ON WHITTAKER'S REPRESENTATION OF THE CLASSICAL ELECTROMAGNETIC FIELD IN VACUO, PART II: POTENTIALS WITHOUT FIELDS

ABSTRACT

Whittaker represents the classical electromagnetic field in vacuo in terms of two longitudinally directed magnetic fluxes, whose amplitudes are F and G. Each obeys the d'Alembert equation in vacuo, an equation which is derived from the Maxwell-Heaviside vacuum equations by using the Lorenz condition. It is shown in this paper that it is possible to represent the solutions of the classical Maxwell-Heaviside vacuum equations in vacuo using potentials derived from F and G. Under well defined conditions, there are potentials but no fields. It is shown that F and G are gauge invariant and physical quantities, and that the four potential representation in vacuo is also physical and gauge invariant. This is contrary to Heaviside's view and in agreement with that of Maxwell and Faraday - that the vector and scalar potentials are physical on the classical level. Therefore it is proven that there is no gauge freedom in electrodynamics represented as a U(1) gauge theory in contemporary language. Since Whittaker's F and G are moduli of magnetic flux vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ in the propagation (longitudinal) direction in vacuo, there must be for every physical situation a magnetic flux density. This is the Evans-Vigier $\boldsymbol{B}^{(3)}$ of the non-Abelian electrodynamics, a Yang-Mills gauge theory with a physical O(3) symmetry internal gauge space. The $\boldsymbol{B}^{(3)}$ field is represented directly in terms of the physical F and G of Whittaker. It is also shown that time-like and longitudinal photons are physical, and that canonical quantization to photons takes place straightforwardly from the d'Alembert equations for F and G, since these are massless Klein-Gordon equations for scalar classical fields.

INTRODUCTION

Sir Edmund Whittaker {1, 2} has represented the electromagnetic field in vacuo in terms of two magnetic fluxes, which are vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ directed longitudinally in the axis of propagation of the electromagnetic field in vacuo as represented by the Maxwell-Heaviside equations. The moduli or magnitudes of $\boldsymbol{f}$ and $\boldsymbol{g}$ are denoted by the scalar magnetic fluxes F and G, from which the contemporary four-potential A^{μ} can be derived; and from which all the electric and magnetic field components present in vacuo can be derived. In this paper, it is proven that F and G are gauge invariant and physical, so that A^{μ} also has this property. Upon canonical quantization of the d'Alembert equations for F and G, which are massless Klein-Gordon equations, the concept of a spin one massless boson is retrieved straightforwardly. This is the photon. The latter is shown to have physical time-like and longitudinal components as well as physical transverse components. Therefore a careful development of Whittaker's work leads to conclusions which are contrary to several of the assertions in the contemporary literature, and contrary to Heaviside's opinion that the vector and scalar potentials are classically unphysical. The gauge freedom of the U(1) representation of the classical electromagnetic field is shown to be a flawed concept. Finally, the existence of longitudinal magnetic fluxes f and g in vacuo implies the existence of a longitudinal magnetic flux density in vacuo under every physical circumstance. This is the Evans-Vigier field $\boldsymbol{B}^{(3)}$, whose proper theoretical development must take place within a different gauge theory from U(1). This is a Yang-Mills theory with a physical internal gauge space of O(3) symmetry. The $\boldsymbol{B}^{(3)}$ field is undefined in Maxwell-Heaviside electrodynamics in the vacuum. Therefore electrodynamics is enriched considerably by a development of Whittaker's work {1, 2} and by a development of gauge theory {3-10} applied to electrodynamics as a direct logical consequence of Whittaker's work.

PROOF OF THE GAUGE INVARIANCE OF THE TRANSVERSE PART OF A^{μ}

The electric and magnetic components of the electromagnetic field are defined by Whittaker under any circumstance, including the classical vacuum, as:

$$\boldsymbol{B} = -\nabla \times (\nabla \times \boldsymbol{g}) + \frac{1}{c}\nabla \times \dot{\boldsymbol{f}} \tag{1}$$

$$\boldsymbol{E} = c\nabla \times (\nabla \times \boldsymbol{f}) + \nabla \times \dot{\boldsymbol{g}} \tag{2}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are directed in the propagation axis, the longitudinal (Z) axis under all circumstances, including the vacuum. Eqn (1) is invariant under:

$$\boldsymbol{g} \rightarrow \boldsymbol{g} + \nabla a; \qquad \nabla \times \boldsymbol{g} \rightarrow \nabla \times \boldsymbol{g} + \nabla b \tag{3}$$

where a and b are arbitrary. This implies that:

$$\nabla \times \boldsymbol{g} \rightarrow \nabla \times \boldsymbol{g} + \nabla \times (\nabla a) = \nabla \times \boldsymbol{g} \tag{4}$$

