NOTES ON THE SAGNAC EFFECT

For infinitesimal displacements, the effect is:

$$-ig\Delta S^{\mu\nu}G_{\mu\nu}$$

Holonomy is:

$$\exp\left(-ig\iint G_{\mu\nu}dS^{\mu\nu}\right).$$

The vector ψ is rotated in the internal space. In the opposite direction, the holonomy is:

$$\exp\left(ig\iint G_{\mu\nu}dS^{\mu\nu}\right).$$

The rotation in the internal space (1), (2), (3) is the rotation of the light beam inside the optical fiber.

The holonomy is:

$$\exp\left(\mp ig\iint G^{a}_{\mu\nu}dS^{\mu\nu}\right)=\exp\left(\mp ig\iint\left(\partial_{\mu}A^{a}_{\nu}-\partial_{\nu}A^{a}_{\mu}-ig\varepsilon_{abc}A^{b}_{\mu}A^{c}_{\nu}\right)dS^{\mu\nu}\right)$$

$S^{\mu\nu}$ is the surface enclosed by the round trip.

In U(1), the holonomy is:

$$\exp\left(\mp ig\iint\left(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}\right)dS^{\mu\nu}\right)$$

and the ordinary Stokes theorem is obtained.

So

$$\exp\left(\mp ig\iint\left(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}\right)dS^{\mu\nu}\right)=\exp\left(\mp ig\oint A_{\mu}dx^{\mu}\right)$$
$$=1 \quad \text{for integration around a circle.}$$

This is because in U(1), the radiation gauge is obtained, and

$$A_{\mu}dx^{\mu}=\boldsymbol{A}^{(1)}\cdot d\boldsymbol{r}=0$$

because $\boldsymbol{A}^{(1)}\perp\boldsymbol{r}$.

Therefore:

$$\exp\left(\mp ig\iint\left(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}\right)dS^{\mu\nu}\right)=1$$

and

$$\exp\left(\mp ig\iint\left(\partial_{\mu}A^{(1)}_{\nu}-\partial_{\nu}A^{(1)}_{\mu}\right)dS^{\mu\nu}\right)=1$$

$$\exp\left(\mp ig\iint\left(\partial_\mu A_\nu^{(2)} - \partial_\nu A_\mu^{(2)}\right)dS^{\mu\nu}\right) = 1$$

$$\exp\left(\mp ig\iint\left(\partial_\mu A_\nu^{(3)} - \partial_\nu A_\mu^{(3)}\right)dS^{\mu\nu}\right) = 1$$

We now consider the holonomy:

$$\exp\left(\mp g^2\iint \varepsilon_{abc} A_\mu^b A_\nu^c dS^{\mu\nu}\right).$$

Consider the special case:

$$\exp\left(\mp g^2\iint\left(\varepsilon_{(3)(1)(2)} A_X^{(1)} A_Y^{(2)} + \varepsilon_{(3)(2)(1)} A_X^{(2)} A_Y^{(1)}\right)dS^{XY}\right)$$

$$= \exp\left(\mp g^2\iint\left(\underbrace{A_X^{(1)}A_Y^{(2)} - A_X^{(2)}A_Y^{(1)}}_{(a)}\right)\underbrace{dS^{XY}}_{(b)}\right)$$

where (a) is the components in the internal space ((1), (2), (3)) and (b) is the area of the Sagnac loop.

By definition:

$$A^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(ii + j)e^{i\phi};$$

$$A^{(2)} = \frac{A^{(0)}}{\sqrt{2}}(-ii + j)e^{-i\phi}.$$

So

$$A_X^{(1)}A_Y^{(2)} - A_X^{(2)}A_Y^{(1)} = iA^{(0)3}.$$

By definition:

$$\boldsymbol{B}^{(3)} = -igiA^{(0)2} = \kappa A^{(0)} = B^{(0)}$$

because:

$$g = \frac{\kappa}{A^{(0)}}$$

The holonomy is therefore:

$$\exp\left(\mp ig\iint B^{(3)}dS^{XY}\right) = \exp\left(\mp ig^2\iint A^{(0)2}dAr\right); \qquad Ar \equiv S^{XY}$$

$$= \exp\left(\mp igB^{(3)}Ar\right)$$

$$= \exp\left(\mp i\kappa^2 Ar\right)$$

The holonomy difference $= \exp\left(-2i\kappa^2 Ar\right)$

The O(3) phase difference $= \cos\left(2\kappa^2 Ar \pm 2\pi n\right)$

The U(1) phase difference $= 0$.

