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Chapter 1. The Conjugate Product in Dirac's Electron Theory

In this third volume some of the major practical advantages of the Evans-Vigier field, $\boldsymbol{B}^{(3)}$ [1—15] are described through the concept of fermion resonance, familiar in contemporary NMR and ESR. The key to much of our development is the empirical evidence for the existence of the optical conjugate product, $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, [16—21], where $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ is the complex vector potential in vacuo of electromagnetic radiation. It may be shown using Dirac's electron theory [22—25] that the classical electromagnetic field *always* interacts with the half integral fermion spin to generate the energy

$$En_{int} = -\frac{e}{m}\left(\frac{\hbar}{2}\boldsymbol{\sigma}^3\right)\cdot\boldsymbol{B}^{(3)*}, \tag{1}$$

where

$$\boldsymbol{B}^{(3)*} = \boldsymbol{B}^{(3)} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{2}$$

is the Evans-Vigier field developed in Vol. 1 and 2 [1,2] using the circular basis (1), (2), (3)) of those volumes. This result is a new general theorem of field-fermion interaction when the field is classical, and is equally valid in quantum electrodynamics, where there are only electrons and photons. In Eq. (1) and (2), which use S.I. units, e is the charge of the fermion, m its mass, $\hbar$ is the Dirac constant, $\boldsymbol{\sigma}^{(3)}$ the third Pauli spinor,

$$\boldsymbol{\sigma}^{(3)} = \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

and $\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(3)*}$ is real and physical, a magnetic flux density [1,2] which is free of electromagnetic phase but which, nevertheless, propagates in vacuo at the speed of light,

c. Clearly, the $\boldsymbol{B}^{(3)}$ field can always be detected experimentally through its interaction with half integral fermion spin, and one of the most sensitive detection methods is resonance, akin to NMR and ESR. The $\boldsymbol{B}^{(3)}$ field is directly proportional in Eq. (2) to the optical conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and is gauge independent and real, because the conjugate product itself is pure imaginary [1,2]. Equations (1) and (2) describe the transfer of energy and angular momentum from the electromagnetic field, considered classically, to the quantized fermion, for example an electron, proton or neutron. Since $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ clearly has magnetic symmetry from Eq. (2), this is a process closely akin to the original description by Dirac [26] of the motion of the electron in a classical, static magnetic field, leading to his famous explanation of the anomalous Zeeman effect and the Stern-Gerlach experiment. It is well known that this, the original relativistic quantum field theory, leads to NMR and ESR, which are essentially absorption spectroscopies [27—31] in the radio frequency and microwave regions of the spectrum.

Therefore in this first chapter we describe the origin of the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in the first principles of relativistic quantum field theory, using the minimal prescription [1,2,22—25] to describe the conservation of energy and linear momentum between fermion and field. It is well known that the minimal prescription is a consequence of special relativity through local gauge invariance and charge-current conservation. In this picture, the elementary charge e is a conserved quantity which scales (or determines the magnitude and sign of) the interaction of a gauge field, the electromagnetic field, with matter. Linear momentum is generated through the product $e\boldsymbol{A}$, and potential energy through the product ecA_0, where A_0 is the scalar part of a four-vector of which $\boldsymbol{A}$ is the space component [1,2]. Conservation laws in field fermion interaction are therefore built up from a consideration not of the electric and magnetic field components of the electromagnetic wave, but from its potential four-vector A_μ. The Aharonov-Bohm effects [32—37] confirm that A_μ is physically meaningful in contemporary orthodoxy, which replaces the original nineteenth century perception of A_μ as being a mathematical intermediary leading to the d'Alembert wave equation [22—25]. It has been known since the work of Weyl [38] and others [39] that A_μ is physically meaningful in quantum mechanics and therefore in relativistic quantum field theory. From this twentieth century perspective, electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) fields are derivatives of A_μ, and are related to it in such a way as to produce force equations such as that of Lorentz [22—25]. It is well known that these results can be obtained from the principle of least action applied to field-matter interaction. Therefore, Eq. (1), although new to physics, is an entirely natural outcome of the role of the vector potential in conserving energy and momentum in field-fermion interaction. In the last analysis, it is itself a statement of conservation, based on the Noether Theorem [1,2,22—25].

In terms of beam intensity (I, W m^{-2}), otherwise known as power density, and beam angular frequency (ω, rad s^{-1}), the $\boldsymbol{B}^{(3)}$ field from Eq. (1) is [14,15]

$$\boldsymbol{B}^{(3)} = \frac{e\,\mu_0 c}{\hbar}\frac{I}{\omega^2}\boldsymbol{e}^{(3)} = 5.723 \times 10^{17}\frac{I}{\omega^2}\boldsymbol{e}^{(3)}, \tag{4}$$

where μ_0 is the permeability in vacuo in S.I. units $\left(\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Js^2C^{-2}m^{-1}}\right)$ and where $\boldsymbol{e}^{(3)}$ is a unit vector in the (3) axis of frame ((1), (2), (3)) [1,2]. Equation (4) is derived in Chap. 2 using the well-known classical relation

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{5}$$

between $\boldsymbol{A}$ and $\boldsymbol{B}$, a relation which originates in the principle of least action [22—25]. Thus if $\boldsymbol{A}$ is a plane wave in vacuo, so are $\boldsymbol{B}$ and $\boldsymbol{E}$, the latter being interrelated through Maxwell's equations. The plane wave $\boldsymbol{A}$ is in general complex because as such it is a solution of d'Alembert's equation in vacuo [1,2,22—25]. If $\boldsymbol{A}$ were pure real the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ would vanish, and this in itself is enough to show that $\boldsymbol{B}^{(3)}$ cannot be a *static* magnetic field, because a static magnetic field is defined as the curl of a pure real $\boldsymbol{A}$, as used by Dirac in his original theory of the electron [26].

For a given intensity, I, therefore, $\boldsymbol{B}^{(3)}$ is inversely proportional to the square of the electromagnetic beam's frequency, $f = \omega/(2\pi)$ (hertz or cycles s^{-1}). This inverse square frequency dependence of $\boldsymbol{B}^{(3)}$ means that it is orders of magnitude more intense (in tesla) for a given I at radio frequencies (MHz) than at visible frequencies (100 THz region). For I, for example, of 10 watts per square cm (10^5 W m^{-2}) the $\boldsymbol{B}^{(3)}$ field reaches an order of magnitude of nanotesla at 5,000 cm^{-1} (inverse centimeters or wavenumbers [40]) in the visible. In the visible under these conditions $\boldsymbol{B}^{(3)}$ is therefore one hundred thousand times weaker than the earth's mean magnetic field, but for a 10.0 MHz radio frequency field it becomes 14.5 *mega*tesla, causing proton resonance from Eq. (1) in the *infra-red* at about two thousand wavenumbers [14,15]. This result is derived in this opening chapter using the original Dirac electron theory [26] but with a complex $\boldsymbol{A}$ instead of a real $\boldsymbol{A}$. This is the only difference between our theory and Dirac's well known original [26].

Proton resonance in the infra-red is entirely out of reach in contemporary NMR practice, which is based [27—31] on permanent magnets. The highest contemporary resonance frequency in conventional NMR is typically about 0.5 GHz (500 MHz). Using Eq. (1), it is clear that NMR can be practiced at will in the much more accessible infra-red-visible frequency range, resulting in an enormous resolution enhancement achieved without the use of expensive superconducting magnets and with an ordinary Fourier transform absorption spectrometer. These advantages result directly from the appearance in Eq. (1) of the Pauli spinor $\boldsymbol{\sigma}^{(3)}$. Whenever a classically described electromagnetic field interacts with a fermion such as a proton, the $\boldsymbol{B}^{(3)}$ field produces

two observable energy states whose existence can be traced to the topological properties of $\boldsymbol{\sigma}^{(3)}$ [2,22—25]. Proton resonance occurs when a photon, $\hbar\omega_{res}$, of a separate probe field is absorbed to induce a change between the lower and upper energy states defined by the mathematical properties of the spinor $\boldsymbol{\sigma}^{(3)}$ of Eq. (1). Electron resonance occurs similarly when the probe photon is tuned to the right frequency. These resonance processes parallel exactly those occurring in everyday NMR or ESR, but with the permanent magnetic field replaced by $\boldsymbol{B}^{(3)}$ as defined in Eq. (1). Therefore the permanent magnet of a contemporary NMR or ESR spectrometer is replaced by a circularly polarized electromagnetic field from an ordinary radio frequency or microwave generator. Not only is this process of vast potential utility, but it is also an excellent method of proving the existence of $\boldsymbol{B}^{(3)}$ experimentally. Experiments such as these would, if positive, also prove the physical nature of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ and underline the contemporary view that the potential four-vector is physical in nature. There are compelling reasons, therefore, to pursue this type of investigation, not least of which is the empirical, i.e., experimental, evidence for the existence of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ already available from magneto-optics [16—21].

Dirac's original methods [26] are used in Sec. 1.1 of this chapter to derive Eq. (1), whose key feature is the interaction of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ with the spinor $\boldsymbol{\sigma}^{(3)*}$. Using Eq. (2), Eq. (1) is identical in structure with the equation first derived by Dirac [26] of the anomalous Zeeman effect. This equation gives rise to NMR and ESR, in which the fermion is respectively a nucleon (e.g. a proton), and an electron. The factor two which gives rise to the everyday term "half integral spin", is the result of a non-relativistic approximation, to be demonstrated in Sec. 1.1 below, and the fundamental reason for the existence of NMR and ESR is the topology of space itself, which allows a global distinction between SU(2) and O(3) [1,2,22—25]. The same topology allows the existence of equation (1), which gives rise to an entirely new absorption spectroscopy based on the Dirac equation itself, and allows nucleon resonance to be observed to great advantage over contemporary NMR and ESR. It is advantageous always to bear in mind that Dirac derived his equation purely from the general principles of relativity and quantum theory. These considerations force the use of four by four anti-commuting Dirac matrices [26] of which the Pauli matrices are two by two components. Therefore the fermion intrinsic spin has a deeper meaning than angular momentum, and it is well known that the fermion spin cannot be pictured classically (e.g. as a spinning object in space) because it is essentially a consequence of the topology of space itself.

There is no reason, therefore, to assume that NMR and/or ESR must always be practiced with static magnetic fields, or that a Pauli spinor must always interact with a static magnetic field, and Sec. 1.1 will show that the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ can be derived from first principles. This is an advance on the original phenomenological inference of Pershan [41], which first indicated that there might be an inverse Faraday effect (phase free magnetization by light [16—21]), and also an advance on approximate, semi-classical theories [42] involving the related product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$. It is well established experimentally [16—21] that the optical conjugate product

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = \frac{c^2}{\omega^2}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = \frac{1}{\omega^2}\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}, \tag{6}$$

produces observable effects, prominent among which is the inverse Faraday effect. Therefore Sec. 1.1 confirms that relativistic quantum field theory produces the inverse Faraday effect. This is a reassuring result both for fundamental field theory and experimental magneto-optics.

The theory in Sec. 1.1 is based on a classical view of the electromagnetic field, whereas more rigorously, there are radiative corrections due to quantum electrodynamics (QED), on which there is an extensive literature [22—25]. In respect of electron resonance, QED leads to a 1% correction in the factor 2 appearing in Eq. (1), and so is not of central importance in resonance due to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ or the $\boldsymbol{B}^{(3)}$ field, being a minor correction of the classical view. In the delicate understanding of photon-electron interaction however, QED is essential, and it is necessary to show that the conjugate product has a precise meaning in QED. This is left to later chapters of this volume. The experimental effects, now known to great accuracy [22—25], of the anomalous magnetic moment of the electron, first discussed by Schwinger [43], should become observable also with $\boldsymbol{B}^{(3)}$, because $\boldsymbol{B}^{(3)}$ is a physical magnetic field. The original experimental measurements, ably described by Dirac [26,44], should be repeated with a radio frequency, circularly polarized, electromagnetic field rather than a static magnetic field.

This line of reasoning can be extended to all magnetic effects in which a free fermion spin is present in a sample, and there are many of these effects now known. In each case the static magnetic field of the measuring apparatus is replaced by $\boldsymbol{B}^{(3)}$ of an ordinary radio frequency generator, provided that the radiation is circularly polarized. Finally, in Sec. 1.2 of this chapter, the original Dirac theory of Sec. 1.1 is updated and reworked consistently in contemporary notation and standard representation [2,22—25].

1.1 $\boldsymbol{B}^{(3)}$ FROM THE ORIGINAL ELECTRON THEORY OF DIRAC

Dirac's famous theory of the electron [26] is recovered exactly if $\boldsymbol{A}$ is pure real in what follows. In other words, Dirac assumes that $\boldsymbol{A}$ is real in order to recover a theory of the fermion in a static magnetic field, and to explain the elegant experiment of Gerlach and Stern [45] and the anomalous Zeeman effects [46] then known. In this section Dirac's clear and powerful description [26] is deliberately and closely followed to show the existence of $\boldsymbol{B}^{(3)}$ from the first principles of this, the original relativistic quantum field theory. Dirac's development is based [26] on the relativistic quantum wave equation

$$\left(p_0 + eA_0 - \rho_1(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A})) - \rho_3 mc\right)\psi = 0, \tag{7}$$

for a fermion (in his case an electron) in a classical electromagnetic field. Equation (7) is written in S.I. units whereas Dirac uses Gaussian units, and this is the only difference in notation. The momentum four-vector used by Dirac is, in S.I. units

$$p_\mu := (p_0, \boldsymbol{p}), \tag{8}$$

and the potential four-vector in S.I. units is

$$A_\mu := (A_0, \boldsymbol{A}). \tag{9}$$

The matrices ρ_1 and ρ_3 are [1]

$$\rho_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \rho_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{10}$$

and ψ is a column vector, described in contemporary terms [2,22—25] as the Dirac four-spinor. In his original account [26], Dirac does not use parity to interconvert components of the four-spinor, as is the contemporary practice [2,22—25].

The Hamiltonian for a classical electron in a classical electromagnetic field is now used by Dirac as a guideline to the properties of Eq. (7). The wave equation expected from analogy [26] with the classical theory is

$$\left((p_0 + eA_0)^2 - (\boldsymbol{p}+e\boldsymbol{A})^2 - m^2c^2\right)\psi = 0, \tag{11}$$

and was written from the outset [26] for a *real* $\boldsymbol{A}$. For complex $\boldsymbol{A}$ Eq. (11) becomes

$$\left((p_0 + eA_0)\left(p_0 + eA_0^*\right) - (\boldsymbol{p}+e\boldsymbol{A})\cdot(\boldsymbol{p}+e\boldsymbol{A}^*) - m^2c^2\right)\psi = 0, \tag{12}$$

because mathematically real, and therefore *physical*, quantities may always be formed from the product of a mathematically complex quantity with its complex conjugate. In order to make his theory of the electron resemble Eq. (11) as closely as possible, Dirac multiplies Eq. (7) by the factor

$$p_0 + eA_0 + \rho_1(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A})) + \rho_3 mc, \tag{13}$$

which for a complex potential four-vector becomes

$$p_0 + eA_0^* + \rho_1(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A}^*)) + \rho_3 mc, \tag{14}$$

giving the product

$$\begin{aligned}&\left(\left(p_0 + eA_0^*\right)(p_0 + eA_0) - (\boldsymbol{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A}^*))(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A})) - m^2c^2\right.\\ &\left. - \rho_1\left(\left(p_0 + eA_0^*\right)(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A})) - (\boldsymbol{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A}^*))(p_0 + eA_0)\right)\right)\psi = 0.\end{aligned} \tag{15}$$

This, in our theory, replaces Eq. (31), Chap. 11, of Ref. 26. The product (15) contains several terms which are developed as follows.