The transverse part of the vector potential is defined by {11}:

$$\boldsymbol{A}_T = -(\nabla \times \boldsymbol{g}) \tag{5}$$

and so is gauge invariant and physical. The transverse $\boldsymbol{A}_T$ has no gauge freedom, in other words under a U(1) gauge transformation:

$$\boldsymbol{A}_T \rightarrow \boldsymbol{A}_T \tag{6}$$

which means that $\boldsymbol{A}_T$ is physical and observable on the classical level. Several experiments illustrating this conclusion are given by Barrett {12}.

From eqn. (1), the vector potential in general is given by:

$$\boldsymbol{A} = -\nabla \times \boldsymbol{g} + \frac{1}{c}\dot{\boldsymbol{f}} \tag{7}$$

and the Stratton potential {11} by:

$$\boldsymbol{S} = -c\nabla \times \boldsymbol{f} - \dot{\boldsymbol{g}}. \tag{8}$$

Therefore both $\boldsymbol{A}$ and $\boldsymbol{S}$ have transverse and longitudinal components in general. An argument similar to the above shows that the transverse space-like and time-like parts of the Stratton four potential S^{μ} are also physical and measurable quantities, contrary to Heaviside's opinion.

The magnetic flux moduli F and G in vacuo obey the Klein-Gordon equation for a massless particle, i.e. the d'Alembert equation for a classical field:

$$\Box F = \Box G = 0. \tag{9}$$

If the U(1) gauge condition is applied to these equations we obtain:

$$\Box(\nabla a) = \Box(\nabla c) = 0 \tag{10}$$

so a and c are not arbitrary as required by the received opinion about U(1) gauge transformation. Therefore f and g are physical and observable, as demonstrated by Whittaker {1, 2}. Since f and g are physical, A_T is physical and so the transverse part of A^μ is physical.

PHYSICAL NATURE OF THE LONGITUDINAL PART OF A^μ.

The received opinion {13} asserts that there is no physical longitudinal vector potential in vacuo and no concomitant time-like part of the longitudinal A^μ. To disprove this opinion we start from the equations:

$$\boldsymbol{A} = -\nabla \times \boldsymbol{g} + \frac{1}{c}\dot{\boldsymbol{f}} \tag{11}$$

$$\boldsymbol{S} = -c\nabla \times \boldsymbol{f} - \dot{\boldsymbol{g}} \tag{12}$$

and the Lorenz condition used by Whittaker in deriving the d'Alembert equation from the Maxwell-Heaviside equation:

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{13}$$

where ϕ/c is the time-like part of A^μ. Differentiating eqn. (11):

$$\dot{\boldsymbol{A}} = -\nabla \times \dot{\boldsymbol{g}} + \frac{1}{c}\ddot{\boldsymbol{f}} \tag{14}$$

and from eqn. (13):

$$\dot{\phi} = c^2\nabla \cdot (\nabla \times \boldsymbol{g}) - c\nabla \cdot \dot{\boldsymbol{f}}. \tag{15}$$

Therefore

$$\begin{aligned} A^\mu &= (\phi, c\boldsymbol{A}) \\ &= \left(c^2 \int \nabla \cdot (\nabla \times \boldsymbol{g}) dt - c\nabla \cdot \boldsymbol{f}, -c\nabla \times \boldsymbol{g} + \dot{\boldsymbol{f}}\right) \\ &= \left(-c^2 \int \nabla \cdot \boldsymbol{A} dt - c\nabla \cdot \boldsymbol{f}, -c\nabla \times \boldsymbol{g} + \dot{\boldsymbol{f}}\right) \\ &= \left(\phi - c\nabla \cdot \boldsymbol{f}, c\boldsymbol{A} + \dot{\boldsymbol{f}}\right) \\ &= \left(\phi_T, c\boldsymbol{A}_T\right) + \left(\phi_L, c\boldsymbol{A}_L\right). \end{aligned} \tag{16}$$