This result is consistent with the fact that a U(1) Yang-Mills gauge field theory is the Maxwell-Heaviside theory, which is free space invariant under motion reversal symmetry, T, which generates A from C. The Maxwell-Heaviside theory cannot describe the Sagnac effect with platform at rest. In free space, it is gauge invariant and cannot describe the Sagnac effect with platform in motion.

The O(3) Phase Difference With Platform at Rest.

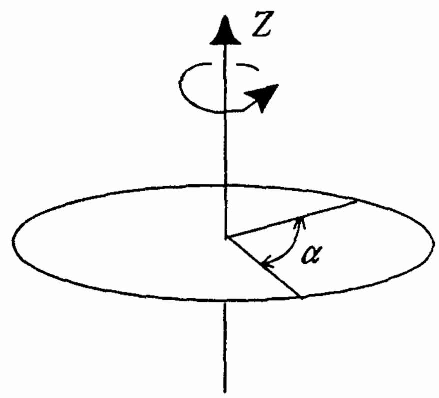
$$\Delta\phi = \cos\left(2\frac{\omega^2}{c^2}Ar \pm 2\pi n\right)$$

The O(3) Phase Difference With Platform in Motion.

In O(3) Yang-Mills theory, this is a change of phase which is a change in orientation in the internal space due to $\psi \to S\psi$, where S is a rotation generator.

To rotate the platform about Z, the rotation generator is:

$$S = \exp\left(iJ_Z\alpha\left(x^\mu\right)\right).$$



The effect of κ_μ, in condensed rotation, is:

$$\kappa_\mu \to S\kappa_\mu S^{-1} - i\left(\partial_\mu S\right)S^{-1}$$

where:

$$\kappa^\mu = \kappa^{\mu(1)}e^{(1)} + \kappa^{\mu(2)}e^{(2)} + \kappa^{\mu(3)}e^{(3)}.$$

We are interested in the effect on $\kappa^{\mu(3)}$, which is the wave vector of the light propagating around the Sagnac loop. The effect is:

$$\kappa_\mu^{(3)} \to \kappa_\mu^{(3)} \pm \partial_\mu\alpha$$

where:

$$\kappa_\mu^{(3)} = \left(\frac{\omega}{c}, \kappa\right)$$

Now consider the index $\mu = 0$, and:

$$\frac{\omega}{c} \to \frac{\omega}{c} \pm \frac{1}{c}\frac{\partial\alpha}{\partial t}$$

$$\omega \to \omega \pm \Omega; \qquad \Omega = \frac{\partial\alpha}{\partial t}.$$

The extra effect of the rotation is therefore to shift the phase by:

$$\Delta\Delta\phi = \frac{Ar}{c^2}\left((\omega + \Omega)^2 - (\omega - \Omega)^2\right)$$

In O(3):

$$\boxed{\Delta\Delta\phi = \cos\left(r\frac{\omega\Omega Ar}{c^2}\right)}$$

This is precisely the correct result.

In U(1), we have:

$$\boxed{\Delta\Delta\phi = 0}$$

Finally, the holonomy difference can be expressed as:

$$\phi = \exp\left(\mp i\oint \kappa_Z^{(3)} dZ\right)$$

in the internal space, where:

$$\oint dZ = 2\pi\, Rx$$

i.e. is some multiple of the circumference $2\pi R$ of the Sagnac loop.

This result is expressed by Barrett in eqn. (51), page 295 of Barrett and Grimes. The quantity:

$$g_m = \frac{1}{V}\iint B^{(3)} dAr = \frac{1}{V}\iint \kappa\, A^{(0)} dAr$$

has the units of "magnetic charge", but is not a point magnetic monopole. It is a **topological** magnetic monopole.