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REPLY TO COMAY'S "RELATIVITY VERSUS THE LONGITUDINAL MAGNETIC FIELD OF THE PHOTON"

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Poincaré group electrodynamics is $\hat{C}\hat{P}\hat{T}$ conserving and Lorentz covariant under all conditions by definition. Examples are given of these properties. Comay's comment is incorrect: any $\hat{C}\hat{P}\hat{T}$ conserving field theory that is Lorentz covariant is consistent with special relativity, whose underlying group is the Poincaré group.

Key words: Poincaré group electrodynamics, special relativity, $\boldsymbol{B}^{(3)}$ field.

1. POINCARÉ GROUP ELECTRODYNAMICS

Electrodynamics must be developed within the underlying symmetry group: the Poincaré group for special relativity; the Einstein group for general relativity. This was the view of both Einstein and Pauli [2], figures on the opposite sides of the Copenhagen-Realist debates in quantum mechanics; probably the most profound dialogue in twentieth century natural philosophy. It has recently been argued in this

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journal and elsewhere that there exist novel relations between electromagnetic field components in the vacuum, e.g.,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \quad \text{et cyclicum}, \tag{1}$$

where $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, $\boldsymbol{B}^{(3)}$ are magnetic flux density components in the basis ((1), (2), (3)) [3-11], and in vacuum

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{e}^{(3)}, \tag{2}$$

where $\boldsymbol{e}^{(3)} = \boldsymbol{k}$ is a unit vector in the propagation axis Z. Here $B^{(0)}$ is the scalar magnitude of $\boldsymbol{B}^{(3)}$, which is directed in the axis of propagation of the beam if $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ is a plane wave. More generally, $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are relations between space components that are proportional to rotation generators of the Poincaré group, a result which is independent of any equation of motion and depends only on the structure of spacetime. The received view [12,13] however, asserts that the group of electrodynamics is U(1), and in consequence there can be no longitudinally (i.e., Z) directed fields in vacuo. Dirac [14] has argued forcibly that the U(1) sector symmetry cannot accommodate the Coulomb law, formulated in 1785, and it is now clear that it cannot accommodate the Evans-Vigier field $\boldsymbol{B}^{(3)}$, first inferred in 1992 and observable empirically in magneto-optics [15–17].

Comay [1] argues that Eq. (1) is inconsistent with special relativity. We first examine this claim in the vacuum (Sec. 2), then in field-matter equilibrium (Sec. 3).

2. COVARIANCE IN VACUO OF THE B CYCLICS

Let us accept, for the sake of argument, the received view that the photon is massless; that the classical electromagnetic field's signal velocity in vacuo is identically c, the speed of light. Accordingly, the signal velocity in *all* Lorentz frames is c, and the notion of transformation from one frame to another is meaningless in this condition. There is no rest frame for the massless photon. Equations (1) are written for the vacuum electromagnetic field, and it is meaningless to apply the Lorentz transform to them in vacuo. Comay's equations (1) to (9) (and his conclusions based thereon) are not applicable to the vacuum cyclic relations (1). The latter form a part of the Lie algebra [18] of the Poincaré group of special relativity and are therefore

Lorentz covariant and therefore $\hat{C}\hat{P}\hat{T}$ conserving [19]. Equation (1) demonstrates the algebra of rotation generators of the Poincaré group [3–7]. On the classical level in which Eq. (1) is written, there is conservation of $\hat{C}\hat{P}\hat{T}$, $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, $\hat{P}\hat{T}$, $\hat{C}$, $\hat{P}$ and $\hat{T}$. The corollary of the $\hat{C}\hat{P}\hat{T}$ theorem then asserts that since Eq. (1) is $\hat{C}\hat{P}\hat{T}$ conserving, it is Lorentz covariant, and therefore consistent with special relativity. This is unsurprising because Eq. (1) represents an algebra of rotation generators, a sub-group of the Poincaré group underlying special relativity. This argument is alone sufficient to show the erroneous nature of Comay's claim that Eq. (1) is inconsistent with special relativity. In the remaining part of this reply we will demonstrate why Comay reached this erroneous conclusion.