There exists in vacuum classical electrodynamics a physical transverse:

$$A_T^\mu = (\phi_T, c\boldsymbol{A}_T) \tag{17}$$

and a physical longitudinal:

$$A_L^\mu = (\phi_L, c\boldsymbol{A}_L) = (-c\nabla \cdot \boldsymbol{f}, \dot{\boldsymbol{f}}) \tag{18}$$

because f is physical. Therefore on canonical quantization, there exist physical longitudinal photons and their concomitant physical time-like parts. By definition:

$$A_L^\mu = \left(-c\frac{\partial F}{\partial Z}, \frac{\partial F}{\partial t}\boldsymbol{k} \right). \tag{19}$$

In the special case where the transverse A_T^μ consists of plane waves in the radiation zone in vacuo, there is a direct relation between F and G:

$$F = iG \tag{20}$$

$$A_L^\mu = -\frac{A^{(0)}}{\sqrt{2}}\omega\,(X - iY)e^{i\Phi}(1, \boldsymbol{k}) \tag{21}$$

where the electromagnetic phase is defined by

$$\Phi = \omega t - \kappa Z. \tag{22}$$

Here ω is the angular frequency at time t and κ the wavevector at position Z in the frame (X, Y, Z). In this case:

$$\partial_\mu A_L^\mu = \nabla \bullet \boldsymbol{A}_L + \frac{1}{c^2}\frac{\partial \phi_L}{\partial t} \tag{23}$$

which obeys the Lorenz condition:

$$\partial_\mu A_L^\mu = 0 \tag{24}$$

as required.

The longitudinal vector potential in vacuo is lightlike:

$$A_{L\mu} A_L^\mu = 0 \tag{25}$$

and may be written as:

$$A_L^\mu = (\phi_L, c\phi_L \boldsymbol{k}) \tag{26}$$

where $\boldsymbol{\phi}_L = \phi_L \boldsymbol{k}$. The potential $\boldsymbol{\phi}_L$ obeys the Klein-Gordon equation for the classical field:

$$\Box \phi_L = 0. \tag{27}$$

Canonical quantization proceeds in a well known manner {13} from eqn. (27), giving physical longitudinal and concomitant time-like photons. The Lagrangian for eqn. (27) is obtained from the Noether Theorem and is:

$$\mathcal{L} = \frac{1}{2} g^{\kappa\lambda} (\partial_\kappa \phi_L)(\partial_\lambda \phi_L^*) \tag{28}$$

from which is obtained the energy momentum tensor:

$$\theta_\nu^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_L)} \partial_\nu \phi_L^* - \delta_\nu^\mu \mathcal{L} \tag{29}$$

and the Hamiltonian

$$H_L = \int \theta^{00} d^3x. \tag{30}$$

In S.I. units, the Hamiltonian must be of the positive definite form:

$$H_L = \frac{1}{\mu_0 R^2} \int (\partial_0 \phi_L^* \partial_0 \phi_L + \nabla \phi_L^* \bullet \nabla_L \phi) dV \tag{31}$$

where $R^2 = X^2 + Y^2$. Therefore R can be interpreted as the radius of a light beam. Using the relations:

$$\phi_L = -\frac{A^{(0)}}{\sqrt{2}} \omega (X - iY) e^{i\Phi} \tag{32}$$

$$\partial_0 \phi_L = -\frac{1}{c} i\omega^2 (X - iY) e^{i\Phi} \frac{A^{(0)}}{\sqrt{2}} \tag{33}$$

$$\nabla \phi_L^* = i\frac{A^{(0)}}{\sqrt{2}} \kappa\omega (X + iY) e^{-i\Phi} \tag{34}$$

$$\nabla \phi_L = -i\frac{A^{(0)}}{\sqrt{2}} \kappa\omega (X - iY) e^{i\Phi} \tag{35}$$

the Hamiltonian is the positive definite quantity:

$$H_L = \frac{1}{\mu_0} \int B^{(0)2} dV \tag{36}$$

which in terms of the Evans-Vigier field $\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}$ is:

$$H_L = \frac{1}{\mu_0} \int \boldsymbol{B}^{(3)} \bullet \boldsymbol{B}^{(3)} dV \tag{37}$$

for a given beam radius in the vacuum. This is an improvement on the concept of plane waves, with undefined lateral extent. A physical light beam has a defined lateral extent, i.e. a defined radius R.

Therefore Whittaker's work, when extended to the O(3) level {11} clearly indicates the existence of the $\boldsymbol{B}^{(3)}$ field, which is the magnetic flux density associated with the magnetic fluxes f and g. The only way to define a magnetic flux without a magnetic flux density is when the area R is infinite. Plane waves allow for this possibility, but every beam of light has finite R. Therefore $\boldsymbol{B}^{(3)}$ exists in every physical situation, including the vacuum, as a direct result of Whittaker's work {1, 2}.