Firstly, the conjugate product term leading to Eq. (1) originates in

$$e^2(\boldsymbol{\sigma}\cdot\boldsymbol{A}^*)(\boldsymbol{\sigma}\cdot\boldsymbol{A})\psi. \tag{16}$$

As shown by Dirac [26], if $\boldsymbol{B}$ and $\boldsymbol{C}$ are any two three-dimensional vectors that commute with what is now known as the Pauli spinor $\boldsymbol{\sigma}$, then

$$(\boldsymbol{\sigma}\cdot\boldsymbol{B})(\boldsymbol{\sigma}\cdot\boldsymbol{C}) = \boldsymbol{B}\cdot\boldsymbol{C} + i(\boldsymbol{\sigma}\cdot\boldsymbol{B}\times\boldsymbol{C}). \tag{17}$$

For a pure real $\boldsymbol{A}$ there is only one term on the right hand side of Eq. (17), but for complex $\boldsymbol{A}$, there enters into the Dirac theory of the electron an all-important new term, which describes the interaction of the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ with the Pauli spinor component $\boldsymbol{\sigma}^{(3)}$. This term is inherently electromagnetic in origin, and does not exist in a static magnetic field.

Following the development by Dirac, but allowing now for complex $\boldsymbol{A}$, we set

$$\boldsymbol{p} \to -i\hbar\nabla, \tag{18}$$

and obtain the terms

$$1)\ (\boldsymbol{p}\times\boldsymbol{p})\psi = (i\hbar)^2\nabla\times\nabla\psi = 0, \tag{19}$$

2) $(eA \times p)\psi = 0 = -i\hbar eA \times \nabla\psi = 0,$ (20)

3) $(ep \times A^*)\psi = -i\hbar e\nabla \times A^*\psi - i\hbar e\nabla\psi \times A^* = -i\hbar eB^*\psi.$ (21)

The last of these uses the quantum prescription (18) to describe the interaction of the intrinsic spin of the fermion with the ordinary magnetic component of the electromagnetic field, usually described as a plane wave,

$$B^* = \nabla \times A^*. \tag{22}$$

For a static magnetic field, term (3) leads to the famous result [26] that the intrinsic angular momentum of a fermion is half integral in the non-relativistic (but non-classical) limit [2,22—25]. In our development, however, B^* is a plane *wave*, and averages to zero over many cycles of the field. Term three is therefore of no further interest in the derivation of Eq. (1) from first principles. We follow Dirac in discarding term one, the cross product $p \times p$. The term $eA \times p$ is also discarded, as usual [2,22—26], because it is a classical quantity multiplying a del operator on ψ. (In contrast, the term $ep \times A^*$ becomes $-ie\hbar\nabla \times A^*$ which is a del operator on A^*, and this is non-zero.)

Continuing in this way, we find that Dirac's original Eq. (34) of Chap. 11 of Ref. 26 is replaced for complex A by

$$\begin{aligned}\left((p_0 + eA_0^*)(p_0 + eA_0) - (p + eA^*)\cdot(p + eA)\right.\\ \left. - ie^2\sigma \cdot A^* \times A - m^2c^2 - e\hbar\sigma \cdot B^* + i\rho_1 e\hbar\sigma \cdot E\right)\psi = 0,\end{aligned} \tag{23}$$

in which there appears the conjugate cross product term, and in which products involving A are conjugate products. The term in $\sigma \cdot E$ in our Eq. (23) can contain a real part if E is a complex plane wave, but the symmetry of this term is $\hat{T}$ and $\hat{P}$ negative [1,2] and does not appear to be physical. (This is the same kind of reasoning that is used to explain why there is no electric analogue of the Faraday or inverse Faraday effects.) For this reason we take no further interest here in this term. It is notable that Dirac [26] also discards this term as unphysical but for different reasons.

So the final result of our calculation is

$$\begin{aligned}\left((p_0 + eA_0^*)(p_0 + eA_0) - (p + eA^*)\cdot(p + eA) - m^2c^2\right.\\ \left. - ie^2\sigma \cdot A^* \times A\right)\psi = 0,\end{aligned} \tag{24}$$

in which there appears the gauge independent term $-ie^2\sigma \cdot A^* \times A$ which allows

resonance to occur between the two physically distinct energy states generated by the topological properties of the spinor σ. The conjugate cross product $A^* \times A$ plays the role of an entirely novel type of magnetic field, the Evans-Vigier field $B^{(3)}$ of Eq. (1). This term vanishes if A is real, and the original Dirac theory is recovered because B^* becomes a conventional static magnetic field.

The non-relativistic, non-classical, limit of Eq. (24) can be obtained using the standard approximations introduced by Dirac [26],

$$En \sim mc^2, \qquad p \sim 0, \tag{25}$$

in which case the eigenvalue of the interaction energy between field and electron is

$$W := En - mc^2 \sim \frac{e^2c^2(\sigma \cdot A)(\sigma \cdot A^*)}{En + mc^2 + ecA_0} - ecA_0. \tag{26}$$

In the radiation gauge, A_0 is zero, and in the Coulomb gauge A_0 is a constant which can be zero [22—25], and if A_0 is zero,

$$W \sim \frac{e^2}{2m}(A \cdot A^* + i\sigma \cdot A \times A^*). \tag{27}$$

The resonance term is the second term on the right hand side of this equation, and using Eq. (2), which was derived in Vol. 2 [2], we recover Eq. (1) from this resonance term.

This development from original first principles shows that the $B^{(3)}$ field *always* exists in field-fermion interaction provided that the field is an electromagnetic wave. In other words, we reach a new theorem which states that the electromagnetic wave interacts with intrinsic fermion spin as if the wave were a phase free magnetic flux density, the Evans-Vigier field, $B^{(3)}$.

1.2 CONTEMPORARY DEVELOPMENT

1.2.1 MINKOWSKI NOTATION

In contemporary notation [2,22—25] it is convenient to rework Sec. 1.1 from the Dirac equation,

$$\gamma_\mu(p_\mu + eA_\mu)\psi(p) = -mc\psi(p)\cdot \tag{28}$$

Here ψ is Dirac's four-spinor, γ_μ is Dirac's matrix in standard representation, and $p_\mu + eA_\mu$ is the energy-momentum of the fermion in the electromagnetic field. We develop the theory in this section with Minkowski notation, as for example in the classic text by Jackson [47]. We continue to use S.I., standard, contemporary units. In vector notation, Eq. (28) becomes [2,22—25]

$$\left(\gamma_0\left(En + ecA_0\right) - c\boldsymbol{\gamma}\cdot\left(\boldsymbol{p} + e\boldsymbol{A}\right)\right)\psi = mc^2\psi, \tag{29}$$

where

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{30}$$

Here u and v are two-spinors (column vectors with two components) interconvertible by parity inversion, $\hat{P}$. Therefore Eq. (29) splits into two equations,

$$\begin{aligned} \left(En + ecA_0\right)u - c\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} + e\boldsymbol{A}\right)v &= mc^2u, \\ -\left(En + ecA_0\right)v + c\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} + e\boldsymbol{A}\right)u &= mc^2v, \end{aligned} \tag{31}$$

which describe the dynamics of the fermion in the field.

It is convenient to work in the circular basis [1,2],

$$\boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)*} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}), \qquad \boldsymbol{e}^{(3)} = \boldsymbol{k}, \tag{32}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are real, Cartesian, unit vectors respectively in X, Y, and Z of the laboratory frame. The basis (32) [1,2], has O(3) symmetry

$$\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)*}, \text{ et cyclicum}, \tag{33}$$

and is therefore a representation of the rotation group in three space dimensions. In this basis, the vector potential of the electromagnetic wave can be written as the plane wave

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \tag{34}$$

and so, from Eq. (5), the magnetic field can be written as

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}. \tag{35}$$

Here ϕ is the electromagnetic phase [1,2]

$$\phi = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}, \tag{36}$$

where ω is its angular frequency at instant t, and $\boldsymbol{\kappa}$ its wave vector at point $\boldsymbol{r}$.

Using Dirac's original approximations (25), i.e., working in the non-relativistic limit with a very slow fermion, [2,22—25], Eqs. (31) become

$$\begin{aligned} \left(En + ecA_0 - mc^2\right)u &= ec\boldsymbol{\sigma}\cdot\boldsymbol{A}v, \\ \left(En + ecA_0 + mc^2\right)v &= ec\boldsymbol{\sigma}\cdot\boldsymbol{A}u, \end{aligned} \tag{37}$$

and their Hermitian transposed counterparts become

$$\begin{aligned} u^+\left(En + ecA_0 - mc^2\right) &= ecv^+(\boldsymbol{\sigma}\cdot\boldsymbol{A})^+, \\ v^+\left(En + ecA_0 + mc^2\right) &= ecu^+(\boldsymbol{\sigma}\cdot\boldsymbol{A})^+. \end{aligned} \tag{38}$$

In these equations the spinors are row vectors and are operated upon to the left [2]. The superscript "+" denotes *Hermitian transpose* [2], which is transposition with simultaneous complex conjugation of elements. It is convenient to develop (Appendix A) the Pauli spinors in the same circular basis ((1), (2), (3)):

$$\boldsymbol{\sigma}^{(1)} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)+} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{39}$$

giving the SU(2) commutative algebra

$$\left[\frac{\boldsymbol{\sigma}^{(1)}}{2}, \frac{\boldsymbol{\sigma}^{(2)}}{2}\right] = -\frac{\boldsymbol{\sigma}^{(3)+}}{2}, \text{ et cyclicum}. \tag{40}$$

The product algebra of

Appendix B, leading to

$$(\sigma^{(1)} \cdot e^{(2)})(\sigma^{(2)} \cdot e^{(1)}) = e^{(1)} \cdot e^{(2)} + i\sigma^{(3)} \cdot e^{(1)} \times e^{(2)}, \tag{41a}$$

$$(\sigma^{(1)} \cdot e^{(2)}) = (\sigma^{(2)} \cdot e^{(1)})^{+}. \tag{41b}$$

We can therefore write the Dirac equations (37) and (38) as

$$\begin{aligned} (En + ecA_0 - mc^2)u &= ec\sigma^{(2)} \cdot A^{(1)}v, \\ (En + ecA_0 + mc^2)v &= ec\sigma^{(2)} \cdot A^{(1)}u, \end{aligned} \tag{42a}$$

$$\begin{aligned} u^{+}(En + ecA_0 - mc^2) &= ecv^{+}\sigma^{(1)} \cdot A^{(2)}, \\ v^{+}(En + ecA_0 + mc^2) &= ecu^{+}\sigma^{(1)} \cdot A^{(2)}, \end{aligned} \tag{42b}$$

in which

$$\sigma^{(2)} \cdot A^{(1)} = (\sigma^{(1)} \cdot A^{(2)})^{+}, \qquad \sigma^{(1)} \cdot A^{(2)} = (\sigma^{(2)} \cdot A^{(1)})^{+}. \tag{43}$$

It is convenient to solve these equations using firstly the transverse gauge, in which $A_0 = 0$, and secondly the Dirac condition [48,49], in which $A_0 = |A|$. In the transverse (or radiation) gauge, we obtain, with $En \sim mc^2$,

$$Wu = \frac{e^2}{2m}(\sigma^{(2)} \cdot A^{(1)})^2 u, \tag{44a}$$

$$u^{+}W = \frac{e^2}{2m}u^{+}(\sigma^{(1)} \cdot A^{(2)})^2. \tag{44b}$$

Multiplying Eq. (44a) by Eq. (44b),

$$u^{+}W^2u = u^{+}\left(\frac{e^2}{2m}\right)^2(\sigma^{(1)} \cdot A^{(2)}\sigma^{(2)} \cdot A^{(1)})^2 u. \tag{45}$$

The eigenvalue of the square of the energy of interaction of fermion and field is therefore

$$W^2 = \left(\frac{e^2}{2m}\sigma^{(1)} \cdot A^{(2)}\sigma^{(2)} \cdot A^{(1)}\right)^2, \tag{46}$$

giving the possibility of positive and negative energy expectation values [2,22—25],

$$W = \pm\frac{e^2}{2m}\sigma^{(1)} \cdot A^{(2)}\sigma^{(2)} \cdot A^{(1)}. \tag{47}$$

This leads to the prediction of anti-fermions as is well known and well verified experimentally [2,22—26]. For our purposes the positive expectation value of energy can be developed with the spinor algebra

$$\sigma^{(1)} \cdot A^{(2)}\sigma^{(2)} \cdot A^{(1)} = A^{(1)} \cdot A^{(2)} + i\sigma^{(3)} \cdot A^{(1)} \times A^{(2)}, \tag{48}$$

and the second term on the right hand side leads to equation (1) as we set out to show. The original and contemporary approaches to the Dirac theory produce the same, potentially very useful, result, that fermions interact with the electromagnetic wave through the $B^{(3)}$ field.

In the radiation gauge, A_μ is not completely covariant, because A_0 is zero in one special Lorentz frame. This unsatisfactory feature of the radiation gauge has been discussed in the recent literature [1,2]. Roy and Evans [50] have recently suggested a novel gauge theory which reconciles non-zero photon mass with type two gauge invariance

$$A_0 \overset{F\dot{A}.\dot{P}.P.}{\dashrightarrow} |A|, \tag{49}$$

so that

$$A_\mu A_\mu \overset{F\dot{A}.\dot{P}.P.}{\dashrightarrow} 0. \tag{50}$$

In this notation, *FAPP* denotes *for all practical purposes*, and Eq. (50) is a limiting form of, and an excellent approximation to, the condition introduced by Dirac [48,49],

$$A_\mu A_\mu = \text{constant}. \tag{51}$$

We refer to Eq. (49) as the Dirac condition. It has the advantage of rendering A_μ completely covariant and of being compatible with the Lorentz gauge [1,2,22]. It also allows the Proca equation [1,2] to be derived from a suitable Lagrangian, and so has

useful properties.

The theory of electromagnetism developed in Vol. 1 and 2 augments the vacuum Maxwell equations with a system of which Eq. (2) is an example. Maxwell's original derivatives are replaced [2] by gauge invariant counterparts, and these are the same as those of the Dirac electron theory being developed in this chapter. Non-Maxwellian terms are developed within O(3) gauge theory [2] and are quadratic in A_μ, as illustrated in Eq. (2). Terms such as this are derivable in Evans-Vigier field theory [1,2] from O(3) gauge theory and are quadratic in the six field $F_{\mu\nu}$. These terms imply the use of complex fields and potentials as in de Broglie's theory of the photon [51]. The Dirac condition (49), which we are about to develop further, is an intrinsic part of the Evans-Vigier theory, some parts of which [1,2] border on the theory of finite photon mass.

The self-consistency and physical good sense of the Dirac condition can be illustrated as follows by incorporating into it Eq. (1), which becomes

$$E_{int} = \frac{e^2c^2}{En + mc^2 + ecA_0}\sigma^{(1)} \cdot A^{(2)}\sigma^{(2)} \cdot A^{(1)} - ecA_0. \tag{52}$$

This equation defines the non-relativistic limit of the interaction energy of a fermion in a circularly polarized electromagnetic field using a completely covariant A_μ defined by Eqs. (49) and (50), and which is light-like in vacuo for all practical purposes. It can be shown as follows that Eq. (52) makes sense by considering the limit

$$ecA_0 \gg (En + mc^2), \tag{53}$$

which represents a very intense, relatively low frequency, electromagnetic wave interacting with a fermion. In the approximation $En \sim mc^2$, Eq. (53) becomes [1,2]

$$\omega \ll \frac{e}{2m}B^{(0)}. \tag{54}$$

In this limit, Eq. (52) becomes

$$E_{int} \rightarrow \frac{ec}{A_0}(A^{(1)} \cdot A^{(2)} + i\sigma^{(3)} \cdot A^{(1)} \times A^{(2)}) - ecA_0, \tag{55}$$

and using

Minkowski Notation

$$A^{(1)} \cdot A^{(2)} = A_0^2, \qquad A^{(1)} \times A^{(2)} = iA_0^2e^{(3)}, \tag{56}$$

we obtain

$$E_{int1} \rightarrow \mp ecA_0. \tag{57}$$

Finally, using the minimal prescription for the free photon [2],

$$eA_0 = \hbar\kappa = \hbar\frac{\omega}{c}, \tag{58}$$

Eq. (57) becomes

$$E_{int} \rightarrow \mp\hbar\omega, \tag{59}$$

which is consistent with energy conservation. The rotational kinetic energy of one photon, $\hbar\omega$, has been transferred to the fermion in an elastic collision. The $\mp$ sign in Eq. (59) is due to the topological properties of the Pauli spinor, and has no classical meaning. That is to say, the fermion can be in one of two energy states, because its intrinsic spin has eigenvalues $\pm\hbar/2$.