3. COVARIANCE OF THE B CYCLICS FOR FIELD-MATTER INTERACTION

Comay's equations (1) to (9) apply the standard linear Lorentz transformation to the standard antisymmetric field tensor (his Eq. (3)). Unsurprisingly, he obtains the equations for the transformation of field components from one frame K to another K' moving at v with respect to K. In order for this orthogonal transform to be physically meaningful there must be two distinct, well-defined, Lorentz frames K and K'. This is possible if and only if there is field-matter interaction, otherwise the field is in vacuo as in Sec. 2, and there is no rest frame, i.e., there is only one well-defined Lorentz frame. Comay has therefore worked out the effect of Lorentz transformation on the B cyclics in field-matter equilibrium.

The results of this procedure do not apply, however, to our Eq. (1), which defines a relation between vacuum field components. If we replace $\boldsymbol{B}$ by the magnetic field strength $\boldsymbol{H}$; and the electric field strength $\boldsymbol{E}$ by the displacement $\boldsymbol{D}$, i.e., introduce field-matter interaction, the Lorentz transform may be applied correctly to each field component. This is demonstrated in the following section, in which we correctly recover Eq. (1) as v goes to c. It is well known that in the presence of field-matter interaction (e.g. in a waveguide) there are longitudinal field components. In this situation, the Poynting vector represents the linear momentum of the field and the $\boldsymbol{B}^{(3)}$ vector is proportional to the field angular momentum about Z. The relative orientation of these vectors before, during and after field matter interaction is determined by conservation of charge; total energy (kinetic and potential) and total momentum (linear and angular). There is no reason to assume that the Poynting vector must always be parallel to $\boldsymbol{B}^{(3)}$ in field-matter equilibrium. Comay shows that the Lorentz transform during field-matter equilibrium has different effects on the vacuum Poynting vector and the vacuum $\boldsymbol{B}^{(3)}$. For plane waves in vacuo, $\boldsymbol{B}^{(3)}$ is parallel to the Poynting vector in vacuo for plane waves [3–11].

4. LORENTZ TRANSFORM OF THE B CYCLICS IN FIELD-MATTER EQUILIBRIUM

Consider an electromagnetic field in a frame K', propagating at c in vacuo. In any other frame K it must also propagate at c, and so in the quantum theory, the photon has no rest frame. This is a counter-intuitive feature of special relativity, but follows from the first principle, which asserts that c is a universal, invariant, constant. This feature must also be compatible with the result of the Lorentz transform applied to the field strength tensor $F_{\mu\nu}$, a process which produces

$$\boldsymbol{B}^{(1)\prime} = \gamma\left(\boldsymbol{B}^{(1)} - \frac{\boldsymbol{v}\times\boldsymbol{E}^{(1)}}{c^2}\right),$$

$$\boldsymbol{B}^{(2)\prime} = \gamma\left(\boldsymbol{B}^{(2)} - \frac{\boldsymbol{v}\times\boldsymbol{E}^{(2)}}{c^2}\right), \tag{3}$$

$$\boldsymbol{B}^{(3)\prime} = \boldsymbol{B}^{(3)}, \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

with v as the constant speed of K' in the $Z = (3)$ axis with respect to K. For a plane wave, Eqs. (3) produce the result

$$B^{(0)\prime} = \left(\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}\right)^{1/2} B^{(0)}, \tag{4}$$