The Lagrangian (28) is invariant under the global gauge transformation (type one gauge transform):

$$\phi_L \rightarrow e^{i\Lambda}\phi_L; \qquad \phi_L^* \rightarrow e^{i\Lambda}\phi_L^* \tag{38}$$

but is not invariant in general under the local gauge transform of the second type {13}:

$$\phi_L \rightarrow e^{i\Lambda(x^\kappa)}\phi_L \tag{39}$$

$$\phi_L^* \to e^{-i\Lambda(x^\lambda)}\phi_L^* \tag{40}$$

So the longitudinal potential ϕ_L is physical and there exist physical longitudinal and time-like photons from the well known canonical quantization of eqn. (27). This conclusion is contrary to the received view, which bases canonical quantization on the Lorenz gauge interpreted with the Gupta-Bleuler condition, a procedure which results in the view that only admixtures of time-like and longitudinal photons are physical {13}.

The neglect of Whittaker's F and G is a major fault of twentieth century electrodynamics because longitudinal waves and concomitant time-like waves are always present in the electromagnetic entity. The objects known as "transverse fields" are derivatives of F and G. Under certain circumstances, there can be potentials without fields, but there can be no fields without potentials. So the potentials are more basic. The most basic entities are F and G.

FURTHER PROOF OF THE PHYSICAL NATURE OF f AND g.

The longitudinal four potential is defined by:

$$A_L^\mu = (-c\nabla \cdot f, \dot{f}) \tag{41}$$

and if we try the usual U(1) gauge transform rule {13}:

$$A_L^\mu \to A_L^\mu + \partial^\mu x \tag{42}$$

i.e.

$$A_L \to A_L - \nabla x; \qquad \phi \to \phi + \frac{\partial x}{\partial t} \tag{43}$$

then

$$\dot{f} \to \dot{f} - c\nabla x; \qquad \nabla \cdot f \to \nabla \cdot f - \frac{1}{c}\frac{\partial x}{\partial t} \tag{44}$$

and

$$\int \nabla x \, dt = \frac{1}{c^2}\int \frac{dx}{dt} dZ. \tag{45}$$

Therefore the quantity x is not random as required by the rule (42). Eqn. (45) is satisfied for example by:

$$x = x_0 e^{i(\omega t - \kappa Z)} \tag{46}$$

so that:

$$A_L^\mu \to A_L^\mu + A_L^\mu{}' \tag{47}$$

where

$$A_L^\mu{}' = i\omega x_0 e^{i\Theta}(1, \boldsymbol{k}) \tag{48}$$

$$A_L^\mu{}' \equiv (\phi_L{}', cA_L') \tag{49}$$

$$\phi_L^* \to e^{-i\Lambda(x^\lambda)}\phi_L^* \tag{40}$$

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The longitudinal four potential is defined by:

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and if we try the usual U(1) gauge transform rule {13}:

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$$A_L \to A_L - \nabla x; \qquad \phi \to \phi + \frac{\partial x}{\partial t} \tag{43}$$

then

$$\dot{f} \to \dot{f} - c\nabla x; \qquad \nabla \bullet f \to \nabla \bullet f - \frac{1}{c}\frac{\partial x}{\partial t} \tag{44}$$

and

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Therefore the quantity x is not random as required by the rule (42). Eqn. (45) is satisfied for example by:

$$x = x_0 e^{i(\omega t - \kappa Z)} \tag{46}$$

so that:

$$A_L^\mu \to A_L^\mu + A_L^{\mu\,\prime} \tag{47}$$

where

$$A_L^{\mu\,\prime} = i\omega x_0 e^{i\Theta}(1, \boldsymbol{k}) \tag{48}$$

$$A_L^{\mu\,\prime} \equiv (\phi_L{}', cA_L') \tag{49}$$

and

$$\Box A_L^{\mu\,\prime} = 0; \qquad \Box \phi_L' = 0. \tag{50}$$

The net result is:

$$\phi_L \to \phi_L + iwx_0 e^{i\Phi}; \tag{51}$$

if for example

$$ix_0 \equiv -\frac{A^{(0)}}{\sqrt{2}}(X - iY) \tag{52}$$

then

$$\begin{aligned} \phi_L &\to 2\phi_L \\ F &\to 2F \\ G &\to 2G \end{aligned} \tag{53}$$

and a field such as

$$E_X = \frac{\partial^2 F}{\partial X \partial Z} + \frac{1}{c}\frac{\partial^2 G}{\partial Y \partial t} \tag{54}$$

doubles in magnitude. So the gauge transform (42) does not leave the field unchanged. The only possibility is that A_μ^L is physical and gauge invariant and that $x = 0$, since by definition, the usual view means that fields must be unchanged.