In the radiation gauge on the other hand, Eq. (59) can not be obtained, because the limiting condition (53) is never valid. The radiation gauge is therefore inconsistent with the physical result (59) and is inconsistent in this context with the law of conservation of energy. The origin of the inconsistency is that in the radiation gauge the potential four-vector is not consistently covariant. This leads to other well-known difficulties [1,2] in the canonical quantization of the electromagnetic gauge field. The failure of the radiation gauge in Dirac's theory of the electron becomes apparent, furthermore, only in the limit (53). In the opposite limit, which is almost always used in the development of the non-relativistic limit,

$$ecA_0 \ll (En + mc^2), \tag{60}$$

there is no practical difference between the Coulomb gauge and the Dirac condition, because $A_0 \sim 0$ is equivalent to Eq. (60). In the rest frame approximation $En \sim mc^2$, Eq. (60) becomes

$$\omega \gg \frac{e}{2m} B^{(0)}, \tag{61}$$

in which Eq. (1) is valid in both gauges. In the opposite limit, Eq. (54), Eq. (1) is no longer gauge independent in this way, meaning that the radiation gauge can no longer be used because it does not lead to energy conservation as just shown. Equation (55) is an experimentally verifiable test of the validity of Dirac's condition (49). As such, it is also important for the theory of finite photon mass [1,2], and is a prediction of the Evans-Vigier theory [1,2].

1.2.2 FEYNMAN SLASH NOTATION

In contemporary Dirac theory, field-fermion interaction is described with Feynman slash notation [22—25], which allows a compact description of Dirac algebra. In this section we demonstrate Eq. (1) using these contemporary methods, in which reduced units $\hbar = c = 1$ are used. The field free Dirac equation in this notation is

$$(i\not{\partial} - m)\psi = 0, \tag{62}$$

where $i\not{\partial} = \not{p} := \gamma^\mu p_\mu$, the reduced energy momentum four-vector. The Hermitian transpose of Eq. (62) is, in contemporary notation [22—25],

$$\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0. \tag{63}$$

The product of Eqs. (62) and (63) produces

$$\bar{\psi}(\not{p}\not{p} - m^2)\psi = 0 \tag{64}$$

in which the quantity between the spinors $\bar{\psi}$ and ψ is a scalar. We first show that Eq. (64) produces the Einstein equation of special relativity through the equivalence principle. It is convenient to proceed by expanding the product $\not{p}\not{p}$,

$$\not{p}\not{p} = p^\mu p_\mu - i\sigma_{\mu\nu} p^\mu p^\nu, \tag{65}$$

where, as usual, $\sigma_{\mu\nu}$ is the commutator of Dirac matrices [22—25],

$$\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{66}$$

If it is assumed that there is no spin contribution to $\not{p}\not{p}$, then,

$$\not{p}\not{p} = p^\mu p_\mu = En^2 - p^2, \tag{67}$$

where En is energy and p is linear momentum in reduced units of $\hbar = c = 1$. Therefore Eq. (64) becomes

$$\bar{\psi}(En^2 - p^2 - m^2)\psi = 0, \tag{68}$$

which in S.I. units is

$$\bar{\psi}(En^2 - p^2c^2 - m^2c^4)\psi = 0. \tag{69}$$

This is the quantum mechanical version (Klein Gordon equation) of the Einstein equation of classical relativistic dynamics,

$$En^2 = p^2c^2 + m^2c^4, \tag{70}$$

and Eq. (69) means that the expectation value of the quantity between the spinors is zero.

In the del representation, Eq. (69) is

$$\bar{\psi}(\Box + m^2)\psi = 0, \tag{71}$$

where $\Box$ is the d'Alembertian operator [1,2]. If interpreted as

$$\Box\psi_i = -m^2\psi_i, \qquad i = 1, \ldots, 4, \tag{72}$$

Eq. (72) becomes another form of the Dirac equation with Klein-Gordon components [2] as required. If we interpret the spinor as the column vector,

$$\psi = \begin{pmatrix} E \\ iB \end{pmatrix}, \tag{73}$$

where E is electric field strength and B is magnetic flux density, and if we assert that the mass, m, is zero, we recover the equations

$$\Box E = \Box B = 0, \tag{74}$$

which become the d'Alembert equation in vacuo [52] if $|A|$ is proportional to $|B|$ (and thus to $|E|$) in vacuo. This proportionality is satisfied by Eq. (5) which gives,

$$|A| = \frac{c}{\omega}|B| = \frac{1}{\omega}|E|, \tag{75}$$

for plane waves in S.I. units. The representation (73) allows the Maxwell equations to be expressed in the form of a Dirac equation [22—25].

The interaction of fermion and classical field is described in contemporary slash notation using the minimal prescription in the form

$$i\partial \!\!\!/ \to i\partial \!\!\!/ - eA\!\!\!/; \qquad -i\partial \!\!\!/ \to -i\partial \!\!\!/ - eA\!\!\!/^*. \tag{76}$$

Since $p\!\!\!/ = i\partial \!\!\!/$, and ∂ is real, the complex conjugate of $p\!\!\!/ - eA\!\!\!/$ is given in general by

$$(p\!\!\!/ - eA\!\!\!/)^* = -p\!\!\!/ - eA\!\!\!/^*. \tag{77}$$

The Hermitian transpose [2,22—25] of the Dirac equation of a fermion in a classical electromagnetic field is therefore obtained from the original by (Appendix C)

$$(p\!\!\!/ - eA\!\!\!/ - m)\psi = 0, \tag{78a}$$

$$\Downarrow$$

$$\bar{\psi}(-p\!\!\!/ - eA\!\!\!/^* - m) = 0. \tag{78b}$$

Multiplying Eq. (78a) and (78b) gives

$$\bar{\psi}(p\!\!\!/ + eA\!\!\!/^* + m)(p\!\!\!/ - eA\!\!\!/ - m)\psi = 0. \tag{79}$$

The expectation value of the scalar quantity which appears between the spinors $\bar{\psi}$ and ψ is zero. The equivalence principle shows that the classical relativistic version of Eq. (79) is

$$En^2 = p^2 + m^2 + e^2A\!\!\!/^*A\!\!\!/ + em(A\!\!\!/^* + A\!\!\!/) + e(p\!\!\!/A\!\!\!/ - A\!\!\!/^*p\!\!\!/), \tag{80}$$

and it is seen that the electromagnetic field has added several terms to the Einstein equation (70).

The resonance phenomenon which we want to demonstrate arises from the conjugate product term $e^2A\!\!\!/^*A\!\!\!/$, which is phase free [1—15] and non-zero after averaging over many cycles of the electromagnetic field at any frequency. It is not, however, a *static* property, because it propagates in vacuo at the speed of light. Terms such as $emA\!\!\!/^*$ average to zero over many field cycles and we take no further interest in them here, as in the previous section. We note that the conjugate product term has appeared from the principle of local gauge invariance, meaning that the standard minimal prescription leads to it as a consequence of current conservation and Noether's Theorem. Not surprisingly, therefore, the conjugate product term is gauge invariant, and therefore physical [1—15]. It has been assumed only that A_μ is complex, as discussed for example by Barut [52], and on countless other occasions. The term $e^2A\!\!\!/^*A\!\!\!/$ is therefore a direct outcome of the Dirac equation and for this reason should produce observable experimental effects.

In reduced units we obtain [22—25]

$$A\!\!\!/^*A\!\!\!/ = -i\sigma^{(3)} \cdot A^{(1)} \times A^{(2)} + A_0^2 - A^{(1)} \cdot A^{(2)}, \tag{81}$$

where A_0 is the scalar component of the four-vector A_μ. For a non-zero A_0 in general we obtain

$$En^2 = m^2 + p^2 + e^2A_0^2 - e^2(i\sigma^{(3)} \cdot A^{(1)} \times A^{(2)} + A^{(1)} \cdot A^{(2)}). \tag{82}$$

In order to proceed we use the approximation

$$(m + eA_0)^2 \sim m^2 + e^2A_0^2, \tag{83}$$

which is valid if: 1) $m \gg eA_0$, 2) $eA_0 \gg m$. The first limit is the non-relativistic one,

discussed in several standard texts [22—25]. Using this approximation, and the related approximation $En \sim m$, $p \sim 0$ produces, in reduced units,

$$W := En - m \sim -i\frac{e^2}{2m}\sigma^{(3)} \cdot A^{(1)} \times A^{(2)} + eA_0 - \frac{e^2}{2m}A^{(1)} \cdot A^{(2)}. \tag{84}$$

In S.I. units, this equation becomes

$$W := En - mc^2 \sim -i\frac{e^2}{2m}\sigma^{(3)} \cdot A^{(1)} \times A^{(2)} + ecA_0 - \frac{e^2}{2m}A^{(1)} \cdot A^{(2)}, \tag{85}$$

for the extra energy added to the Einstein equation of the fermion by the electromagnetic field. The expected resonance term is $-i(e^2/2m)\sigma^{(3)} \cdot A^{(1)} \times A^{(2)}$ and is real, physical, and gauge independent as expected.

In Chap. 2, it is described how this result leads to a new resonance technique which if developed, has the capability of making unnecessary the use of superconducting magnets in NMR and ESR. This is the result of the exact solutions to the Dirac equation that we have been discussing in this chapter.

Chapter 2. Limits and Resonance Conditions

The resonance method of detecting features outlined in Chap. 1 is potentially of great practical utility, because, as explained in this chapter, it can lead theoretically to ultra-high resolution NMR and ESR without magnets. In contemporary terms this is a seemingly impossible aim, the best available superconducting magnets produce about 20 tesla at most, resulting in resonance at about 0.5 GHz [20]. The basic reason for using high intensity magnetic fields in NMR is the need for high resolution. It will be shown that it is possible in theory to remove the magnets and at the same time achieve a much higher resolution than hitherto attainable. Furthermore, this aim can be achieved in principle with mature contemporary radio frequency technology.

2.1 LIMITS FROM THE DIRAC EQUATION

Before developing the theory of resonance which will possibly form the basis of NMR and ESR in the future, it is instructive to infer the physical meaning represented by mathematical limits of the Dirac theory of Chap. 1. In particular, it is necessary to explain how the equation self-consistently deals with seemingly disparate definitions of $B^{(3)}$, namely,

$$1)\ B^{(3)*} = B^{(0)}e^{(3)} = -\frac{i}{B^{(0)}}B^{(1)} \times B^{(2)}, \tag{86}$$

$$2)\ B^{(3)*} = -i\frac{e}{\hbar}A^{(1)} \times A^{(2)}. \tag{87}$$

Definition (1) [1,2] simply puts $B^{(3)}$ proportional to the amplitude of the magnetic component of vacuum electromagnetism, meaning that $B^{(3)}$ is proportional to the square root of intensity (I). Definition (2) [2], which was derived rigorously in Chap. 1, has $B^{(3)}$ proportional to the conjugate product $A^{(1)} \times A^{(2)}$ and thus to intensity itself. How can definitions (1) and (2) both apply; which should be used to describe the field-fermion interaction; and how can one definition be equivalent to the other? The answers are obtained straightforwardly as follows from a simple inspection of limits.

Definition (2) represents the usual experimental limit, under which all magneto-optic results known to us have been obtained to date. These results appear always to show that the magneto-optic phenomenon is proportional to the intensity (I) of the

pump laser or electromagnetic field. Some of the key references to the available experimental data are collected in [16-21]. The lowest frequency circularly polarized pump field used to date was at three GHz, by Deschamps *et al.* [17]. In this experiment the inverse Faraday effect [16] was detected experimentally using pulsed, circularly polarized, microwaves, and the effect found to be proportional to I. The definition (2), also has $\boldsymbol{B}^{(3)}$ proportional to I, through a universal constant $e/\hbar$, and inversely proportional to the square of ω. It is now realized that this result can be shown from the first principles of relativistic quantum theory, as demonstrated in Chap. 1. In Vol. 2 it was inferred from O(3) (non-Abelian) gauge theory.

It corresponds to the limit

$$\hbar\omega \gg ecA_0, \tag{88}$$

where A_0 is the time-like component of A_μ, and where ω is the angular frequency of the field in which the fermion is immersed. In the radiation, or transverse, gauge [47], it is asserted that $A_0 = 0$, so that in this gauge the limit (88) is an exact result for finite ω. In this limit, Eq. (27) shows that the energy transferred to the fermion is second order in the vector potential, i.e., first order in the intensity, I. This result is quantum relativistic in origin, although obtained in a non-relativistic approximation as described in Chap. 1, and although the electromagnetic field is still treated classically. Classical physics cannot account for the existence of the spinor term (e.g. in Eq. (27)), nor can it account for the definition (2) of $\boldsymbol{B}^{(3)}$, a definition which includes $\hbar$, the Dirac constant of quantum mechanics [22-25]. Conservation of energy-momentum is represented in the transverse gauge by an equation such as (27), but the gauge is not manifestly covariant [22-25] and produces well known difficulties in canonical quantization of the field. As discussed in Chap. 1, conservation of energy-momentum in the transverse gauge is also not entirely satisfactory. These caveats apply, however, to a well accepted gauge assumption [47] rather than to the method being used, and it can be shown (next section) that the classical limit of an equation such as (1) is precisely that obtained by Talin *et al.* [54] for the inverse Faraday effect and is precisely that predicted by the classical, but relativistic, Hamilton-Jacobi equation of one classical electron or proton in a field [1]. This is discussed further in Sec. 2.2, where the Dirac equation is transformed into a Pauli equation, and finally into the Hamilton-Jacobi equation [1]. The inverse Faraday effect in the limit (88) is proportional to I as observed [16-21] to date and can be understood through $\boldsymbol{B}^{(3)}$ in definition (2). This high field frequency definition should almost always be used in experimental applications.

The definition (1) of $\boldsymbol{B}^{(3)}$ was the first to be inferred heuristically [1,2], however, and from the work of Chap. 1 it is clear that it applies in the limit,

$$ecA^{(0)} \to \hbar\omega, \qquad A^{(0)} = (\boldsymbol{A}\cdot\boldsymbol{A}^*)^{1/2} \to A_0. \tag{89}$$

In this limit, nearly all the energy-momentum of the photon (anticipating the quantized interpretation of the field) is transferred to the fermion, which is thereby accelerated very close to the speed of light. As discussed in Chap. 1, and in Vol. 1 and 2 [1,2], this is a low frequency, high intensity limit,

$$\omega \ll \frac{e}{2m}B^{(0)}, \tag{90}$$

and since A_0, the time-like component of A_μ, cannot be zero in this view, this limit cannot be attained in the transverse or radiation gauge. Using it in Eq. (1) produces, with definition (1), the result,

$$En_{int} \xrightarrow[ecA_0 \to \hbar\omega]{} -\frac{eB^{(0)}}{m}\frac{\hbar}{2}\sigma^{(3)}\cdot e^{(3)*}, \tag{91}$$

which shows that the energy formed between field and fermion in the limit (89) is proportional to the square root of I and ultimately becomes independent of ω. This is experimentally distinguishable from Eq. (1), which was obtained in the opposite limit of the Dirac equation, although no experimental investigations have been made to our knowledge under the required conditions of very low pump frequency, very high pump intensity. These investigations would show experimentally whether or not the Dirac condition is more representative of free space electromagnetism than the transverse gauge.