where we have used $E^{(0)} = cB^{(0)}$, a relation which holds only at the speed of light in vacuo. The phase of the plane wave is a Lorentz invariant [5], and so the amplitude of the plane wave would gradually diminish to zero in frame K' if v were allowed to approach c. If the hypothetical rest frame K' could be defined in this way and if the plane wave continued to propagate at c with respect to K', there would be no radiation, a reduction to absurdity. The root of this paradox is found in the fact that the plane wave is already propagating at c in frame K', and in consequence frame K' cannot move with respect to K at any velocity, (other than 0 or c), without changing the value of c measured with respect to frame K, and thus violating the first principle of relativity, that c is a *Lorentz-frame invariant* universal constant. The only possible solution of the paradox is $v = 0$ in Eq. (4), leading to $B^{(0)\prime} = B^{(0)}$. This means that the plane wave propagates at c in any frame, and has no rest frame. It is frequently stated that the massless photon has no rest frame.

There follows the important result that the B cyclic theorem, when applied to a plane wave propagating at c in vacuo, is also a Lorentz invariant construct. It remains the same in any Lorentz frame of reference, and is therefore an invariant feature of the Poincaré group of special relativity, forming an invariant Lie algebra. It is therefore automatically Lorentz covariant in vacuo, and a $\hat{C}\hat{P}\hat{T}$ conserving field equation. It is concluded that in standard special relativity, $\boldsymbol{B}^{(3)}$ is a bona fide magnetic flux density. If the converse were true the B cyclic theorem would violate the $\hat{C}\hat{P}\hat{T}$ theorem, probably the most fundamental theorem in physics. Note that this conclusion follows directly from application of the Lorentz transform to the field strength tensor of electromagnetic radiation propagating at c in vacuo. Similarly [5–7], the B cyclic theorem conserves the other six symmetry combinations, i.e., $\hat{C}$, $\hat{P}$, $\hat{T}$, $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, $\hat{P}\hat{T}$, and at the classical level violates no discrete symmetry in physics. The $\boldsymbol{B}^{(3)}$ field in vacuo is phase free and Lorentz invariant, as indicated directly by Eq. (3).

In the received view the field $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ conserves the seven symmetry combinations of physics, but it is asserted that the cross product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ does not produce another field. At this point, the received view violates the $\hat{C}\hat{P}\hat{T}$ theorem, which, as we have just seen, asserts that $\boldsymbol{B}^{(3)}$ must be a field.

If for some reason the electromagnetic radiation does not propagate at c in frame K, e.g. if the medium of propagation in frame K is in general magnetizable and polarizable, then $\boldsymbol{B}$ must be replaced by the magnetic field strength $\boldsymbol{H}$, and $\boldsymbol{E}$ by the displacement $\boldsymbol{D}$. The Lorentz transform (3) retains its form only if there is no magnetization ($\boldsymbol{M}$) or polarization ($\boldsymbol{P}$), so that

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0}, \qquad \boldsymbol{D} = \epsilon_0 \boldsymbol{E}, \tag{5}$$

and in frame K' there exists a relation

$$\left(B_1' + E_2'\right)^{(1)} \left(B_2' + E_1'\right)^{(2)} = \frac{i}{2} B^{(0)} B_3^{(3)*} = B_1^{(1)} B_2^{(2)} , \tag{6}$$

and so forth in $c = 1$ units, where [12]:

$$B_1' = \gamma B_1 , \quad B_2' = \gamma B_2 , \quad B_3' = B_3 = B^{(0)} , \tag{7}$$

$$E_1' = -\beta \gamma B_2 , \quad E_2' = \beta \gamma B_1 , \quad E_3' = 0 . \tag{8}$$

In Jackson's notation [12], Eq. (6) is a Z axis Lorentz transform to frame K' of the K frame cyclics

$$\epsilon_{ijk} B_j^{(1)} B_k^{(2)} = i B^{(0)} B_i^{(3)*} . \tag{9}$$

Equation 6 has the same form as, and is therefore *automatically covariant* with Eq. (9) and the B cyclics are Lorentz covariant. This is hardly surprising because the cyclics transform as spin angular momentum. The complete theory of relativistic angular momentum requires the Pauli-Lubanski vector formalism as is well known. This is the correct way to carry out the Lorentz transform of the B cyclics, and the result is not surprising because $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = i B^{(0)} \boldsymbol{B}^{(3)*}$ is a physical observable [2–5].