It is concluded that the magnetic fluxes F and G are gauge invariant and physical.

PHYSICAL POTENTIALS WITHOUT U(1) FIELDS

In the special case of transverse plane waves:

$$\boldsymbol{f} = i\boldsymbol{g} \tag{55}$$

and the electric and magnetic fields can be represented by:

$$\boldsymbol{E} = i\nabla \times (\nabla \times \boldsymbol{g}) + \frac{1}{c}\nabla \times \dot{\boldsymbol{g}} \tag{56}$$

$$\boldsymbol{B} = \frac{i}{c}\nabla \times \dot{\boldsymbol{g}} - \nabla \times (\nabla \times \boldsymbol{g}) \tag{57}$$

Under the condition

$$\nabla \times (\nabla \times \boldsymbol{g}) = \frac{i}{c}\frac{\partial}{\partial t}(\nabla \times \boldsymbol{g}) \tag{58}$$

then all components of E and B vanish. This condition is satisfied by the equation:

$$\nabla \times \boldsymbol{A}_T = \frac{i}{c}\frac{\partial \boldsymbol{A}_T}{\partial t} \tag{59}$$

whose solution is

$$\boldsymbol{A}_T = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)}. \tag{60}$$

This gives $\boldsymbol{E} = \boldsymbol{B} = 0$;

$$\boldsymbol{A}_L = -\kappa\frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{-i(\omega t - \kappa Z)}\boldsymbol{k} \tag{61}$$

$$\phi_L = -\omega\frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{-i(\omega t - \kappa Z)} \tag{62}$$

$$G = \frac{F}{i} = \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{-i(\omega t - \kappa Z)}. \tag{63}$$

So there can be both transverse and longitudinal physical potentials, and their concomitant time-like parts, while there are no U(1) fields present. On the O(3) level it has been shown {11} that

$$\begin{aligned}\boldsymbol{B}^{(3)} &= -i\frac{\kappa}{A^{(0)}}\left(\frac{dG^*}{dY}\frac{dG}{dX} - \frac{dG}{dY}\frac{dG^*}{dX}\right)\boldsymbol{k} \\ &= -i\frac{\kappa}{A^{(0)}}\left(\frac{dF^*}{dY}\frac{dF}{dX} - \frac{dF}{dY}\frac{dF^*}{dX}\right)\boldsymbol{k}\end{aligned} \tag{64}$$

so $\boldsymbol{B}^{(3)}$ is the **only** field present under condition (58), being the magnetic flux density associated with $\boldsymbol{f}$ and $\boldsymbol{g}$ in the vacuum for finite R.

LONGITUDINAL STANDING WAVES IN THE VACUUM.

If we choose the solution:

$$G = \frac{A^{(0)}}{\sqrt{2}}(X - iY)(e^{i(\omega t - \kappa Z)} + e^{-i(\omega t + \kappa Z)}) \tag{65}$$

to the d'Alembert equation: $\qquad \square G = 0 \qquad$ (66)

the real part of eqn. (65) is:

$$\mathrm{Re}(G) = \frac{2}{\sqrt{2}}A^{(0)}(X\cos\omega t\cos\kappa Z + Y\cos\omega t\sin\kappa Z) \tag{67}$$

which is a standing wave {14} in the vacuum, along the propagation axis. Such waves do not exist in the received view of electrodynamics {13}, but were pointed out by Barrett {12}. The overall result is:

$$\boldsymbol{g} = 2\frac{A^{(0)}}{\sqrt{2}}(X\cos\omega t\cos\kappa Z + Y\cos\omega t\sin\kappa Z)\boldsymbol{k} \tag{68}$$

which is a longitudinal standing wave which is a solution to the vibrating string problem. Since Whittaker's work originates in the Maxwell-Heaviside equations, these also give such waves, contrary to the contemporary opinion {13}. The idea that electromagnetic waves must be transverse is clearly erroneous. For example there can be interference between physical potentials when no fields are present. There can be interference between transverse and longitudinal physical potentials and their time-like parts. This has important consequences for physics and technology.

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