It has been demonstrated theoretically, therefore, that limiting forms of the Dirac equation produce both the heuristic definition of $\boldsymbol{B}^{(3)}$ (definition (1)), and the more recent [15] definition (2) appearing in Eq. (1) from fundamental first principles of relativistic quantum mechanics. Clearly, definition (1) of $\boldsymbol{B}^{(3)}$ can only be used under the right conditions, represented by Eq. (89); and $\boldsymbol{B}^{(3)}$ cannot be defined at all under these conditions in the transverse gauge. Self consistently, therefore [1,2], the definition (1) of $\boldsymbol{B}^{(3)}$ requires a finite time-like component of A_μ ($= (A_0, \boldsymbol{A})$). Definition (2) can however be arrived at if the time-like component is zero. In both cases it is possible to retain the useful condition

$$A_\mu A_\mu \xrightarrow[FAPP]{} 0, \tag{92}$$

recently suggested by Roy and Evans [55] as a means of reconciling the theory of type two gauge invariance (which we support) and the theory of finite photon mass [1,2]. Under the Dirac condition (of which Eq. (92) is a limit [55]),

$$\boldsymbol{A}^{(1)}\cdot\boldsymbol{A}^{(1)*} + \boldsymbol{A}^{(2)}\cdot\boldsymbol{A}^{(2)*} + \boldsymbol{A}^{(3)}\cdot\boldsymbol{A}^{(3)*} = A_0^2, \tag{93}$$

can always be obtained if the magnitude of the space-like and time-like components of A_μ are equal. For all practical purposes A_μ is a fully covariant, *light-like*, four-vector in this view. In the transverse gauge, the Roy-Evans condition (92) applies if

$$A^{(0)2}\left(e^{(1)}\cdot e^{(2)*}+e^{(2)}\cdot e^{(2)*}\right)+A^{(3)2}\left(e^{(3)}\cdot e^{(3)*}\right)=0,$$
$$A^{(3)}=\sqrt{2}\,iA^{(0)}, \tag{94}$$

i.e., if a pure imaginary and phase free third component is introduced for the space part of A_μ, there being by definition no time-like component [47]. The longitudinal component of $\boldsymbol{A}$ in this view is unphysical, and so the usual assertion that the radiation gauge [47] is pure transverse is retained in the usual physical applications. Therefore we arrive at the rather unexpected conclusion that a *massless* particle may be described mathematically with three states of polarization, provided that only two of these are physical. In the usual Wigner theory [56], there can be only two physical polarization states, and these must be transverse. The Wigner theory does not appear to prohibit a mathematically imaginary longitudinal state of polarization, even for a *massless* particle.

The very concept of a *massless* particle, however, is fraught with great difficulty, because mass is a Lorentz invariant scalar [1,2]. A good text such as that by Marion and Thornton [57] explains clearly that, for this reason, mass does not change upon Lorentz frame transformation any more than it changes from one Cartesian frame to another. The term *rest mass* is misleading therefore because mass in the rest frame is the same as mass in any other Lorentz frame [57]. It is well known that mass is one of the two *invariants* of the Poincaré group [23], the other being loosely referred to as *spin*. The commonly held notion that mass must go to zero for a particle travelling at c is therefore a gross misconception. It is truer to assert that for a particle mathematically translating at c there is no rest frame, and so the meaning of transformation from one frame to another is lost. In other words, there is only one frame possible, and the Lorentz transformation is not defined from a frame in which a particle is moving at less than c to a frame in which it is moving at c, because if the mass is invariant, the transformation produces a zero in the denominator of some transformed quantities. A dynamical quantity can become mathematically infinite if Lorentz transformed into a frame moving at c and such a transformation cannot be physically meaningful. This is easily checked by identifying the relative frame velocity, v, appearing in the transformation [1,2] with c. Once the concept of massless particle is abandoned, derivative deductions such as a massless particle having two degrees of polarization also become physically meaningless. Therefore if a photon is indeed a particle (which is open to serious doubt [6]) it is massive and has three degrees of polarization in three dimensional space.

The artificial nature of the radiation gauge [47] is also discussed in Ref. 1 and 2 and Chap. 1. Its use in canonical quantization leads to serious difficulties because it is not fully covariant. The gauge is an assertion [47] made for convenience. In Dirac's

original development of his electron theory [26] it is not used, because the time-like component of A_μ is finite. The Roy-Evans condition (92) is always valid in the Dirac electron theory if

$$A^{(0)2}\left(e^{(1)}\cdot e^{(1)*}+e^{(2)}\cdot e^{(2)*}\right)\to A_0^2-A^{(3)2}\left(e^{(3)}\cdot e^{(3)*}\right), \tag{95}$$

and this is always possible, not surprisingly, because the Roy-Evans condition (92) is a limiting form of the Dirac condition itself [48,49],

$$A_\mu A_\mu=\text{constant}. \tag{96}$$

The condition

$$A_\mu A_\mu=0, \tag{97}$$

should not therefore be referred to as *light-like* in general because it can apply to *any* four-vector. This mathematical result shows that the concept of zero mass for a particle travelling at the speed of light is meaningless.

Our discussion of limits is summarized in Table 1, which is also a summary of the properties of fermion spin in a classical, circularly polarized, electromagnetic field. The Table shows that the intrinsic spin of the fermion always interacts with the $\boldsymbol{B}^{(3)}$ field in one form or another. The most general result is (I), and (II) and (III) are well defined limits of the same fundamental equation. Limit (II) is reached when the fermion energy is much greater than the energy ecA_0. In the radiation gauge, A_0 is zero by definition [47], in which case (I) and (II) become identical and (III) can never be reached because in relativity mc^2 is never zero. That $\boldsymbol{B}^{(3)}$ is much more than a mathematical convenience is revealed by a careful consideration of limit (III), in which the energy ecA_0 is much greater than the total fermion energy, including its rest energy, mc^2. In this limit,

$$ecA_0 \gg \left(En+mc^2\right)\sim 2mc^2. \tag{98}$$

TABLE 1

Limits from the Dirac Equation

	Limit	Description	$B^{(3)}$ Field
I	$(En + mc^2) \sim ecA_0$, $\hbar\kappa = eA^{(0)},\ eA_0$	Energy ecA_0 and fermion energy about the same.	$B^{(3)*} \sim -ix\frac{e}{\hbar}A^{(1)} \times A^{(2)}$
II	$(En + mc^2) > ecA_0$ $\hbar\kappa = eA^{(0)} > eA_0$	Initial fermion energy much greater than ecA_0.	$B^{(3)*} \to -i\frac{e}{\hbar}A^{(1)} \times A^{(2)}$
III	$ecA_0 > (En + mc^2)$, $\hbar\kappa = eA^{(0)} \to eA_0$	Energy ecA_0 much greater than initial fermion energy.	$B^{(3)*} \to -\frac{i}{B^{(0)}}B^{(1)} \times B^{(2)}$

$$x = 2mc^2/(En + mc^2 + ecA_0)$$

We have seen in Chap. 1 that in the limit (III), and in the quantized interpretation of the field, the photon energy $\hbar\omega$ is transferred wholly to the fermion for all practical purposes (F.A.P.P.),

$$\hbar\omega \to ecA_0, \tag{99}$$

and if

$$A_0 \to A^{(0)} := (\boldsymbol{A} \cdot \boldsymbol{A}^*)^{1/2}, \tag{100}$$

then

$$\boldsymbol{B}^{(3)*} \to -\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{101}$$

where

$$B^{(0)} = \frac{\omega}{c}A^{(0)}. \tag{102}$$

It is important to note that Eq. (101) *is a result of the Dirac equation*, provided that Eq. (98) and (99) are used as limits. They imply,

$$(\hbar\omega)_{\text{photon}} > (2mc^2)_{\text{fermion}}, \tag{103}$$

which can be interpreted physically to mean that the energy acquired by the fermion in a F.A.P.P. elastic collision with a photon is much greater than twice the fermion's initial rest energy. If we prefer to retain the classical interpretation of the field, Eq. (103) is

$$ecA_0 > 2mc^2. \tag{104}$$

Equation (101) has also been deduced heuristically [1], and implies that $\boldsymbol{B}^{(3)}$ is a property of the free photon — the photomagneton operator [1,2] of the quantized field. Table 1 shows that $\boldsymbol{B}^{(3)}$ in this limit can be isolated experimentally only under the appropriate conditions, also discussed in Vols. 1 and 2 [1,2]. These conditions must be such that the electromagnetic field is of very great intensity, so that its ecA_0 is much greater than the initial fermion energy. In order for this to be possible at all, we must follow Dirac [26] and use a *non-zero* A_0 in the theory. Furthermore, we must use $A^{(0)} \to A_0$ as in Eq. (100), and this is precisely the Roy-Evans condition [55], a limiting form of the Dirac condition [48,49] itself.

In classical electrodynamics [47] it is well known that the classical electron, when translating infinitesimally near c, radiates plane waves indistinguishable from transverse electromagnetic waves under this condition. This conclusion can be reached using special relativity [47]. Therefore, in quantum mechanics, if a fermion (electron) were to acquire $\hbar\omega$ as in limit (III), it would *become a photon*. In this state *it would retain the* $\boldsymbol{B}^{(3)}$ *field* defined in limit (III), a definition in which the electron's charge, e, has been incorporated into the scalar amplitude $B^{(0)}$. Conservation of C symmetry in this exchange has been discussed in Ref. 1; a symmetry conservation which requires, consistently, both e and $B^{(0)}$ to be C negative. Conservation of angular momentum [1-15] in classical physics also requires the $\boldsymbol{B}^{(3)}$ of the incoming electromagnetic field to be transferred to the classical electron, and in an elastic exchange the angular momentum is transferred entirely to the electron along with the $\boldsymbol{B}^{(3)}$ field. Any other conclusion for a finite ecA_0 would be in violation of the Dirac equation, because the latter conserves energy and angular momentum.

It can always be asserted that A_0 is zero, as is the accepted practice [47] in the transverse, or radiation, gauge, and this assertion can be used to force us to the flawed conventional wisdom that $\boldsymbol{B}^{(3)}$ *does not exist*. As discussed in Chap. 1, this is in violation of the principle of conservation of energy if it is assumed, reasonably, that a photon's incoming energy is $\hbar\omega$, and that under well defined experimental conditions this can be transferred, F.A.P.P. in its entirety, to a fermion in an elastic collision. However, this same assertion, that A_0 is zero, has caused fundamental difficulties [1,2,22—25] ever since the first attempts were made to apply the principles of canonical quantization [22] to the classical electromagnetic field. Therefore, an attempt to assert

that $B^{(3)}$ in limit (III) is zero on the grounds that A_0 must be zero by assertion will run into exactly the same problems. Such an assertion also spoils the symmetry of Table 1, in that it forces limit (I) to become the same as limit (II), and makes limit (III) impossible if it is accepted that rest energy, mc^2, can never be zero. It also means that A_μ, which is now accepted to have physical meaning in *classical* electrodynamics [58], must always have a time-like component which is zero. This cannot be true for a four-vector in general, because as soon as Lorentz transformation is applied, an initially zero time-like component can become non-zero in another Lorentz frame. It can only be true if the mass associated with A_μ is rigorously zero, so that Lorentz transformation *loses meaning* because there is no rest frame [1,57]. (Alternatively, we can think of a zero mass particle as implying the existence of only one Lorentz frame, so that Lorentz transformation is by definition prohibited.) However, we have already argued that the notion of a zero mass particle is physically *meaningless*, and the photon as particle should be interpreted as the originators of special relativity probably intended, as an aid to understanding. It has been pointed out repeatedly down the years [59,60] that the photon is an enigma. Much of the received wisdom about the photon as particle is flawed, as pointed out, for example, by Einstein [61]. Table 1, based on an elementary development of the Dirac equation of a fermion in the classical field, shows that the assertion that $B^{(3)}$ is zero [62—65] is incorrect.

The conservation of C symmetry in Table 1 leads to an interesting line of thought that will be pursued later in this volume. In agreement with Noether's Theorem [22—25] and the much older Ampère Hypothesis [66] C conservation shows that the symmetry of e and $B^{(0)}$ are both negative. The photon as an idea in natural philosophy must allow for elementary charge to be present in its theoretical framework. This is simply an expression of charge-current conservation and the contemporary orthodoxy that elementary charge is a conserved quantity in nature [22—25]. The notion that the photon is an uncharged particle [67] must therefore be based on the idea that the photon shows no *net* charge, because if charge is elementary and conserved in nature, it cannot be annihilated. In the received view of electron-positron annihilation [67] *two* photons are produced,

$$e^- + e^+ \rightarrow 2\gamma. \qquad (105)$$

The charge on the electron is elementary and negative, that on the positron is elementary and positive, so it is asserted usually that each of the two photons produced on the right hand side of equation (105) must be uncharged. This view means that the charge and mass present on the left hand side of Eq. (105) have disappeared from the right hand side, because conventionally, photons are also claimed to be massless particles [22—25] travelling at exactly the speed of light in vacuo, c.

If mass and charge are elementary and conserved in nature, however, they cannot be annihilated, which leads us to the view that the electron and positron combine to

form a single entity, a new type of photon, which translates infinitesimally close to c, and which is made up of a combination of plus and minus e. This new photon structure therefore rests on equal and opposite elementary charge and allows for the smallest possible unit of elementary mass to be present, the mass of the photon [68]. In this view rest energy is mc^2, as usual, but m can never be zero, because if so, rest energy would vanish entirely, in contradiction to the principles of special relativity. In the conventional view, mass *can* be zero by assertion, and in a *massless particle*, there is zero rest energy and no rest frame. In the conventional view, therefore, there is no notion of elementary mass (an indivisible, constant, universal and scalar unit of mass), because mass is allowed under certain circumstances to disappear. If so, there cannot be an indivisible and constant elementary unit, and in our view, this conclusion is unphysical. Mac Gregor [69] has recently discussed the history of the new type of photon which we wish to develop, a photon which can be shown to be consistent with relativity. For example, French [70] has shown that equal and opposite elementary charges, co-propagating exactly at c, can never attract, so that this photon can never collapse in on itself. In a photon with our very small elementary mass travelling infinitesimally near c, the attraction is very weak and only very gradually will the photon collapse, as it slows below c. This process becomes noticeable [68] only after the photon has travelled millions of light years. It is well known [68] that Tolman described this eventuality many years ago, and coined the term *tired light*. More contemporary texts such as that of Itzykson and Zuber also discuss photons with finite mass [23]. It will be shown later in this volume that this photon implies the existence of the $B^{(3)}$ field if one charge is situated at the origin and the other at the tip of the rotating electric or magnetic field vector of the classical electromagnetic field.

2.2 THE PAULI EQUATION AND OTHER LIMITS OF THE DIRAC EQUATION

By reducing the Dirac equation to the Pauli equation [22], it can be shown that the effect of $A \times A^*$ on a fermion is indistinguishable in general from that of a magnetic field, provided that the latter is interpreted properly as discussed in Sec. 2.1. The Pauli equation is obtained from the Dirac equation (23) as follows, in a non-relativistic approximation. Various limits of the Pauli equation are then discussed in this section.

We consider as starting point equation (23), with

$$En = p_0 c, \qquad (106)$$

and consider that $A_0 = A_0^*$ is the real, scalar, amplitude of the time-like component of the four-vector A_μ. In the transverse gauge this amplitude is zero by assertion, but

more generally it is non-zero, as used by Dirac [26]. Therefore Eq. (23) becomes

$$\left((En - mc^2 + ecA_0)(En + mc^2 + ecA_0) - c^2(p + eA^*)\cdot(p + eA)\right.$$
$$\left. - ec^2\hbar\sigma\cdot B^* - ie^2c^2\sigma\cdot A^*\times A\right)\psi = 0, \tag{107}$$

in which B^* is the oscillating magnetic component of the classical electromagnetic field, and in which $A\times A^*$ is phase free. This is a Dirac wave equation in which the eigenfunction is a four-spinor, ψ. It reduces to a Pauli equation in the non-relativistic limit in which $En \sim mc^2$, with the additional condition,

$$(En + mc^2) > ecA_0, \tag{108}$$

As we have seen, this condition is exact if we choose to work in the transverse gauge, but then we introduce difficulties in canonical quantization and energy conservation as discussed already.