For electromagnetic radiation propagating at c in the vacuum, there is no rest frame, and the Lorentz transform in this condition is compatible with the first principle of special relativity if and only if $v = 0$. This is also true for Lorentz transform in axes other than Z. These conclusions can be arrived at in several ways, one of which is as follows. Consider the Lorentz transformation matrices of a four-vector in the X, Y, and Z axes. For a single Lorentz transform from frame K to K' these are, in the conventional notation $\beta = v/c$; $\gamma = (1 - v^2/c^2)^{-1/2}$

$$L_X := \begin{bmatrix} \gamma & 0 & 0 & i\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\beta & 0 & 0 & \gamma \end{bmatrix}, \quad L_Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & i\gamma\beta \\ 0 & 0 & 1 & 0 \\ 0 & -i\gamma\beta & 0 & \gamma \end{bmatrix},$$
$$L_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix}. \quad (10)$$

If we now consider a double boost, one from X to X', followed by one from X' to X'', the resultant matrix is

$$X_1 X_2 = \begin{bmatrix} \gamma_1\gamma_2(1 + \beta_1\beta_2) & 0 & 0 & i\gamma_1\gamma_2(\beta_1 + \beta_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma_1\gamma_2(\beta_1 + \beta_2) & 0 & 0 & \gamma_1\gamma_2(1 + \beta_1\beta_2) \end{bmatrix}, \quad (11)$$

and this becomes the matrix L_X for a single boost from K to K'' if

$$\gamma := \gamma_1\gamma_2(1 + \beta_1\beta_2), \qquad \beta\gamma := \gamma_1\gamma_2(\beta_1 + \beta_2). \quad (12)$$

These equations give the relativistic velocity addition rule [12]

$$v = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}, \quad (13)$$

where v_1 is the speed of K' relative to K; and v_2 is the speed of K'' relative to K'. Equation (13) gives the result that if $v_1 = v_2 = c$, then $v = c$; and if $v_1 = v_2 = c$, then $v = c$. So if K' moves at c relative to K, K' also moves at c relative to both K and K'. If however, K' moves at c relative to K, and K'' moves at a velocity $v_2 \ll c$ relative to K', then K'' from Eq. (13) must move at $\sim c + v_2$

relative to K. This violates the principle that c is the same in every frame of reference. A field $\boldsymbol{B} = \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)}$ which propagates at c in one frame propagates at c in every other frame and cyclic relations between these field are Lorentz invariant in the vacuum.

If we consider consecutive boosts in orthogonal directions (e.g., a boost in Z followed by one in X), the relevant matrix product is

$$X_1 Z_2 := \begin{bmatrix} \gamma_1 & 0 & 0 & i\gamma_1\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ --i\gamma_1\beta_1 & 0 & 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_2 & i\gamma_2\beta_2 \\ 0 & 0 & -i\gamma_2\beta_2 & \gamma_2 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_1 & 0 & \gamma_1\gamma_2\beta_1\beta_2 & i\gamma_1\gamma_2\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_2 & i\gamma_2\beta_2 \\ -i\gamma_1\beta_1 & 0 & -i\gamma_1\gamma_2\beta_2 & \gamma_1\gamma_2 \end{bmatrix}, \tag{14}$$

and this does not have the structure of an individual boost in X or in Z, the reason being that a commutator of boost matrices is a *rotation* generator in spacetime [5]. For example, the commutator of boosts represented by $XZ - ZX$ is an off diagonal 4 x 4 matrix representing a rotation about Y

$$XZ - ZX = \begin{bmatrix} 0 & 0 & \gamma^2\beta^2 & 0 \\ 0 & 0 & 0 & 0 \\ -\gamma^2\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

