FORMAL PROOF OF THE GAUGE INVARIANCE OF G, F, A AND ϕ

Whittaker shows that:

$$\boldsymbol{B} = -\nabla \times (\nabla \times \boldsymbol{g}) + \frac{1}{c} \nabla \times \dot{\boldsymbol{f}} \tag{1}$$

$$\boldsymbol{E} = c\nabla \times (\nabla \times \boldsymbol{f}) + \nabla \times \dot{\boldsymbol{g}} \tag{2}$$

Eqn. (1) is invariant under:

$$\boldsymbol{g} \to \boldsymbol{g} + \nabla a; \qquad \nabla \times \boldsymbol{g} \to \nabla \times \boldsymbol{g} + \nabla b \tag{3}$$

where a and b are arbitrary. This implies that:

$$\nabla \times \boldsymbol{g} \to \nabla \times \boldsymbol{g} + \nabla \times (\nabla a) = \nabla \times \boldsymbol{g} \tag{4}$$

$$\nabla \times \boldsymbol{g} \to \nabla \times \boldsymbol{g} \tag{5}$$

Now use:

$$\boldsymbol{A} = -\nabla \times \boldsymbol{g} + \frac{1}{c} \dot{\boldsymbol{f}} \tag{6}$$

The transverse part of A is:

$$\boldsymbol{A}_T = -\nabla \times \boldsymbol{g} \tag{7}$$

$$\boldsymbol{A}_T \to \boldsymbol{A}_T \tag{8}$$

The transverse vector potential is **physical**. This overturns Heaviside's assertion of 1895, and supports the point of view of Maxwell and Faraday.

Eqn. (2) is invariant under:

$$\boldsymbol{f} \to \boldsymbol{f} + \nabla c; \qquad \nabla \times \boldsymbol{f} \to \nabla \times \boldsymbol{f} + \nabla d \tag{9}$$

where c and d are arbitrary.

So:

$$\nabla \times \boldsymbol{f} \to \nabla \times \boldsymbol{f} \tag{10}$$

Now use:

$$\boldsymbol{S} = -c\nabla \times \boldsymbol{f} - \dot{\boldsymbol{g}} \tag{11}$$

and the transverse part of Stratton's potential is **physical**.

$$\boldsymbol{S}_T \to \boldsymbol{S}_T \tag{12}$$

In the special case of plane waves:

$$\boldsymbol{S} = ic\boldsymbol{A} \tag{13}$$

$$\boldsymbol{f} = i\boldsymbol{g} \tag{14}$$

$$\dot{\boldsymbol{f}} = i\dot{\boldsymbol{g}} \tag{15}$$

So:

$$\boldsymbol{E} = ic\nabla\times(\nabla\times\boldsymbol{g}) + \nabla\times\dot{\boldsymbol{g}} \tag{16}$$

$$\boldsymbol{B} = i\nabla\times(\nabla\times\boldsymbol{f}) + \frac{1}{c}\nabla\times\dot{\boldsymbol{f}} \tag{17}$$

If, however,

$$\boldsymbol{g} \to \boldsymbol{g} + \nabla a$$

as in eqn. (3), then

$$\nabla\times\boldsymbol{g} \to \nabla\times\boldsymbol{g}; \qquad \nabla\times\dot{\boldsymbol{g}} \to \nabla\times\dot{\boldsymbol{g}}$$

So the longitudinal and transverse parts of both ***E*** and ***B*** are **physical**. This overturns the received view that only the transverse parts of ***E*** and ***B*** may be physical in vacuo.

The fields are given by:

$$E_X = \frac{\partial^2 F}{\partial X \partial Z} + \frac{1}{c}\frac{\partial^2 G}{\partial Y \partial t}; \qquad B_X = \frac{1}{c}\frac{\partial^2 F}{\partial Y \partial t} - \frac{\partial^2 G}{\partial X \partial Z};$$

$$E_Y = \frac{\partial^2 F}{\partial Y \partial Z} - \frac{1}{c}\frac{\partial^2 G}{\partial X \partial t}; \qquad B_Y = -\frac{1}{c}\frac{\partial^2 F}{\partial X \partial t} - \frac{\partial^2 G}{\partial Y \partial Z};$$

$$E_Z = \frac{\partial^2 F}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}; \qquad B_Z = \frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial Y^2}.$$

and if

$$\boldsymbol{g} \to \boldsymbol{g} + \nabla a; \qquad \boldsymbol{f} \to \boldsymbol{f} + \nabla c$$

the fields change in general, so the only possibility is

$$\boldsymbol{g} \to \boldsymbol{g}; \qquad \boldsymbol{f} \to \boldsymbol{f}$$

so ***g*** and ***f*** are **physical**.

This can be seen in another way from:

$$\square G = 0; \qquad \square F = 0$$

If

$$G \to G + \frac{\partial a}{\partial Z}$$