In the limit (108) we obtain the Pauli wave equation in the form

$$(En - mc^2)\psi \sim \left(\frac{1}{2m}(p + eA^*)\cdot(p + eA) - ecA_0\right.$$
$$\left. + \frac{e}{2m}(\hbar\sigma\cdot B^* + ie\sigma\cdot A^*\times A)\right)\psi. \tag{109}$$

This is a wave equation in a four-spinor ψ, whose two two-spinor components are interchangeable by parity inversion [2,22—25]. It is similar in structure to the Schrödinger wave equation and its four components are Klein-Gordon equations [22—25]. The equation (109) differs from the standard Pauli equation only in one respect, that A is complex. Since A is accepted as complex in the literature [47], our new term $ie\sigma\cdot A^*\times A$ will lead in theory to novel resonance effects described later in this chapter.

Equation (109) is a Pauli equation for one fermion in a circularly polarized electromagnetic field, and as such, it is able to describe both the intrinsic spin ($2S$) and orbital (L) angular momentum of the electron through the well known factor $L + 2S$. There are several limits of the Pauli equation which are well discussed in the standard texts [22—25], but which are considered here in order to explore the effect of the new magnetic term in $\sigma\cdot A\times A^*$. The positive energy solutions of the terms

$$(En - mc^2)\psi = \left(\frac{1}{2m}(p + eA^*)\cdot(p + eA) - ecA_0\right)\psi, \tag{110}$$

give the classical Hamiltonian as an expectation value of this wave equation

$$H_{\text{class}} = \frac{1}{2m}(p + eA^*)\cdot(p + eA) - ecA_0. \tag{111}$$

The classical result (111) identifies ecA_0 as a kinetic energy and shows that the vector potentials A and A^* also contribute to the kinetic energy. The rest energy mc^2 has been incorporated into the factor two of the classical Eq. (111) in our non-relativistic limit. The first kinetic energy term in Eq. (111) can be expanded as

$$H_{\text{class}} = \frac{1}{2m}(p\cdot p + e^2A^*\cdot A + e(p\cdot A + A^*\cdot p)), \tag{112}$$

where p is the initial, non-relativistic, fermion linear momentum. Averaging over many cycles of the field leaves

$$\langle H_{\text{class}}\rangle = \frac{1}{2m}(p\cdot p + e^2A^*\cdot A), \tag{113}$$

an equation which has the same dynamical content as the equation first derived by Volkow in 1935, using a different route [22], of the Dirac electron in a linearly polarized plane wave. In Volkow's notation [22],

$$\overline{A^2} = A^*\cdot A = A^{(2)}\cdot A^{(1)}. \tag{114}$$

This checks that our own derivation of the Pauli equation and its above classical limit is consistent with literature derivations. The latter, however, miss the key new term in $A\times A^*$, which in our Pauli equation (109) multiplies the Pauli spinor and therefore gives rise to *two* energy states between which resonance can be made to occur. This is a perfectly general result that applies to all types of fermions, including nucleons, so that $A\times A^*$ can cause nuclear resonance as well as electron resonance.

If we reinstate this term into Eq. (110),

$$(En - mc^2)\psi = \frac{1}{2m}\left(\boldsymbol{p}\cdot\boldsymbol{p} + e^2(\boldsymbol{A}\cdot\boldsymbol{A}^* - i\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*) - ecA_0\right)\psi. \quad (115)$$

The term in $\boldsymbol{B}^*$ has been left out because it vanishes after averaging over many cycles of the field [47], $\boldsymbol{B}^*$ being a classical plane wave component, and not a static magnetic field as in the standard derivations of the anomalous Zeeman effect [22—25]. The term in $\boldsymbol{A}\times\boldsymbol{A}^*$, however, is non-zero after averaging over many field cycles, simply because it is free of the electromagnetic phase [1,2]. Equation (2) of Chap. 1 shows that it is directly proportional to a new type of magnetic field, $\boldsymbol{B}^{(3)}$. Using Eq. (2) reduces Eq. (115) to a new type of Pauli equation [22],

$$(En - mc^2)\psi = \left(\frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \frac{e}{2m}\hbar\boldsymbol{\sigma}^{(3)}\cdot\boldsymbol{B}^{(3)*} + \frac{e^2}{2m}\boldsymbol{A}\cdot\boldsymbol{A}^* - ecA_0\right)\psi, \quad (116)$$

in which the Evans-Vigier field, $\boldsymbol{B}^{(3)}$, is the magnetic field.

Obviously, $\boldsymbol{B}^{(3)}$ in this equation must be interpreted as in Eq. (2), and Table 1 defines the condition under which this interpretation is applicable. The Pauli equation (116) has been obtained from the Dirac equation (23), in which the standard property was used that $\boldsymbol{A}$ is complex [47] in general. The standard Pauli equation can be reduced [22] to the form,

$$(En - mc^2)\psi = \left(\frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} - \frac{e}{2m}(\boldsymbol{L} + 2\boldsymbol{S})\cdot\boldsymbol{B} - ecA_0\right)\psi, \quad (117)$$

in which $\boldsymbol{B}$ is the standard static magnetic field, where $\boldsymbol{L}$ denotes the electron's orbital angular momentum, a classical concept, and in which $2\boldsymbol{S}$ denotes its spin angular momentum with $\boldsymbol{S}$ defined as $\boldsymbol{\sigma}/2$. It is well known that $2\boldsymbol{S}$ does not appear in classical physics. The reduction of Eq. (116) to this general form can be achieved with the Hamilton-Jacobi formalism of Chap. 12 of Vol. 1, a formalism which allows the calculation of the orbital angular momentum of the fermion through the classical

$$L_Z = Xp_Y - Yp_X. \quad (118)$$

Here $\boldsymbol{p}$ is the linear momentum and $\boldsymbol{r}$ $(:= X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k})$ the position of the fermion in three dimensional space. The momentum components are given by [1],

$$p_X = eA^{(0)}\cos\omega t, \quad p_Y = -eA^{(0)}\sin\omega t, \quad (119)$$

and the position components by

$$X = -\frac{ecA^{(0)}}{\gamma\omega}\sin\omega t, \quad Y = -\frac{ec}{\gamma\omega}A^{(0)}\cos\omega t, \quad (120)$$

where

$$\gamma^2 := m^2c^2 + e^2A_0^2, \quad (121)$$

is the relativistic factor. In the non-relativistic limit [1,54],

$$\gamma \to mc, \quad (122)$$

and the angular momentum of the fermion in the field is

$$L_Z = \frac{e^2A^{(0)2}}{m\omega}. \quad (123)$$

Incorporating this into the Pauli equation (116) gives in the limit Eq. (122),

$$W\psi = \left(\frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \frac{1}{2}(1 + \boldsymbol{\sigma}^{(3)*}\cdot\boldsymbol{e}^{(3)})\omega L_Z\right)\psi. \quad (124)$$

The physical meaning of this equation is found through comparison with the work of Talin *et al.* [54], whereupon it becomes clear that the term in $\omega L_Z/2$ describes the *inverse Faraday effect* [1,2] for one fermion. Talin *et al.* [54] left out of consideration the spinor term in $(1/2)\boldsymbol{\sigma}^{(3)*}\cdot\boldsymbol{e}^{(3)}\omega L_Z$ because the conjugate product $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ was unconsidered in their classical theory. The quantum relativistic theory developed here for the first time reveals the existence of the spinor term.

If we use the definition (2) for the field $\boldsymbol{B}^{(3)}$ we obtain

$$L_Z = \frac{\hbar}{\omega}\frac{e}{m}B_Z, \quad \boldsymbol{B}^{(3)} := B_Z\boldsymbol{e}^{(3)}, \quad (125)$$

i.e., the angular momentum of the fermion is proportional directly to $\boldsymbol{B}^{(3)}$. The final

form of the Pauli equation is therefore

$$W\psi = \left(\frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \hbar\frac{e}{2m}(1 + \boldsymbol{\sigma}^{(3)*}\cdot\boldsymbol{e}^{(3)})B_Z\right)\psi, \tag{126}$$

which shows that the Evans-Vigier field is responsible for the inverse Faraday effect, and for a novel, as yet undetected effect due to the fermion's intrinsic spin. The following sections develop resonance theory based on the existence of this new term. We shall see that in theory, this term makes conventional NMR and ESR obsolete, because the $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ property of a radio frequency field can be substituted for the static magnetic field of a permanent magnet, one of the most costly parts of a contemporary NMR spectrometer.

It should be noted that despite the presence of the Dirac constant in Eq. (126), the $\boldsymbol{B}^{(3)}$ field has been defined as proportional to the inverse of $\hbar$, so the all important new term in $\boldsymbol{\sigma}^{(3)*}$ is, overall, independent of $\hbar$. It is not classical, however, because spinors do not appear in the classical understanding [54] of the inverse Faraday effect. This term is new to physics, and has many interesting properties, theoretical and practical.

2.3 RESONANCE EQUATIONS IN NOVEL NMR AND ESR SPECTROSCOPIES

Equation (115) shows that the intrinsic spin of a fermion forms an interaction energy with the optical property $\boldsymbol{A}\times\boldsymbol{A}^*$, and that the classical angular momentum of the fermion also contributes to this fundamental process of field-fermion interaction through $\boldsymbol{A}\cdot\boldsymbol{A}^*$. The latter process was first described by Talin *et al.* [54] when developing the classical theory of the inverse Faraday effect. The conjugate cross product $\boldsymbol{A}\times\boldsymbol{A}^*$ is inversely proportional to the square of the field angular frequency ω. These properties make it ideal for the development of powerful new resonance spectroscopies, and in this section the fundamental equations are given of the optical equivalent of NMR and ESR.

In S.I. units the fundamental equation linking $\boldsymbol{A}$ to a magnetic field $\boldsymbol{B}$ is, in classical electrodynamics [47],

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}. \tag{127}$$

So if $\boldsymbol{A}$ is a plane wave in vacuo then so is $\boldsymbol{B}$ (and its electric counterpart $\boldsymbol{E}$). If the plane wave $\boldsymbol{A}$ is a solution of the vacuum d'Alembert equation then it may be written

as

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}}(\boldsymbol{i}\boldsymbol{i} + \boldsymbol{j})e^{i\phi}. \tag{128}$$

From Eq. (127), the plane wave $\boldsymbol{B}$ is

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = \frac{\omega}{c}\boldsymbol{A}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(\boldsymbol{i}\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \tag{129}$$

and using the classical vacuum Maxwell equation, the plane wave $\boldsymbol{E}$ is

$$\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{i}\boldsymbol{j})e^{i\phi}. \tag{130}$$

Here ϕ is the electromagnetic phase [1,2], $A^{(0)}$, $B^{(0)}$ and $E^{(0)}$ are scalar amplitudes, and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit Cartesian vectors in X and Y, perpendicular to the propagation direction Z of the wave. The following key relations then follow using elementary algebra,

$$\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)} = \frac{c^2}{\omega^2}\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)} = \frac{1}{\omega^2}\boldsymbol{E}^{(1)}\times\boldsymbol{E}^{(2)}, \tag{131}$$

and show that the product $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ is proportional to $\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ divided by the square of the angular frequency. Expressing $\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ in terms of beam intensity or power density (I in W m^{-2})

$$\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)} = i\frac{\mu_0}{c}I\boldsymbol{e}^{(3)*}, \tag{132}$$

where μ_0 is the vacuum permeability in S.I. (Chap. 1).

The basis of the resonance phenomenon to be developed here is that a probe photon $\hbar\omega_{res}$ at a resonance angular frequency ω_{res} can be absorbed under the *resonance condition* [71],

$$\hbar\omega_{res} = \frac{e^2c^2B^{(0)2}}{2m\omega^2}(1-(-1)), \tag{133}$$

defined by a transition from the negative to positive states of the spinor $\sigma^{(3)}$. This process is precisely analogous to ordinary optical absorption, in which terms NMR is a radio frequency spectroscopy, and ESR a microwave spectroscopy. The probe resonance frequency (f_{res}) in this theory is therefore

$$f_{res} = \frac{\omega_{res}}{2\pi} = \left(\frac{e^2\mu_0 c}{2\pi\hbar m}\right)\frac{I}{\omega^2}, \tag{134}$$

and is inversely proportional to the square of the angular frequency ω of the circularly polarized, pump electromagnetic field which takes the place of the superconducting magnet of conventional NMR or ESR [71,27—31].

For ^{1}H (proton) resonance the result (134) is adjusted empirically in our development for the experimentally different Landé factors of the proton and electron [71], respectively 5.5857 and 2.002. A more complete theory must rest on the internal structure of the proton, and similarly for other fermions whose Landé factors differ from that of the electron. In principle the theory developed here gives rise to a means of investigating nuclear properties using readily available radio frequency generators instead of magnets.

For proton resonance our final equation is

$$\omega_{res} = \left(\frac{5.5857\, e^2\mu_0 c}{2.002\,\hbar m}\right)\frac{I}{\omega^2} = 1.532\times10^{25}\frac{I}{\omega^2}, \tag{135}$$

and some data from this equation are given in Table 2, which shows that it becomes possible in theory to practice NMR at much greater resolution than at present using much higher resonance frequencies. The latter can easily be chosen at will by changing the power density of the radiation being used to generate $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$. The relative separation between chemically shifted resonance lines [27—31,71] becomes much greater at, say, visible frequencies than the radio frequencies accessible in contemporary apparatus [27—31], and so the instrumental resolution is increased dramatically. This is the major advantage of our technique in chemical physics, in which contemporary NMR is already used extensively. In ESR the same overall advantage obtains, the equivalent of equation (135) becomes

$$\omega_{res} = 1.007\times10^{28}\frac{I}{\omega^2}, \tag{136}$$

TABLE 2

Resonance Frequencies from Equation (135)

Pump Frequency	$\boldsymbol{B}^{(3)}$	Resonance Frequency
5,000 cm^{-1} (visible)	64.5 nT	0.28 Hz
500 cm^{-1} (infra-red)	6.45 μT	28.0 Hz
1.8 GHz (microwave)	448 T	1.8 GHz*
1.0 GHz (microwave)	1.45 kT	6.18 GHz
0.1 GHz (r.f./micr.)	145 kT	20.6 cm^{-1} [a]
10.0 MHz (r.f.)	14.5 MT	2,060 cm^{-1} [b]
1.0 MHz (r.f.)	1.45 GT	206,000 cm^{-1} [c]

*. auto-resonance, at which the resonance frequency is the same as the applied field frequency
a. far infra-red
b. infra-red
c. ultra-violet

and it is possible, in principle, to practice ESR without magnets at any convenient frequency, say in the visible. In this design the probe radiation can be broad band radiation from the source of a contemporary Fourier transform infra-red spectrometer [40]. The spectrum picked up by the interferometer is then the ESR (or NMR) spectrum, giving a great resolution advantage over contemporary NMR spectra laboriously gathered in the radio frequency range. Fine details of NMR and ESR spectra, currently obscured by lack of resolution, become easily observable in theory using the interaction energy between $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ and the third Pauli spinor $\sigma^{(3)*}$. This novel term is the key that nature provides through fundamental topology [1,2,22].

The magnetic field in Table 2 is calculated directly from Eq. (2) expressed in terms of field intensity and frequency,

$$\boldsymbol{B}^{(3)} = \frac{e\mu_0 c}{\hbar}\frac{I}{\omega^2}\boldsymbol{e}^{(3)} = 5.723\times10^{17}\frac{I}{\omega^2}. \tag{137}$$

The flux density in tesla from this equation is tabulated for various frequencies for an illustrative power density I of ten watts per square centimeter (10^5 W m^{-2}). At visible frequencies it is in the nanotesla range, ten thousand times weaker than the Earth's magnetic field. This produces NMR resonance in the hertz range (Table 2), roughly the same as some experimental data obtained by Warren *et al.* [20], data which support the general validity of our simple one fermion theory.

For a 1.0 GHz *pump field* (by which term we mean the field that produces $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$) the magnetic field defined in Eq. (2) is in the kilotesla range. A 10 MHz pump field of intensity 10 watts per square centimeter produces NMR resonance in the

infra red, in which range it can be picked up in principle by the probe field of an ordinary Fourier transform infra red (FT IR) spectrometer. If this advantage can be realized in practice it would make contemporary NMR obsolete in the sense that expensive superconducting magnets would no longer be needed, and it would be possible to pick up the nucleon resonance with ordinary absorption spectroscopy. As we have seen, these major developments in practice are the result of the Dirac equation, i.e., of rigorous relativistic quantum theory based on the fundamental principles of relativity and quantum mechanics. These principles have been known and used for over sixty years, our contribution has been to use a complex $\boldsymbol{A}$ instead of a real $\boldsymbol{A}$, and to realize that the spinor interacts with $\boldsymbol{A} \times \boldsymbol{A}^*$.

