1/100(17) : Expansion of the First Bianchi Identity

As in GCUFT1, Chap. 17, C Appendix :

$$\partial_\mu T^a_{\nu\rho} + \omega^a_{\mu b} T^b_{\nu\rho} + \partial_\rho T^a_{\mu\nu} + \omega^a_{\rho b} T^b_{\mu\nu} + \partial_\nu T^a_{\rho\mu} + \omega^a_{\nu b} T^b_{\rho\mu}$$

$$:= \left(R^\lambda_{\mu\nu\rho} + R^\lambda_{\rho\mu\nu} + R^\lambda_{\nu\rho\mu} \right) q^a_\lambda \quad - (1)$$

where :

$$T^a_{\nu\rho} = \left(\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\rho\nu} \right) q^a_\lambda \quad \text{etc.} \quad - (2)$$

$$T^b_{\nu\rho} = \left(\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\rho\nu} \right) q^b_\lambda \quad \text{etc.} \quad - (3)$$

Using the Leibnitz Theorem :

$$\partial_\mu T^a_{\nu\rho} = \left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu} \right) q^a_\lambda + \left(\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\rho\nu} \right) \partial_\mu q^a_\lambda$$

$$\text{etc.} \quad - (4)$$

So:

$$\left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu} \right) q^a_\lambda + \left(\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda \right) \left(\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\rho\nu} \right)$$

$$+ \cdots\cdots \quad := \left(R^\lambda_{\mu\nu\rho} + R^\lambda_{\rho\mu\nu} + R^\lambda_{\nu\rho\mu} \right) q^a_\lambda$$

$$- (5)$$

Relabel dummy indices :

$$\left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu} \right) q^a_\lambda + \left(\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma \right) \left(\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu} \right)$$

$$+ \cdots \quad := \left(R^\lambda_{\mu\nu\rho} + R^\lambda_{\rho\mu\nu} + R^\lambda_{\nu\rho\mu} \right) q^a_\lambda$$

$$- (6)$$

Use the tetrad postulate :

2)

$$\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma = \Gamma^\lambda_{\mu\sigma} q^a_\lambda \quad -(7)$$

So:

$$\begin{aligned}
&\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\mu\sigma}\left(\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu}\right) && R^\lambda_{\mu\nu\rho} \\
+&\partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\rho\sigma}\left(\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}\right) &:=& + R^\lambda_{\rho\mu\nu} \\
+&\partial_\nu \Gamma^\lambda_{\rho\mu} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\nu\sigma}\left(\Gamma^\sigma_{\rho\mu} - \Gamma^\sigma_{\mu\rho}\right) && + R^\lambda_{\nu\rho\mu}
\end{aligned} \quad -(8)$$

Finally re-arrange terms in eq. (8):

$$R^\lambda_{\rho\mu\nu} + R^\lambda_{\mu\nu\rho} + R^\lambda_{\nu\rho\mu} \quad -(9)$$

$$\begin{aligned}
:= &\ \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} \\
&+ \partial_\nu \Gamma^\lambda_{\rho\mu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\rho\mu} - \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\nu\mu} \\
&+ \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\rho\nu}
\end{aligned}$$

In shorthand notation this identity is:

$$D \wedge T := R \wedge q \quad -(10)$$

It is an exact identity because by definition:

$$R^{\lambda}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \quad -(11)$$

and so on, and:

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \quad -(12)$$

and so on.

In general:

$$R^{\lambda}{}_{\rho\mu\nu} + R^{\lambda}{}_{\mu\nu\rho} + R^{\lambda}{}_{\nu\rho\mu} \neq 0 \quad -(13)$$

The Bianchi identity (10) can be written as: $-(14)$

$$D_\mu T^{a}{}_{\nu\rho} + D_\rho T^{a}{}_{\mu\nu} + D_\nu T^{a}{}_{\rho\mu} := R^{a}{}_{\mu\nu\rho} + R^{a}{}_{\rho\mu\nu} + R^{a}{}_{\nu\rho\mu}$$

i.e. as:

$$D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}^{a}{}_{\mu}{}^{\mu\nu} \quad -(15)$$

A particular solution is ~~the~~ base ~~to~~ manifold relation:

$$D_\mu \tilde{T}^{\kappa\mu\nu} := \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(16)$$

Therefore eqns. (10), (14), (16) and (9) state the same ~~thing~~, that if a tensor can be defined by eq. (11) and another by eq (12), then ~~the~~ the Bianchi identity follows. It is known that the tensors (11) and (12) follow from a round trip

