100(14): Rotational Equivalent of the Einstein Postulate

The original Einstein postulate of 1915 is well known to be:

$$D^\mu G_{\mu\nu} = k D^\mu T_{\mu\nu} = 0, \quad -(1)$$

where:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \quad -(2)$$

Here $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature, $g_{\mu\nu}$ is the symmetric metric, D^μ indicates covariant derivative, k is Einstein's constant and $T_{\mu\nu}$ is the symmetric canonical energy-momentum density tensor. In eq. (1) the second Bianchi identity is:

$$D^\mu G_{\mu\nu} = 0 \quad -(3)$$

and Noether's Theorem is:

$$D^\mu T_{\mu\nu} = 0. \quad -(4)$$

From eq. (1) Einstein used the particular solution:

$$G_{\mu\nu} = k T_{\mu\nu}. \quad -(5)$$

In paper 88 it was shown that eq. (3) is true if and only if the torsion is identically zero. In the presence of torsion:

$$D_\mu T^{\kappa\mu\nu} = - R^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(6)$$

$$D_\mu \tilde{T}^{\kappa\mu\nu} = - \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(7)$$

Eqs. (6) and (7) are particular solutions of the true Bianchi identity:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(8)$$

In the notation of paper 88 it was shown that eq. (8) is the only Bianchi identity. The usually named second Bianchi identity is a special case of:

$$D \wedge (D \wedge T^a) := D \wedge (R^a{}_b \wedge q^b) \quad -(9)$$

which may be written as:

$$(D \wedge R)_{cyclic} := (D(D \wedge T))_{cyclic} \quad -(10)$$

The special case used by Einstein is:

$$D \wedge (R^a{}_b \wedge q^b) = 0 \quad -(11)$$

i.e.

$$D \wedge R^a{}_b = 0 \quad -(12)$$

which is also the special case considered by Cartan.

In tensor notation eq. (8) is:

$$D_\mu \tilde{T}^{\kappa\mu\nu} = -\tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(13)$$

A well-defined Hodge dual of eq. (8) is:

$$D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b \quad -(14)$$

which in tensor notation is:

$$D_\mu T^{\kappa\mu\nu} = -R^{\kappa\ \mu\nu}_{\ \mu} \quad -(15)$$

It follows from eq. (15) that if $R^{\kappa\ \mu\nu}_{\ \mu}$ is non-zero then the torsion tensor $T^{\kappa\mu\nu}$ is also non-zero.

The Ricci cyclic equation is:

$$R^a{}_b \wedge q^b = 0 \quad -(16)$$

which implies:

$$D \wedge T^a = 0. \quad -(17)$$

Eq (17) does not imply that:

$$T^a = 0, \quad -(18)$$

and as was shown in paper 93, $R^{\kappa\ \mu\nu}_{\ \mu}$ is zero only for Ricci flat spacetimes, i.e. only for:

$$G_{\mu\nu} = 0, \; T_{\mu\nu} = 0. \quad -(19)$$

One can only say that for metrics that obey the Ricci cyclic equation:

$$D_\mu \tilde{T}^{\kappa\mu\nu} = -\tilde{R}^{\kappa\ \mu\nu}_{\ \mu} = 0, \quad -(20)$$

but

$$D_\mu T^{\kappa\mu\nu} = -R^{\kappa\ \mu\nu}_{\ \mu} \neq 0, \quad -(21)$$

in general. However, for the Christoffel connect

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$$T^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} = 0. \quad -(22)$$

This is the special case considered by Einstein and Hilbert. In this special case it is possible to state only that if

$$\tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} = 0 \quad -(22)$$

then

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq 0 \quad -(23)$$

in general. For a Christoffel connection:

$$T^{\kappa\mu\nu} = 0 \quad -(24)$$

$$\tilde{T}^{\kappa\mu\nu} = 0. \quad -(25)$$

Eq. (22) is equivalent to:

$$R^{\kappa}{}_{\mu\nu\sigma} + R^{\kappa}{}_{\sigma\mu\nu} + R^{\kappa}{}_{\nu\sigma\mu} = 0 \quad -(26)$$

and Eq. (23) is:

$$\tilde{R}^{\kappa}{}_{\mu\nu\sigma} + \tilde{R}^{\kappa}{}_{\sigma\mu\nu} + \tilde{R}^{\kappa}{}_{\nu\sigma\mu} \neq 0 \quad -(27)$$

in general.

So the well known EH equation (5) is a very special case that loses information.

5) If a Christoffel connection is used, eq. (6) can only be rigorously true in a Ricci flat spacetime, where:

$$D_\mu T^{\kappa\mu\nu} = -R_\mu{}^{\kappa\mu\nu} = 0. \quad -(28)$$

In general, eq. (28) indicates that the Christoffel connection cannot be used in general, because for a Christoffel connection:

$$T^{\kappa\mu\nu} = 0 \text{ but } R_\mu{}^{\kappa\mu\nu} \neq 0 \quad -(29)$$

resulting in a contradiction.

This contradiction is present in the EH theory of gravitation.

In paper 93, eq. (6) was interpreted as the balance of two different fields, electromagnetic and gravitational:

$$A^{(0)}\left(D_\mu T^{\kappa\mu\nu}\right)_{\text{electromagnetic}} \doteq -A^{(0)}\left(R_\mu{}^{\kappa\mu\nu}\right)_{\text{gravitational}} \quad -(30)$$

so $T^{\kappa\mu\nu}$ and $R_\mu{}^{\kappa\mu\nu}$ refer to two different fields. This is an approximation. In paper 93 the Christoffel connection was used to compute

$$R_\mu{}^{\kappa\mu\nu} \neq 0 \quad -(31)$$

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in general. This means that there is no gravitational torsion present:

$$\left(T^{\kappa\mu\nu}\right)_{gravitation} = 0 \quad -(32)$$

but there is an electromagnetic torsion present.

If we are considering one field only, then eq. (28) prohibits the use of the Christoffel symbol. To describe gravitation correctly, the more general gamma connection must be used, i.e.

$$\boxed{\Gamma^{\kappa}_{\mu\nu} \neq \Gamma^{\kappa}_{\nu\mu}} \quad -(33).$$

It is reasonable then to assume that the more general conservation law is: $\quad -(34)$

$$\boxed{D_\mu T^{\kappa\mu\nu} = k D_\mu J^{\kappa\mu\nu} = -R^{\kappa}{}_{\mu}{}^{\mu\nu}}$$

A particular solution is:

$$\boxed{T^{\kappa\mu\nu} = k J^{\kappa\mu\nu}} \quad -(35)$$

where $J^{\kappa\mu\nu}$ is the canonical angular energy-momentum density / tensor: $\quad -(36)$

$$J^{\kappa\mu\nu} = -\frac{1}{2}\left(T^{\kappa\mu}x^{\nu} - T^{\kappa\nu}x^{\mu}\right)$$