There are other major advantages of the new resonance technique which can be sketched using simple Maxwell-Boltzmann statistics as follows. If the relevant energy of interaction of field and fermion is taken to be, from Eq. (115)

$$\Delta En(m) = En(m+1) - En(m) = \frac{e^2 \mu_0 c}{m} \frac{I}{\omega^2}, \tag{138}$$

then the relevant Maxwell-Boltzmann exponential is

$$\exp\left(-\frac{\Delta En}{kT}\right) \sim 1 - \frac{\Delta En}{kT} + \ldots, \tag{139}$$

and the population ratio [75] of the up and down states of the spinor is

$$\frac{N(m)}{N(m+1)} \sim 1 - \frac{e^2 \mu_0 cI}{mkT}. \tag{140}$$

For a radio frequency-microwave range field of intensity 100 watts per square centimeter at 100 MHz at 293 K, this ratio is 0.87, indicating a 13% difference in spin population between the two fermion states. This compares with about seven parts per million for a contemporary NMR instrument at 293 K, 1.0 tesla magnet. An elementary population analysis reveals another advantage of our new technique, which for want of a more evocative description we refer to as radio frequency NMR (RF-NMR). The advantage is that the applied radio frequency field does not have to be manufactured to any great degree of homogeneity because we can tolerate fluctuations in the 10% range.

In summary therefore, the Dirac equation of one fermion (a proton or an electron for example) in a classical electromagnetic field produces a powerful new spectroscopy which if developed makes obsolete our contemporary NMR and ESR technology. It is convenient to understand the magnetic nature of the new phenomenon through the Evans-Vigier field defined in equation (2) of Chap 1. It is equally straightforward to understand it in terms of the product $\boldsymbol{A} \times \boldsymbol{A}^*$, which forms an interaction energy with

the intrinsic fermion spin, a famous concept in twentieth century physics. In the next chapter we discuss the impact of this term on the Aharonov-Bohm effects, which typify the advances being made in contemporary fundamental field theory.

Chapter 3. Optical Aharonov-Bohm Effects of $A \times A^*$

It has been argued that $A \times A^*$ is a physically observable optical property, which forms an interaction energy with a Pauli spinor, an energy which appears as a term in the Dirac equation. The existence of $A \times A^*$ is a direct consequence of the first principles of relativity and quantum mechanics, because these form the basis of Dirac's original work [26]. Therefore the physical cross product $A \times A^*$ must be gauge invariant [1,2], and its proportionality to the physical $B^{(3)}$ ensures this. In Vol. 2, it was shown [2] that Eq. (2) is, indeed, a consequence of gauge theory provided that this is developed within the O(3) group. This non-Abelian theory of vacuum electrodynamics produces Eq. (2) in what seems to us to be a very natural way, and this development finds a clear echo in the Dirac equation, as we have seen. The heuristic arguments of Vol. 1 and 2 [1,2] were used to propose the existence of an optical equivalent of the Aharonov-Bohm effects [32—37], in which $B^{(3)}$ is the magnetic field. In this chapter we apply relativistic quantum theory to develop the optical Aharonov-Bohm effect (OAB) from the Dirac equation.

The Aharonov-Bohm effects [32—37] are central to contemporary field theory, because they show that the vector potential is physically meaningful in classical electrodynamics. Excellent discussions of the effects are found in texts such as that of Ryder [23], who develops the ideas of non-trivial vacuum topology from the original paper of Aharonov and Bohm [37]. Bohm *et al.* have provided a concise summary [72] quoted as follows:

> "If the price of avoiding non-locality is to make an intuitive explanation impossible, one has to ask whether the cost is not too great."

This statement means that the experimentally verified existence of effects due to the vector potential implies action at a distance in classical electromagnetism. For example, the gauge transformation into the vacuum of the vector potential, A through whose curl we define B, produces another vector potential which is able to affect the trajectory of an electron matter wave. The trajectory is changed physically even though the original field, B, is not present. Therefore action at a distance has occurred and has been observed experimentally. The topology of the vacuum itself [23] must therefore be such as to be able to sustain this physical phenomenon due to gauge *transformation*.

In this chapter we examine whether such an effect exists due to the generation

of $\boldsymbol{A} \times \boldsymbol{A}^*$ by the Dirac equation for a quantized fermion matter wave interacting with a classical electromagnetic wave, and find that in theory it does, being proportional to I and inversely proportional to the *cube* of the angular frequency, ω, of the electromagnetic field used to generate $\boldsymbol{A} \times \boldsymbol{A}^*$. We continue to maintain a classical description of the field, but there are undoubtedly effects of this kind due to quantum field theory. Perhaps there are non-classical effects akin to light squeezing which require careful statistical development [73].

3.1 VACUUM TOPOLOGY NEEDED FOR AN OAB EFFECT

The $\boldsymbol{B}^{(3)}$ field manifests itself in interaction with matter as described by the Dirac equation and contemporary developments thereof [22—25]. The equation is fairly simple in structure and can be solved exactly (Chaps. 1 and 2) when matter is represented by a single fermion, more accurately the de Broglie matter wave of a single fermion. The existence of an OAB effect due to $\boldsymbol{B}^{(3)}$ would mean that the group of vacuum electromagnetism is more naturally O(3), because $\boldsymbol{B}^{(3)}$ is longitudinal, i.e., is an axial vector in the axis of propagation. This makes the transverse gauge seem to be an unnatural assertion of classical electrodynamic theory [47] rather than a property of nature itself. It is clearer to think of electromagnetic waves propagating in vacuo as producing a quantized flux vortex [23], which is identifiable with $\boldsymbol{B}^{(3)}$ [1,2]; a vortex which is stabilized by the doubly connected vacuum topology of the group space of O(3) [1,2,23]. The interaction of $\boldsymbol{B}^{(3)}$ with one fermion then takes place as in Table 1, which defines limits which again emerge from the Dirac equation itself. Under experimentally accessible conditions the analysis summarized in Table 1 shows that $\boldsymbol{B}^{(3)}$ must be interpreted through Eq. (2), i.e., through the conjugate cross product $\boldsymbol{A} \times \boldsymbol{A}^*$. The latter therefore generates an optical Aharonov-Bohm effect (OAB) if $\boldsymbol{B}^{(3)}$ is a physical magnetic field.

This is an understanding based on relativistic quantum theory and is an advance on the heuristic theory of $\boldsymbol{B}^{(3)}$ pursued in Vol. 1 and 2. That theory must be interpreted in appropriate limits of the Dirac equation and is the high field limit, the applied electromagnetic field is of such very high intensity as to accelerate the fermion infinitesimally close to the speed of light. In the opposite case it has now become clear that Eq. (2) follows from the Dirac equation in the low field limit, and the Dirac equation must be solved under the circumstances appropriate to the sample being used. For example, if the sample has no free electron spin, Eq. (2) is not directly applicable. In such circumstances a non-relativistic, semi-classical theory such as that of Woźniak, Evans and Wagnière [42] must be developed with the Dirac equation, a completely non-trivial task, even for the fermions of the hydrogen atom. The most appropriate methods to use are based on RF-NMR as developed in the previous chapter. Experiments such as those of Rikken [74], carried out under inappropriate circumstances, will, unsurprisingly, produce a negative result. To interpret such a result as implying the non-existence of $\boldsymbol{B}^{(3)}$ implies a rejection of the Dirac equation itself.

Vacuum topology is now thought to be highly non-trivial, and to be responsible for the existence of the various Aharonov-Bohm effects now known [32—37]. They can exist only in certain types of group space, such as the Abelian U(1) or non-Abelian O(3). They cannot exist in the SU(2) group space because the latter is singly connected. The topological arguments to date have been applied to the original Aharonov-Bohm effect, and have assumed that the group space of vacuum electromagnetism is the Abelian U(1) = O(2). In Vol. 2 [2] however, we have argued that electromagnetism in vacuo exists in the non-Abelian O(3) group space, because of the existence in vacuo of the new field $\boldsymbol{B}^{(3)}$. Equation (2) was derived [2] using O(3) gauge theory, and e interpreted in vacuo as a constant of this non-Abelian gauge theory, meaning that the momentum magnitude of a photon in the vacuum is both $\hbar\kappa$ and $e\boldsymbol{A}^{(0)}$. The vacuum Maxwell equations are written down [47] with source terms missing, the charge and current being located infinitely far away from the field at a given point in phase space. Despite this, the origin of the fields (an oscillating charge at infinity) betrays itself in the $\hat{C}$ negative symmetry [1] of the amplitudes $\boldsymbol{B}^{(0)}$ and $\boldsymbol{E}^{(0)}$, so charge is, after all, present in the vacuum Maxwell equations. This is a consequence of the conservation of elementary charge, e, i.e., of Noether's theorem and Ampère's hypothesis.

Conservation of charge is the *origin* of the minimal prescription [22—25] through type two gauge invariance, and the minimal prescription defines the additional energy-momentum, $e\boldsymbol{A}_\mu$, acquired by a charged particle in a classical field. If $e\boldsymbol{A}_\mu$ is such as to accelerate the fermion infinitesimally close to c, the fermion becomes essentially indistinguishable from a photon, and F.A.P.P., complete transfer of energy-momentum has occurred from field to fermion. If so, the momentum acquired by the fermion must be both $e\boldsymbol{A}$ and $\hbar\boldsymbol{\kappa}$. This allows the $\boldsymbol{B}^{(3)}$ field in vacuo to be defined in two ways

$$\boldsymbol{B}^{(3)*} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = -\frac{i}{\boldsymbol{B}^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{141}$$

and this definition is the high field limit. It is easily checked that the identity in Eq. (141) produces

$$e\boldsymbol{A}^{(0)} = \hbar\boldsymbol{\kappa}, \tag{142}$$

which was derived independently in Vol. 2 [2] as the charge quantization condition. It is a high field limit because the field intensity needed to accelerate a fermion infinitesimally close to the speed of light is enormous. It demonstrates clearly however that e can be a field property, because it is simply the charge on the fermion accelerated infinitesimally close to c. Under these conditions the transverse gauge does not apply,

as argued in Chaps. 1 and 2, because the highly accelerated fermion must have a time-like component in its energy-momentum four-vector, i.e., must have energy as well as momentum, and part of this energy must be its original rest energy, an invariant of the Poincaré group for a given fermion mass. In other words eA_μ must be a physical energy-momentum, as indicated by the Aharonov-Bohm effects [32—37].

In the opposite limit (II) of Table 1, when the electromagnetic field's intensity is such that the initial fermion energy $En + mc^2$ is much greater than ecA_0, the Dirac equation shows that $\boldsymbol{B}^{(3)}$ must be interpreted as Eq. (2), because the fermion energy-momentum does not begin to resemble that of a photon. In this limit, the identity in Eq. (141) *does not apply*, and the transverse gauge can be used as an aid to calculation, but not, in our opinion, as anything that is physically meaningful. If eA_μ is physical in nature, then in general its time-like component is not zero, as asserted in the transverse gauge. In general however, when a photon collides with a fermion as in the Compton effect [45] a *fraction* of $\hbar\kappa$ is transferred in an inelastic collision, and this can always be put equal to $e\boldsymbol{A}$. There then occurs the well known Compton frequency shift, indicating a loss of momentum by the photon. This is a gain of $e\boldsymbol{A}$ by the fermion according to the minimal prescription. In this sense, the charge quantization condition *always* applies because it is a consequence of Noether's theorem and the quantum hypothesis. Therefore $\boldsymbol{B}^{(3)}$ as defined by Eq. (2) in this situation is a fraction of the free space $\boldsymbol{B}^{(3)}$.

The simplest theory of electron-photon collision [45] can be used to illustrate the above discussion with equations. Consider the traditionally massless photon colliding with an initially stationary electron of rest energy mc^2. The electron acquires momentum (p) and its final energy is [45] $(p^2c^2 + m^2c^4)^{\frac{1}{2}}$ in the observer's frame of reference. The change in energy of the electron is, according to the minimal prescription [1,2,22—25], ecA_0. Therefore

$$ecA_0 = (p^2c^2 + m^2c^4)^{\frac{1}{2}} - mc^2 \tag{143}$$

is the energy gained by the electron, where

$$\boldsymbol{p} = e\boldsymbol{A}, \qquad p^2 = e^2\boldsymbol{A}\cdot\boldsymbol{A}^*. \tag{144}$$

The energy lost by the photon is the Compton frequency shift in its simplest form, $\hbar(\omega_i - \omega_f)$. So,

$$ecA_0 = \hbar\Delta\omega = \hbar(\omega_i - \omega_f), \tag{145}$$

where ω_i and ω_f are the initial and final angular frequencies. Conservation of energy therefore leads to

$$\hbar\Delta\omega = (e^2c^2\boldsymbol{A}\cdot\boldsymbol{A}^* + m^2c^4)^{\frac{1}{2}} - mc^2, \tag{146}$$

which, for the electron, is a classical but relativistic equation with no spinors. The left hand side is a quantum mechanical representation of the change in electromagnetic energy, whereas the right hand side contains a classical representation of the same thing through the minimal prescription. Equation (146) is therefore an expression of the equivalence principal, as well as the result of conservation of energy.

The initial electron momentum is zero and its final momentum is $\boldsymbol{p} = e\boldsymbol{A}$. The momentum gained by the electron is the same as that lost by the photon, $\hbar(\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_f)$. Conservation of momentum leads to

$$\hbar\Delta\boldsymbol{\kappa} := \hbar(\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_f) = e\boldsymbol{A}. \tag{147}$$

Physically, this equation means that the (classical) electron is put into a helical trajectory [75] by a travelling, circularly polarized, plane wave. The Lorentz force on the electron is

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \tag{148}$$

where

$$\boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t} - c\nabla A_0, \tag{149}$$

is the rotating electric field component of the plane wave and where

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \tag{150}$$

is the rotating magnetic field component. Here $\boldsymbol{v}$ is the velocity of the electron in the field. The transverse momentum, $\hbar\boldsymbol{\kappa}$, of the photon is used on the left hand side of Eq. (147) because $e\boldsymbol{A}$ is the classical transverse momentum of the electron in its helical trajectory. Again, the intrinsic spin of the electron has not been considered in Eq. (147).

Equation (142) is recovered from Eq. (147) if $\boldsymbol{\kappa}_f$ is zero, and where the amplitude of $\boldsymbol{\kappa}_i$ is the scalar $\kappa = \omega/c$ of the free photon. The scalar magnitude $\boldsymbol{A}^{(0)}$ in Eq. (142)

is

$$A^{(0)} = (\boldsymbol{A} \cdot \boldsymbol{A}^*)^{\frac{1}{2}}. \tag{151}$$

In these calculations, $\boldsymbol{A}$, $\boldsymbol{E}$ and $\boldsymbol{B}$ are complex plane waves of the circularly polarized classical field in vacuo, given by Eqs. (128) to (130) respectively. Equation (142) is therefore the limit in which all the photon's transverse momentum is given to the electron. If so, the free photon has *also* lost all its energy, the angular frequency of the incoming field has been changed from ω_i to 0, and the photon (the field's quantum of energy) has been lost entirely, i.e., *annihilated*. In this limit Eq. (146) becomes

$$\hbar\omega := \hbar\omega_i = (e^2c^2\boldsymbol{A} \cdot \boldsymbol{A}^* + m^2c^4)^{\frac{1}{2}} - mc^2, \tag{152}$$

and this can be consistent with Eq. (142) if and only if

$$A_0 \to A^{(0)} = (\boldsymbol{A} \cdot \boldsymbol{A}^*)^{1/2} \gg \frac{mc}{e}. \tag{153}$$

This is limit III of Table 1 when we make the theory more precise through use of the Dirac equation of an electronic matter wave interacting with a classical electromagnetic wave. Obviously, Eq. (141) applies only in limit (153), but nevertheless shows that $\boldsymbol{B}^{(3)}$ emerges from the Dirac equation in this limit in the form

$$\boldsymbol{B}^{(3)*} \to -\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}. \tag{154}$$

The $\boldsymbol{B}^{(3)}$ field is therefore fundamental in nature [1,2]. The electromagnetic A_μ of the plane wave always interacts with one fermion as if the electromagnetic plane wave were a magnetic field $\boldsymbol{B}^{(3)}$. Since this is a fundamental interaction between field and fundamental particle, it occurs in matter such as atoms and molecules.