Comparing the (0,3) elements of matrices (14) and (15) we see that the extra (0,3) element in Eq. (14) is caused by rotation, and

$$\beta = (\beta_1\beta_2)^{1/2}, \qquad v = (v_1 v_2)^{1/2}, \tag{16}$$

is a relativistic velocity multiplication rule equivalent to the addition rule given by Eq. (13). If $v_1 = v_2 = c$ then again, $v = c$ from Eq. (16), and if $v_1 = v_2 = 0$, $v = 0$. If however, we try $v_1 = c$ and $v_2 < c$, the resultant v becomes again different from c. This violates the principle of special relativity that c is the same in every Lorentz frame of reference. Again, we arrive at the conclusion that a field moving at c in one Lorentz frame moves at c in all Lorentz frames. Therefore the B cyclics in vacuo form a Lorentz invariant field theory. This is an important result because it follows that the B cyclics must form a $\hat{C}\hat{P}\hat{T}$ conserving theory of fields. Therefore $\boldsymbol{B}^{(3)}$ is a physical magnetic flux density in the vacuum. In matter, the H cyclics are Lorentz covariant, where $\boldsymbol{H}$ is magnetic field strength, but no longer invariant. Covariance is sufficient to show that the H cyclics also form a $\hat{C}\hat{P}\hat{T}$ conserving field theory or structure, and a theory which is invariant is also covariant.

Equation (1) shows that there exists a longitudinal current density $\boldsymbol{j}^{(3)}$ in the vacuum, related to $\boldsymbol{B}^{(3)}$ through

$$\boldsymbol{j}^{(3)} = \frac{1}{\mu_0}\frac{\omega}{c}\boldsymbol{B}^{(3)} , \tag{17}$$

and this current density can be expressed as part of the j cyclics in the vacuum

$$\boldsymbol{j}^{(1)} \times \boldsymbol{j}^{(2)} = i j^{(0)} \boldsymbol{j}^{(3)*} , \text{ et cyclicum,} \tag{18}$$

where $\boldsymbol{j}^{(1)}$ and $\boldsymbol{j}^{(2)}$ are transverse current densities

$$\boldsymbol{j}^{(1)} = \boldsymbol{j}^{(2)*} = \frac{j^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi} . \tag{19}$$

Therefore the magnitude $B^{(0)}$ is proportional to the magnitude of the current density

$$B^{(0)} = \mu_0 \frac{c}{\omega} j^{(0)} = \mu_0 c \frac{e}{Ar} , \tag{20}$$

and if the photon area is $1/\kappa^2$, we recover, for every photon

$$j^{(0)} = \omega \frac{e}{Ar} = \omega e \kappa^2 = c\rho^{(0)} , \tag{21}$$

where $\rho^{(0)}$ is the charge density of the photon. This type of radiated vacuum current needs for its existence a finite ω, and a finite, constant tangential velocity, defined by $v = \omega r$ where r is a radius. In the vacuum, the radius is $1/\kappa$ and the constant tangential velocity is c. The forward velocity is also c, and the resultant velocity is c as we have shown already using the relativistic multiplicative and addition rules.

5. DETAILED REPLIES TO COMAY

Comay's detailed comments are corrected as follows. His points A to E of his introduction are applied to our Eq. (1), which defines $\boldsymbol{B}^{(3)}$ to be irrotational in the vacuum. Therefore,

$$\nabla \times \boldsymbol{B}^{(3)} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{B}^{(3)} = 0 . \tag{22}$$

The energy density associated with $\boldsymbol{B}^{(3)}$ is defined through the rotational Poynting theorem in Ref. 5. The $\boldsymbol{B}^{(3)}$ component of the field propagates at c in vacuo. The real part of the component $\boldsymbol{E}^{(3)}$ is zero, but formally,

$$\nabla \times \boldsymbol{E}^{(3)} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{E}^{(3)} = 0 . \tag{23}$$