+) on a vector V^ρ with the commutator $[D_\mu, D_\nu]$ (paper 99). They are the Riemann tensor and torsion tensor for any connection. They are anti-symmetric by definition:

$$R^\lambda_{\ \rho\mu\nu} = -R^\lambda_{\ \rho\nu\mu} \quad -(17)$$

$$T^\lambda_{\ \mu\nu} = -T^\lambda_{\ \nu\mu} \quad -(18)$$

so by definition have well defined Hodge duals. These Hodge duals denoted $\tilde{R}^\lambda_{\ \rho\mu\nu}$ and $\tilde{T}^\lambda_{\ \mu\nu}$ can be defined in terms of a connection denoted $\Lambda^\lambda_{\ \nu\rho}$:

$$\tilde{R}^\lambda_{\ \rho\mu\nu} = \partial_\mu \Lambda^\lambda_{\ \nu\rho} - \partial_\nu \Lambda^\lambda_{\ \mu\rho} + \Lambda^\lambda_{\ \mu\sigma}\Lambda^\sigma_{\ \nu\rho} - \Lambda^\lambda_{\ \nu\sigma}\Lambda^\sigma_{\ \mu\rho} \quad -(19)$$

$$\tilde{T}^\lambda_{\ \mu\nu} = \Lambda^\lambda_{\ \mu\nu} - \Lambda^\lambda_{\ \nu\mu} \quad -(20)$$

where in eq. (20), $\Lambda^\lambda_{\ \mu\nu}$ is the Hodge dual of $\Gamma^\lambda_{\ \mu\nu}$. Strictly speaking $\Lambda^\lambda_{\ \mu\nu} - \Lambda^\lambda_{\ \nu\mu}$ is the Hodge dual of $\Gamma^\lambda_{\ \mu\nu} - \Gamma^\lambda_{\ \nu\mu}$ because the connection is not a tensor, but the difference of two connections is a tensor. The Hodge dual is defined in 4-D on the anti-symmetric indices μ and ν.

It therefore follows from (19) and (20) that

$$D \wedge \tilde{T} := \tilde{R} \wedge q \quad -(21)$$

This can be expressed in the base manifold as:

$$\boxed{D_\mu T^{\kappa\mu\nu} := R^{\kappa}{}_{\mu}{}^{\mu\nu}.} \quad -(22)$$

Eq. (22) invalidates the EH theory because it is incompatible with the use of the Christoffel connection:

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \quad -(23).$$

The existence of the definition (21) can therefore be traced to the fact that there exists a Hodge dual of the commutator $[D_\mu, D_\nu]$ or $[D^\mu, D^\nu]$

$$[D^\mu, D^\nu] = \frac{1}{2}\|g\|^{1/2}\epsilon^{\mu\nu\rho\sigma}\widetilde{[D_\rho, D_\sigma]} \quad -(24)$$

$$\widetilde{[D_\rho, D_\sigma]} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\rho\sigma\mu\nu}[D^\mu, D^\nu] \quad -(25)$$

where

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda_{\mu\nu}D_\lambda V^\rho \quad -(26)$$

so there exist Hodge duals of $R^\rho{}_{\sigma\mu\nu}$ and $T^\lambda_{\mu\nu}$

6)

We may also define:

$$[D^\mu, D^\nu] V^\rho = R^\rho{}_\sigma{}^{\mu\nu} V^\sigma - T^{\lambda\mu\nu} D_\lambda V^\rho \quad -(27)$$

and the Hodge duals of both sides of eq (27) produce:

$$[D_\alpha, D_\beta] V^\rho = \tilde{R}^\rho{}_{\sigma\alpha\beta} V^\sigma - \tilde{T}^\lambda{}_{\alpha\beta} D_\lambda V^\rho \quad -(28)$$

The Bianchi identity is an exact identity because it is a cyclic sum of definitions of the Riemann tensor. The Hodge dual Bianchi identity (21) is an exact identity because it is a cyclic sum of definitions of the Hodge dual of the Riemann tensor. These definitions occur from the commutator $[D_\mu, D_\nu]$ acting on a vector V^ρ. The commutator is antisymmetric and so has a well defined Hodge dual generator, another antisymmetric commutator in four dimensions. If we denote this Hodge dual commutator by $[D_\alpha, D_\beta]$ it acts on a vector V^ρ to produce $\tilde{R}^\rho{}_{\sigma\alpha\beta}$ and $\tilde{T}^\lambda{}_{\alpha\beta}$ as in eq. (28). It follows that the Christoffel connection cannot be used in the theory of gravitation.