Using $A^{(0)} = cB^{(0)}/\omega$, Eq. (153) becomes

$$B^{(0)} \gg \frac{m}{e}\omega, \tag{155}$$

which is Eq. (411) of Vol. 1 [1], obtained from an independent analysis based on the classical but relativistic Hamilton-Jacobi equation of the classical electron in the field. Equation (411) of Vol. 1 becomes recognizable in its quantized form as limit III of

Table 1. Using $I = cB^{(0)2}/\mu_0$ shows that Eq. (155) is equivalent to

$$I \gg \left(\frac{cm^2}{\mu_0 e^2}\right)\omega^2 = 7.72 \times 10^{-9}\omega^2, \tag{156}$$

for the electron. For the proton

$$I \gg 0.026\,\omega^2. \tag{157}$$

We refer to these as equations of *the strong field limit*. For a visible frequency laser as used by Rikken [74], with $\omega = 1.77 \times 10^{16}\ rad\ s^{-1}$ (10,640 cm^{-1}), Eq. (156) gives

$$I \gg 2.4 \times 10^{22}\ W\,m^{-2}. \tag{158}$$

The peak intensity used in the experiment [74] was only $5.5 \times 10^{12}\ W\,m^{-2}$ and the sample used was benzene, with no free electrons. Under these conditions, Eq. (141) does not apply because: 1) the field intensity is at least ten orders of magnitude too low; 2) there are no free electrons.

In the opposite limit, limit II of Table 1, or Eq. (414) of Vol. 1 [1], the field energy ecA_0 is much smaller than the rest energy mc^2 of the fermion. In the transverse gauge it is asserted [47] that A_0 is zero, in which case Eq. (145) shows that there can be no Compton effect, contrary to experiment [71]. In the limit $A_0 \to 0$, however, $\omega_f \to \omega_i$, and there can be a small but non-zero Compton shift. From Eq. (146) in this limit,

$$mc^2 \gg e^2\boldsymbol{A} \cdot \boldsymbol{A}^*, \tag{159}$$

which in terms of intensity becomes

$$I \ll \left(\frac{cm^2}{\mu_0 e^2}\right)\omega^2. \tag{160}$$

We refer to this, purely for convenience, as *the weak field limit*. A glance at Eq. (160) shows that most, if not all, experiments in magneto-optics to date have been carried out well within the weak field limit, under which $\boldsymbol{B}^{(3)}$ is defined through Eq. (2) as a result

of the Dirac equation. Even in this limit, however, the experimental results by Rikken [74] cannot be compared with Eq. (1), because the sample used, benzene, has no free electron spin. There *is* however, an optical (*B term*) Faraday effect in benzene due to perturbation of the polarizability by $\boldsymbol{A} \times \boldsymbol{A}^*$ [76—78]. The experiment failed to detect this effect, but it had been detected already by Flytzannis *et al.* [79] under more appropriate conditions, and was first predicted by Kielich [80]. Our demonstration of the origins of the fundamentally *non*-zero $\boldsymbol{A} \times \boldsymbol{A}^*$ in relativity (Chaps. 1 and 2) shows the incorrectness of any claim that $\boldsymbol{B}^{(3)}$, as defined by Eq. (2), is zero. The only way to sustain such a view is to assert, arbitrarily, that the left and right hand sides of Eq. (2) cannot be identified. There is no known symmetry argument or dimensionality argument that supports this assertion, and no known experimental data.

We have therefore illustrated the emergence of the condition $e\boldsymbol{A}^{(0)} = \hbar\kappa$ in the strong field limit. This relation was obtained independently in Vol. 2 [2] from O(3) gauge theory. The primary theme of this section is to show that vacuum topology [23] allows the existence of the OAB in the O(3) group space as well as that of U(1). In order for any Aharonov-Bohm effect to exist, vacuum topology must define an appropriate group space, a space that cannot be singly connected [23]. The fundamental topological arguments have been applied to date mainly to the original Aharonov-Bohm effect [23,32—37] and have assumed that the group space of vacuum electromagnetism is the Abelian U(1) = O(2). In both the weak and strong field limits, however, there exists a magnetic field, $\boldsymbol{B}^{(3)}$, perpendicular to the Abelian plane defining $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$. Self-consistently, therefore, relations such as Eq. (2) must be considered in group theory as being *non*-Abelian, because they contain three space indices, (1), (2) and (3). Proceeding on this basis, Eq. (2) was derived in Vol. 2 [2] from general O(3) gauge theory [23], with ideas adapted from general relativity.

Within the O(3) group space in vacuo the field equations of Evans and Vigier [1,2] supplement the Maxwell equations in vacuo with terms quadratic in $\boldsymbol{A}_\mu$, of which Eq. (2) is an example. The theory [2] shows that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is gauge invariant within the O(3) group space. Within the U(1) = O(2) group space [23] $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is still gauge invariant but is not put equal to a magnetic field in axis (3). This is a key conceptual difference between Evans-Vigier and Maxwell electrodynamics in vacuo, and leads to many interesting ramifications in the theory of finite photon mass [1,2]. It also has consequences in the elementary (i.e., fundamental) nature of charge, e, because in the O(3) gauge e can be a *field* property. We have illustrated this by accelerating a charged fermion infinitesimally close to the speed of light, whereupon it takes the properties of a photon, F.A.P.P. This idea has been neatly illustrated by Jackson [47] in classical electrodynamics, and can be traced [81] to the time of Bragg, Bateman and contemporaries.

In the conventional U(1) gauge theory of electromagnetism in vacuo [22—25], there is also to be expected an OAB, because $\boldsymbol{A} \times \boldsymbol{A}^*$ and $\boldsymbol{A} \cdot \boldsymbol{A}^*$ exist in U(1) as well as O(3). As we have shown, however, the U(1) group space does not allow the left and

right hand sides of Eq. (2) to be identified. This is because fields are defined in a plane in U(1), and $\boldsymbol{B}^{(3)}$ is a field that is obviously not in this plane, being perpendicular to it. However, there is nothing in fundamental symmetry [1] or dimensionality that prohibits the identity in Eq. (2), which asserts, quite naturally, that magneto-optical effects are magnetic in nature. The equation (2) emerges rigorously as a *field equation* [2] in the group space O(3) and so $\boldsymbol{B}^{(3)}$ is a property of this group space. Experimentally, the magnetic effect of $\boldsymbol{B}^{(3)}$ can be distinguished from that of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in the strong field limit, because the effect becomes proportional to the square root of I in this limit, as argued in Vol. 1 and 2 [1,2]. Evidence for photon mass [82] is also evidence for $\boldsymbol{B}^{(3)}$. In the weak field limit, the magnetic effect of $\boldsymbol{B}^{(3)}$ must be understood in terms of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and is proportional to beam intensity I, as known experimentally [16—21]. All magneto-optic experiments to date have been carried out well within the weak field limit and the I proportionality always found in consequence.

Therefore we expect that the OAB (as yet unobserved) will be due to $\boldsymbol{B}^{(3)}$ defined in the weak field limit by Eq. (2) because this is the only experimentally accessible limit at present. The only consistent way to view a positive result would be to assert that vacuum electromagnetism is non-Abelian in nature, and described within the O(3) group space, because Aharonov-Bohm effects are always ascribed to a *magnetic* field's vector potential [32—37]. The theory of vacuum topology [23] must allow this result therefore. Topological arguments are existence arguments, giving very general conditions which must be fulfilled in order that solutions similar to $\boldsymbol{B}^{(3)}$, such as solitons or stable vortices [23] can exist. Vacuum topology is thought to set the stage upon which any subsequent argument must be played out. They show for example [23] that string-like solutions to spontaneously broken gauge theories [2] cannot exist in a group space such as SU(2) because $\pi_1(S^3)$ is trivial, where the group space of SU(2) is S^3 and where π_1 is the first homotopy group. The mapping $\pi_1(S^3)$ is trivial because every closed curve S^1 on S^3 may be shrunk to a point: boundary conditions may be shrunk to a trivial constant condition ϕ = const., and no stable vortices [23] exist. It follows that no field equations can produce the stable vortex $\boldsymbol{B}^{(3)}$ within the group space of SU(2); and conversely if $\boldsymbol{B}^{(3)}$ exists in vacuo, the group space needed to sustain it cannot be that of SU(2).

The O(3) group on the other hand is doubly connected topologically [23]. There is a 2 : 1 mapping of SU(2) on to O(3), and the group space of O(3) is obtained from that of SU(2) by identifying opposite points on the three space, S^3, since these points correspond to the same O(3) transformation. The group space of O(3) being doubly connected, there are two types of closed path S^1, those homotopic to a point and those homotopic to a line. The O(3) group space sustains, in consequence, one non-trivial, stable, vortex [23], which in the Evans-Vigier field equations becomes [2] $\boldsymbol{B}^{(3)}$. The O(3) group space is a vacuum group space, so $\boldsymbol{B}^{(3)}$ is sustained topologically in vacuo.

The vacuum in this contemporary view is a complicated entity, anything but trivial in nature.

The Evans-Vigier field equations [1,2] are also those of particles, through the de Broglie principle which is the basis of matter waves [83]. The O(3) group is that of particles with integral isospin, the SU(2) that of particles with half-integral isospin. In the field equations of Vol. 1 and 2, the isospin indices are identified with the circular indices, (1), (2) and (3), of three dimensional space itself. The non-trivial vortex line sustained by O(3) is $\boldsymbol{B}^{(3)}$ itself and there is a non-Abelian OAB effect, meaning that solitons [23] exist in O(3). Therefore the existence and stability of $\boldsymbol{B}^{(3)}$ is ensured by the doubly connected O(3) vacuum topology. The particles with integral isospin become photons with mass, and the Evans-Vigier field equations become equations describing three dimensional photons with mass, such as the Proca field-particle equation [1,2]. Thus, $\boldsymbol{B}^{(3)}$ is a solution of the vacuum Proca equation [1,2].

In the conventional U(1) framework, in which photons are asserted to have identically zero mass, and to propagate in vacuo identically at c, solitons are also considered to create a vortex line *perpendicular* [23] to the U(1) plane. This view is obviously inconsistent with itself if $\boldsymbol{B}^{(3)}$, a physical field, can be described in terms of this vortex line in vacuo. The Lagrangian used conventionally in building up this result is that of the Higgs model [2,23] with spontaneous symmetry breaking (SSB). It is structurally identical with the famous Landau-Ginzburg Lagrangian that leads to the Proca wave equation itself. The soliton flux, if identifiable with $\boldsymbol{B}^{(3)}$, must be a *quantized* flux in vacuo, or in matter must be exemplified by observable quantized flux lines such as those of Abrikosov in type II superconductors. The *same* Lagrangian is used for superconductor and vacuum flux, but it is asserted [23], arbitrarily in our view, that the superconductor flux (Abrikosov line) exists but the vacuum flux ($\boldsymbol{B}^{(3)}$) does not. Ryder, for example, gets out of this quandary by stating blandly that a superconductor is not a vacuum [84]. The problem, we suggest, is that fields exist in both media and must be described consistently in both, especially if we are using the same Lagrangian to do so.

In summary therefore, the conventional theory, although a flat U(1) theory, produces the Proca equation, which is the wave equation for a boson with mass. It is nevertheless asserted that the boson known as the photon is massless identically, an assertion which is inconsistent with the existence of a stable vortex line in vacuo *perpendicular* to the U(1) plane. The O(3) Evans-Vigier field equations in contrast are free of this logical strain in that they recognize that $\boldsymbol{B}^{(3)}$ can be an intrinsic component of vacuum electromagnetism, whatever that may turn out to be in future. Finally, Roy and Evans [50] have adapted the Dirac condition [48,49] to show that the existence of photon mass can be reconciled straightforwardly with gauge invariance of the second kind (which we accept because it is the result of a fundamental conservation law of physics, the Noether theorem). As described already the Roy-Evans condition leads to

$$A_\mu A_\mu \xrightarrow[F.A.P.P.]{} 0, \tag{161}$$

and this condition can be satisfied by $\boldsymbol{B}^{(3)}$ in *both* the high and low field limits. There appears to be no internal inconsistency in the theory of $\boldsymbol{B}^{(3)}$.

3.2 THE ORIGIN OF THE OAB

In Chap. 1 and 2 the Dirac equation was used to show that there is a novel interaction energy (En_2) generated between $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ and the intrinsic fermion spin, represented as usual by a Pauli spinor, $\boldsymbol{\sigma}^{(3)}$. This interaction produces two energy levels between which resonance can be made to occur. The energy En_2 divided by ω has the units of angular momentum, which are also the units of action [1,2]. The action En_2/ω divided by $\hbar$ is unitless, and as we shall show, is responsible for the OAB through a phase shift in the matter wave of a fermion such as an electron. This phase shift occurs in regions where there is no electromagnetic radiation and no $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and is due to a gauge transformation of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ into the structured vacuum, a vacuum whose topology is not trivial. Accordingly, the expected phase shift is proportional to I/ω^3 where I is intensity and ω is beam angular frequency, because En_2 is proportional to I/ω^2 as we have seen. The OAB should therefore be much more readily observable at low frequencies and depends for its existence on non-locality in the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ that defines the Evans-Vigier field in the accessible weak field limit. Therefore electromagnetic radiation acts at a distance because of the structure of the vacuum: the OAB is caused by a gauge transformation, in other words, of the type

$$\boldsymbol{A}^{(1)} \to \boldsymbol{A}^{(1)} + \nabla^{(1)}\chi, \qquad \boldsymbol{A}^{(2)} \to \boldsymbol{A}^{(2)} + \nabla^{(2)}\chi, \tag{162}$$

into regions of the vacuum in which the original $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is excluded experimentally. A narrowly collimated microwave beam, for example, which is circularly polarized and directed in the shadow of two interfering fermion beams, should produce an OAB due to the multi-valued (periodic) nature of the gauge transformed $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, i.e., due to periodicity [23] in $\chi^{(1)}$ and $\chi^{(2)}$ of Eq. (162). This periodicity is possible only because the group space of O(3) is doubly-connected: the vacuum topology needed to sustain $\boldsymbol{B}^{(3)}$ also sustains an OAB due to $\boldsymbol{B}^{(3)}$. Due to its implication of action at a distance in electromagnetic waves, the observation of the OAB might catalyze work within the interesting framework of deterministic quantum mechanics proposed by Bohm [85] after discussions with Einstein.

The original Aharonov-Bohm effect [37] was inferred theoretically from a quantum relativistic theory, in which action, S, was expressed in terms of A_μ for static fields. In the Dirac equation, the wave functions are four-spinors which can be described as plane waves [22—25] with phase exponent $\exp(S/\hbar)$ where S is a relativistically defined action, so a fully relativistic theory of the Aharonov-Bohm effect requires a description in these terms of the observed phase shift [32—37]. In a simplified view of the original effect [37], the observable shift in a fringe pattern due to interfering fermion matter waves is attributed [23] to a change in the phase,

$$\alpha_1 \rightarrow \alpha_1 + \frac{1}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}, \tag{163}$$

where $\boldsymbol{p}$ is the non-relativistic limit of the fermion linear momentum at a point $\boldsymbol{r}$ in three dimensional space. The phase α_1 is an action or angular momentum divided by the quantized unit of action or angular momentum, $\hbar$. The intrinsic angular momentum ($\boldsymbol{S}^{(3)} = (\hbar/2)\boldsymbol{\sigma}^{(3)}$) of the fermion does not appear in the picture because $\boldsymbol{S}^{(3)}$ is a relativistic concept. Reinstating $\boldsymbol{S}^{(3)}$ produces the *spin phase of the fermion*

$$\alpha_S = \frac{1}{\hbar}\boldsymbol{S}^{(3)}\cdot\boldsymbol{\sigma}^{(3)}, \tag{164}$$

which exists due to topology. In Eq. (164), the angular momentum $\boldsymbol{S}^{(3)}$ is analogous with the linear momentum $\boldsymbol{p}$ in Eq. (163), and the coordinate $\boldsymbol{\sigma}^{(3)}$ is analogous with the coordinate $\boldsymbol{r}$.