The continuity equations for $\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ in vacuo are [3–7]

$$\frac{1}{c}\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} + \frac{\partial \boldsymbol{B}^{(3)}}{\partial Z} = 0 , \tag{24}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{E}^{(3)}}{\partial t} + \frac{\partial \boldsymbol{E}^{(3)}}{\partial Z} = 0 . \tag{25}$$

Therefore,

$$\nabla \times \boldsymbol{B}^{(3)} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(3)}}{\partial t} = \mathbf{0}, \tag{26}$$

$$\nabla \times \boldsymbol{E}^{(3)} = -\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \mathbf{0} \tag{27}$$

are vacuum *Maxwell equations* for $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$. Since $\nabla \times \boldsymbol{E}^{(3)}$ is irrotational it does not induce $-\partial \boldsymbol{B}^{(3)}/\partial t$

through the vacuum Faraday law. Similarly, $\nabla \times \boldsymbol{B}^{(3)}$ does not induce $(1/c^2)\partial \boldsymbol{E}^{(3)}/\partial t$ through the vacuum Ampère-Maxwell law. These precise results correct Comay's claims A to E.

Comay's Eqs. (1) to (9) apply the standard Lorentz transform but assume that there are two well defined frames, K and K'. In Sec. 4 we have given a correct derivation of the B cyclics in vacuo using Comay's own methods. Comay's conclusions are corrected as follows:

(1) The fact that the Poynting vector (field linear momentum) after Lorentz transform from K to K' is no longer parallel to $\boldsymbol{B}^{(3)}$ (field angular momentum in Z) after Lorentz transform from K to K' is not in conflict with Eq. (1) of this reply. It is, rather, the result of Comay's assumption that there are two distinct Lorentz frames. In other words, this finding is the result of field matter interaction and standard special relativity.

(2) The appearance of the uniform electric field is, again, a normal outcome of the ordinary Lorentz transform from K to K' in a state of field- matter equilibrium. Comay's result means that if $\boldsymbol{B}^{(3)}$ is in $\boldsymbol{k}$ and a Lorentz boost occurs in $\boldsymbol{i}$, an electric field appears in $\boldsymbol{j}$. This is a Biot-Savart law in the presence of charged matter.

In a revised version of his comment, Comay makes a few additional remarks which are rebutted as follows:

(i) The magnitude of $\boldsymbol{B}^{(3)}$ is a scalar, $B^{(0)}$, which is not in general a Lorentz invariant, as can be seen from Eq. (4) of this reply. Therefore Comay has confused $B^{(0)}$ with a Lorentz invariant of the electromagnetic field. If there were nothing that could be described as $B^{(0)}$, then presumably $B^{(0)}$ would be identically zero and there would be no field. The rest of Comay's comment here is sequentially erroneous.

(ii) The B cyclic relations are angular momentum relations, and the fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, and $\boldsymbol{B}^{(3)}$ are related by angular momentum commutator theory, whose energy properties are well worked out in quantum mechanics. Therefore Comay's argument here is a purely subjective assertion. The equivalent of the Poynting Theorem for angular momentum is given in the third volume of *The Enigmatic Photon* [5]. Comay makes a basic error in asserting that the Pauli-Lubanski four-vector is not Lorentz covariant. It is well known that the second Casimir invariant of the Poincaré group is defined in terms of this vector, which is clearly derived from the antisymmetric four-tensor by multiplication by the fully antisymmetric four-tensor and the generator of the space-time translation. This is why the Pauli-Lubanski four-vector is Lorentz covariant, and this is why the classical four-vector $A^{(0)}(B^{(0)}, \boldsymbol{B}^{(3)})$ is Lorentz covariant.

