Gamma^{\lambda}_{\ \mu\sigma}. \quad -(23)$$

Eq. (21) proves that the Christoffel symbol:

$$\Gamma^{\kappa}_{\ \mu\nu} = \Gamma^{\kappa}_{\ \nu\mu} \quad -(24)$$

is self-inconsistent because for eq (24), and for various exact solutions of the Einstein field equation, $R^{\kappa\ \ \mu\nu}_{\ \ \mu}$ is not zero but $T^{\kappa\mu\nu}$ is zero.

Conclusion

The Einstein field equation is in general self-inconsistent because of this result, so no physical inference can be based on it. ECE theory is consistent with eq. (21)