(iii) It is by now well established that the B cyclics are Lorentz covariant and fully compatible with special relativity and the Maxwell equations. Thus $B^{(0)}$ is the scalar component of the Pauli-Lubanski vector $A^{(0)}(B^{(0)}, \boldsymbol{B}^{(3)})$ and the B cyclics are defined in a complex, circular basis ((1), (2), (3)). If $\boldsymbol{B}^{(3)}$ is defined in the Z axis, then the real parts of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are in the X and Y axes. Due to the Lorentz invariance of the electromagnetic phase, the B cyclics behave under Lorentz transformation as

$$B^{(0)2}\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = B^{(0)2}(i\boldsymbol{e}^{(3)*}), \tag{28}$$

where $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$ and $\boldsymbol{e}^{(3)}$ are the unit vectors of the basis ((1), (2), (3)). Since $B^{(0)2}$ is a common factor on both sides of Eq. (28), consideration of Lorentz transformation comes down to the behavior of the frame *itself* under Lorentz transformation, i.e., of

$$\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)*}, \tag{29}$$

and this is covariant by definition. If frames of reference were not covariant, no laws of physics could be Lorentz covariant.

The counter-example proposed by Comay is based on the ordinary Lorentz transform of the field tensor, in which $\boldsymbol{B}^{(3)}$ is automatically aligned in Z. This means that $\boldsymbol{B}^{(3)\prime} = \boldsymbol{B}^{(3)}$ after Lorentz transformation. We have shown that this result is compatible with the covariance of the B cyclics, because after Lorentz transformation, $\boldsymbol{B}^{(1)\prime} \times \boldsymbol{B}^{(2)\prime}$ is proportional to $\boldsymbol{B}^{(3)\prime}$ defined as $B^{(0)\prime}\boldsymbol{e}^{(3)}$. Therefore Comay's counter-example is incorrect, and he does not understand that an O(3) gauge is necessary to define $\boldsymbol{B}^{(3)}$, i.e., $\boldsymbol{B}^{(3)*} := -i(e/\hbar)\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, as in standard O(3) gauge theory.

In a note added to his original paper, Comay makes *further* erroneous assertions which suggest that his claims are monotonously subjective: never moderated in response to reasoned reply or available scholarship. Equations (6) and (9) are covariant, this fact is alone sufficient to refute his claim that the B cyclic theorem is incompatible with special relativity. The theorem is covariant in the same way, tautologically, as spin angular momentum. The complete Pauli-Lubanski theory has been developed in Ref. 11b and expanded in Ref. 3. This shows that $(B^{(0)}, \mathbf{B}^{(3)})$ *multiplied by a normalized linear momentum* [3,11b] forms a classical Pauli-Lubanski pseudo four vector, which transforms under parity as such. Had Comay read Refs. 3 and 11b, this should have been as clear to him as every other scholar. Therefore his new claim that $(B^{(0)}, \mathbf{B}^{(3)})$ is neither a four-vector nor a pseudo four-vector is as trivially in error as the rest of his work. The claim seems to be based on an unwillingness to read or understand the literature and a basic unfamiliarity with relativistic angular momentum theory.

The correct commutation relations for the Pauli-Lubanski pseudo four-vector is as follows:

$$[W^{\mu}, W^{\nu}] = -i\epsilon^{\mu\nu\sigma\rho}A_{\sigma}W_{\rho}, \tag{30}$$

where A_{σ} is a *fully covariant* potential four-vector [6]. The correct parity law for W^{μ} is, on the classical level,

$$\hat{\mathbf{P}}(W^{0}, \mathbf{W}) = (-W^{0}, \mathbf{W}), \tag{31}$$

where

$$W^{0} := -B^{1}A_{1} - B^{2}A_{2} - B^{3}A_{3}, \tag{32}$$

is the scalar helicity, a pseudo-scalar quantity negative under parity. For plane waves,

$$W^{0} = -B^{3}A_{3} = B_{Z}A_{Z}, \tag{33}$$

in the required non-Abelian gauge theory [6].

Doctor Comay has chosen to ignore the work of several authors who have recently deduced *independently* [6–9] the existence of longitudinal modes of vacuum electromagnetic radiation; in my opinion this demonstrates a troubling contempt for contemporary scholarship. Despite the rapid emergence of several corroborating theories, Comay has refused repeatedly to respond to or even to refer to this new work. His critical manuscript remains the same as originally submitted, about eighteen months ago at the time of writing (March 18th 1997). It remains a trivially erroneous mocking of the serious minded and legitimate scientific progress currently being made on several fronts. My replies to the critical papers cited by Comay have also been available for some months, for example my reply to the experiment by Rikken: but again we find in Comay's claims no reasoned reference to these replies, merely a litany of criticism already shown to be deeply flawed.

Our conclusion is vividly illustrated in at least two ways: through the link between standard non-Abelian electrodynamic gauge theory [18] and the B cyclic theorem; and secondly through the invariance of the latter under a Z axis Lorentz boost. It has already been shown repeatedly [4] that the general electrodynamical gauge field defined in Ryder's Eq. (3.166) reduces to the B cyclic theorem from a consideration of the element

$$B_3 = G_{12} = -ig[A_1, A_2^*] , \tag{34}$$

remembering that A_μ is a complex operator. The isospace indices a, b, and c used by Ryder are, respectively, (1), (2) and (3) of the B cyclic theorem. Therefore the latter is an example of non-Abelian gauge theory which is not only standard, but highly developed. For example, the general non-Abelian field tensor $G_{\mu\nu}$ is gauge invariant and the general A_μ transforms as in Ryder's Eq. (3.162). The non-Abelian homogeneous Maxwell equations become a Jacobi identity of the underlying symmetry group, Ryder's Eq. (3.173). The quantized non-Abelian theory is fully renormalizable to all orders in the absence and presence of spontaneous symmetry breaking. The relevant Lagrangian is defined in Ryder's Eq. (7.55), the coupling to spin half matter fields in his Eq. (7.59). The latter gives rise to the well known Faddeev-Popov ghosts in the Feynman rules; the self energy operator; vertex functions and Slavnov-Taylor identities, the non-Abelian equivalents of the Ward identities. The $\boldsymbol{B}^{(3)}$ field shows the need to apply this well developed non-Abelian theory in non-linear optics. It is utterly erroneous to claim, as does Comay, that this well known standard theory is not covariant. The proper way to develop the Maxwell equations is well known (Ryder's Eq. (3.173)) and so forth. Furthermore, the non-Abelian theory conserves $\hat{C}\hat{P}\hat{T}$, $\hat{P}$, and $\hat{C}$, showing once again that Comay is in error. This author has no access to Comay's *Physica B* reference, but it is anticipated that the claims made there are again erroneous, for the same reasons as presented in this reply. The B cyclic theorem is a standard, covariant construct of non-Abelian gauge theory, and is, in fact, contained within, and is a special case of, Ryder's Eq. (3.166).

Finally, adopting Jackson's [12] notation and standard Z boost transformation, we obtain Eq. (6), showing that the B cyclics are *invariant* under the Z axis boost. They are therefore automatically covariant. This result is of course consistent, once more, with standard non-Abelian gauge theory. Comay's work can be seen to be a catalogue of elementary errors and half-baked commentary. The field $\boldsymbol{B}^{(3)}$ happens to be a solution of the Abelian Maxwell equations (because it is phase free), but is defined through the B cyclic theorem, meaning that the non-Abelian Maxwell equations must be used. This is discussed fully in Ref. 3, a fact which has been pointed out to Dr. Comay, but which he has chosen to ignore in his Comment and in several other recent works.

6. SUMMARY

Comay's comment is based on his use of two distinct Lorentz frames K and K', whereas our Eq. (1), the object of his criticism, is written in the vacuum in which there is no rest frame. Comay makes a series of sequential claims which are corrected in our reply.

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