The OAB can now be traced to the effect of a circularly polarized electromagnetic field on the spin phase α_S of one fermion's matter wave. Following Talin *et al.* [54] we write the field-fermion interaction energy En_2 in the form

$$En_2 = \omega J = \omega S, \tag{165}$$

which originates in the conservation laws of angular momentum (J) and energy. The energy of a free photon is $\hbar\omega$ by definition, and this is a limit in which J has been replaced by the quantized unit of angular momentum (or action) of an electromagnetic or matter wave, the Dirac constant $\hbar$. In the presence of an electromagnetic field the spin phase of one fermion is changed to

$$\alpha_S \rightarrow \alpha_S + \frac{En_2}{\hbar\omega}, \tag{166}$$

which is the action En_2/ω generated by the field-fermion interaction divided by $\hbar$. This is the change in energy, En_2, caused by fermion-field interaction divided by the photon $\hbar\omega$, the quantized, elementary, unit of electromagnetic or matter-wave energy. Alternatively the phase change can be viewed as $J/\hbar$, i.e., as the change in angular momentum, J, generated by the interaction of the fermion and field divided by $\hbar$, the quantized, elementary, unit of electromagnetic or matter-wave angular momentum. It can be checked that if there is no change in energy or angular momentum, i.e., no *interaction* energy or angular momentum, there is no phase shift, and vice-versa. The OAB therefore requires an interaction between a fermion property and a *gauge transformed* field property if the electromagnetic field is excluded from direct interaction with the interfering fermion matter waves.

For one fermion, the change in phase is, from Eq. (1),

$$\Delta\alpha_S = \frac{ie^2}{2m_0\hbar\omega}\boldsymbol{\sigma}^{(3)}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}, \tag{167}$$

and is proportional to the conjugate product in the weak field limit, or in the transverse gauge. In the strong field limit this result is modified as discussed in section (3.1). Finally, the phase change (167) is expressible as

$$\Delta\alpha_S = \mp\left(\frac{e^2c\mu_0}{2m_0\hbar}\right)\frac{I}{\omega^3}, \tag{168}$$

and is proportional to I/ω^3 as we set out to show. If there is no direct field-fermion interaction, $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ in Eq. (167) must have been gauge transformed from an original electromagnetic beam. If there is direct field-fermion interaction the field property $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ is used without gauge transformation.

3.3 GAUGE TRANSFORMATION AND THE OAB

Equation (168) shows that a circularly polarized electromagnetic field changes the spin phase of a fermion by an amount which is proportional to I/ω^3. The effect can be positive or negative depending on the spin state of the fermion. The inverse cubed dependence on ω means that the OAB should be orders of magnitude greater for a given I at radio or microwave frequencies than at visible frequencies. The OAB is inversely proportional to the fermion mass and the lightest accessible fermion, the electron, should be used experimentally to maximize the effect. Theoretically, the OAB

also exists for other fermion matter waves, such as those of protons, neutrons and certain atoms and molecules with net-odd total of protons, neutrons and electrons (e.g. ^{3}He but not ^{4}He). An array of OAB (and AB) effects is possible in principle using various interfering beams and resonance techniques. Clearly, the simple one fermion theory will not be adequate in detail for interfering atomic and molecular beams and must be developed appropriately. The OAB is closely related to RF-NMR and other magnetic effects of light as described in Chaps. 1 and 2.

In order to demonstrate the OAB experimentally it is necessary to show that the phase shift occurs in regions where there is no direct field-fermion interaction. The only physical possibility of an OAB in this experimental configuration is through gauge transformation. Since $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are both vector potentials of the electromagnetic field in the complex representation they each undergo, by definition [22—25], type two gauge transformation as follows,

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \rightarrow \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} + \nabla^{(1)}\chi \times \nabla^{(2)}\chi, \tag{169}$$

and since the original cross product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is physical, the gauge transformed cross product, $\nabla^{(1)}\chi \times \nabla^{(2)}\chi$, must be such that gauge invariance of the original $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is maintained. This is analogous with the well known requirement of gauge invariance in the curl, $\nabla \times \boldsymbol{A}$, of a physical magnetic field. It is therefore convenient to think of the OAB as the original AB effect [37] with the magnetic field of that effect replaced by $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. The Evans-Vigier field $\boldsymbol{B}^{(3)}$ plays this role precisely if defined as in Eq. (2) in the experimentally accessible weak field limit.

The OAB is therefore an AB effect due to the Evans-Vigier field.

Gauge transformation into the vacuum [23] can therefore be symbolized in exactly the same way as in the original AB theory, bearing in mind that $\boldsymbol{B}$ of the original theory is replaced by $\boldsymbol{B}^{(3)}$ of Eq. (2). The OAB can equally well be understood in terms of the gauge transformation into the vacuum of the *physical* conjugate product, physical because it is known [16—21] to produce physical magneto-optic effects. In order for an electromagnetic conjugate product (OAB) or a static magnetic field (AB) to be physical, they must each be gauge invariant. This is the requirement that leads to non-trivial vacuum topology [23] in both cases. The only contemporary difference is that the AB has been observed [32—37] while the OAB has not yet been observed.

In the original AB effect [23] gauge transformation into the vacuum can be summarized by

$$\boldsymbol{O} \rightarrow \boldsymbol{O} + \nabla\chi, \tag{170}$$

because initially there is no magnetic field or vector potential present in the vacuum, thus $\boldsymbol{O}$ on the left hand side. The right hand side of Eq. (170) symbolizes the fact [23] that for an AB effect to occur $\nabla \times \nabla\chi$ must be *non-zero in the vacuum*. (In the

vacuum there is no $\boldsymbol{B}$ and no $\boldsymbol{A}$ because $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ by definition.) In order for $\nabla \times \nabla\chi$ to be non-zero, χ must be periodic as discussed by Ryder [23] and this can only be so if the vacuum itself possesses a non-trivial topology. In classical electrodynamics of the pre-AB era, $\boldsymbol{B}$ was physical, and $\boldsymbol{A}$ was mathematical, because it was thought that the type two gauge transformation,

$$\boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\chi, \tag{171}$$

changed $\boldsymbol{A}$ to $\boldsymbol{A} + \nabla\chi$ but left $\boldsymbol{B}$ unchanged. This assumed implicitly that the quantity $\nabla \times \nabla\chi$ was always zero, implying a trivially structured (or non-structured) vacuum.

Using a combination of the quantum hypothesis $\nabla^{(1)} := i\boldsymbol{p}^{(1)}/\hbar$ and the minimal prescription $\boldsymbol{p}^{(1)} := e\boldsymbol{A}^{(1)}$ we obtain

$$\nabla^{(1)} = \nabla^{(2)*} = \frac{i}{\hbar} e\boldsymbol{A}^{(1)}, \tag{172}$$

and the analogous argument for the OAB can be developed conveniently as follows. We consider the eigenequations,

$$\nabla^{(1)}\chi = i\frac{e}{\hbar}\boldsymbol{A}^{(1)}\chi, \qquad \nabla^{(2)}\chi = i\frac{e}{\hbar}\boldsymbol{A}^{(2)}\chi, \tag{173}$$

where the wavefunction is

$$\chi = \chi_0 e^{i\phi}, \qquad \chi_0 = i\frac{\hbar}{e}, \tag{174}$$

in which ϕ is the electromagnetic phase [1] itself. Therefore

$$\nabla^{(1)}\chi_0 \times \nabla^{(2)}\chi_0 = -\frac{e^2}{\hbar^2}\chi_0^2 \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{175}$$

and $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is invariant under gauge transformation as required by its physical nature. This result is consistent with Noether's theorem [1,2,23] which implies that invariance of action under space-time translation or rotation produces conservation of energy-linear momentum and energy-angular momentum respectively. The action in our

context is

$$S = \frac{ie^2}{2m_0\omega} A^{(1)} \times A^{(2)} \cdot \sigma^{(3)*}, \tag{176}$$

and, self-consistently, is invariant under type two gauge transformation if and only if $A^{(1)} \times A^{(2)}$ is so invariant. If so, χ must be periodic as in Eq. (174), leading again to the concept of non-trivial vacuum topology as in the AB effect. Furthermore, the periodicity in χ is supplied by the electromagnetic phase itself, and the OAB becomes a phase shift, which in quantum mechanics is a phase shift in the wave function χ. Since S in Eq. (176) is the magnitude of an angular momentum, the OAB is the result of the necessity for conservation of angular momentum when $A^{(1)} \times A^{(2)}$ is gauge transformed into the vacuum. Finally, therefore, if there is no OAB, there can be no conservation of angular momentum and no Noether theorem. It is very likely on these grounds that the OAB will be observed experimentally.

3.4 EXPERIMENTAL INVESTIGATIONS OF THE OAB

There are several elegant experimental methods [32—36] now available to detect the original AB [37] and each can be adapted in principle for the OAB provided the necessary experimental configuration can be attained in practice. Phase shifts in the original Aharonov-Bohm effect can be observed with high precision using a Wien filter. Hasselbach and co-workers [86] have developed very accurate instruments based on electron interferometry. The Wien filter is a device that uses magnetic and electric fields mutually perpendicular to interfering electron beam paths. The electric force on the electron is balanced by the magnetic force when the Wien filter is compensated. Hasselbach *et al.* [86] have discussed the role played by the electric and magnetic Aharonov-Bohm phase shifts in such a device, and have shown that the Aharonov-Bohm effects are responsible for the experimentally verifiable fact that the fringe system does not change its appearance with increasing deflection angle. Thus, the original Aharonov-Bohm quantum phase shifts [37] are responsible for the practical utility of electron-optical elements. The magnetic field in the conventional Wien filter causes an Aharonov-Bohm phase shift due to the difference in magnetic flux enclosed by the two coherent electron beams. Therefore the novel Evans-Vigier field defined in Eq. (2), a physical magnetic flux density, is expected to produce an OAB if the Wien filter is modified to accommodate a circularly polarized radio frequency field mutually perpendicular to the electron beam paths and electric field.

Such an experiment would be of fundamental importance because it would imply that the group space of the electromagnetic sector is O(3), and not U(1), in contemporary string and superstring theory [2]. The OAB can be understood entirely equivalently through the conjugate product $A^{(1)} \times A^{(2)}$, which affects the relativistic spin phase of the

electron wave function, as discussed already. In the modified Wien filter, the shift in electronic spin phase caused by $A^{(1)} \times A^{(2)}$, or equivalently $B^{(3)}$, is observable because it is different for the two interfering electron beams. In other words, the magnetic flux density $B^{(3)}$ enclosed by the two interfering electron beams is different, and this flux density is $A^{(1)} \times A^{(2)}$ multiplied by $-ie/\hbar$. Therefore the phase shift per electron due to $A^{(1)} \times A^{(2)}$, Eq. (42), is non-zero and observable after appropriate path integration.

The Wien filter thus modified to take a circularly polarized electromagnetic field instead of a magnetic field can now be used to detect the OAB. The experimental method follows the description in Fig (3) of Ref. 86a. Electron wave packets emerge from two coherent sources and reach the modified Wien filter, in which an electric field and circularly polarized radio frequency electromagnetic field are arranged mutually perpendicular to the electron beam baths. In the absence of the modified Wien filter, an electron fringe pattern is observed which is symmetric about the axis of propagation. Application of the electric field renders this pattern asymmetric because the two electron beams traverse different electric field potential regions. The central maximum of the pattern is shifted off axis, and this effect is exactly equivalent to moving back one of the sources by n wavelengths. The central axis can be brought back on to the symmetry axis by application of the magnetic field in the conventional Wien filter, or by switching on the circularly polarized radio frequency field in the modified Wien filter. This automatically shows the presence of $B^{(3)}$ in the radio frequency beam, or equivalently the presence of $A^{(1)} \times A^{(2)}$.

An alternative procedure is to take a conventional Wien filter, consisting of crossed electric and magnetic fields, and to supplement the magnetic field with an additional, circularly polarized electromagnetic field. The Wien filter is first compensated with the electric and magnetic fields as usual, but then the r.f. field is applied. The latter should again make the fringe pattern unsymmetrical, because the total magnetic flux density being applied has been increased by $B^{(3)}$. This effect can be observed in principle with high precision, because shifts of this kind can be measured to small fractions of a wavelength [86]. The primary interference pattern in the two electron beams should also be deflected by the OAB, i.e., the spin phase of the matter wave should be changed. This is a consequence of Eq. (168) for one electron. As for the electric and magnetic AB effects [86], the OAB spin phase shift should keep pace exactly with the path length difference caused by deflection, and the electron fringe system should not change its appearance with increasing deflection angle caused by the radio frequency field. In other words, the difference in magnetic flux density, B, enclosed between the coherent beams compensates the geometric phase shift for every wavelet making up the wave fronts. The additional path length Δ caused by the bending of the electron beams by the additional radio frequency field applied to a compensated Wien filter, and the corresponding change in phase Δ/λ, where λ is the wavelength, are exactly cancelled by the counteracting OAB effect. With increasing deflection angle therefore there should be observed *no* continuous change from bright to dark of a single fringe of the pattern with increasing deflection angle caused by the additional r.f. field.

In order to demonstrate the quantum relativistic nature of the OAB, it is necessary in addition to these experiments, precise as they are, that resonance can be made to occur between the two phase states of Eq. (168), negative and positive respectively. These two states originate in the nature of the Pauli spinor $\sigma^{(3)}$, and this has no counterpart in classical or non-relativistic quantum physics. A complete appreciation of the OAB requires therefore a resonance experiment. It has been demonstrated theoretically in Chaps. 1 and 2 that a circularly polarized radio frequency field produces fermion resonance akin to NMR and ESR. For an electron, the resonance frequency is

$$\omega_{res} = 2\Delta\alpha_S\omega, \tag{177}$$

i.e., is twice the OAB phase shift of Eq. (168) multiplied by the angular frequency of the resonance causing (pump) field. The resonance frequency for a 100 watt cm^{-2} microwave beam interacting with one electron is at about 150 cm^{-1} in the far infra red, and detection of this frequency would also be a demonstration of the OAB and of the Evans-Vigier field $\boldsymbol{B}^{(3)}$. A complete appreciation of the OAB therefore requires a combination of the elegant methods based on the Wien filter [86] and resonance technique. The latter specifically demonstrates the quantum relativistic origins of the OAB.

The OAB as proposed in this chapter would allow action at a distance in electromagnetism, and would signal the existence of the Evans-Vigier field $\boldsymbol{B}^{(3)}$. The Bohm model of quantum mechanics [72] has been much discussed [1—15] as an alternative to the Copenhagen interpretation, but requires at the outset action at a distance. The experimental observation of the OAB might go some way towards explaining action at a distance through vacuum topology, and towards reconciling two strikingly different interpretations of quantum theory, leading to a deeper understanding of both. The discovery of the non-zero $\boldsymbol{B}^{(3)}$ and its implication of non-zero photon mass is another feature which may become useful in a realist interpretation of quantum mechanics, or in efforts to reconcile the realist and Copenhagen points of view.

3.5 OBSERVATIONAL CONDITIONS FOR $\boldsymbol{B}^{(3)}$ IN GENERAL

This chapter has been concerned with the delicate and interesting optical Aharonov-Bohm effect and outlines the conditions under which experiments should be pursued. Recently, an experiment [74] has been reported which did not detect the optical Faraday effect (OFE) [79,80] in liquid benzene. This negative result was used to conclude that the Evans-Vigier field does not exist [74]. The existence, however, of magneto-optic effects [16—21] is well supported by data and the OFE had already been observed [79] under appropriate conditions. The claim that $\boldsymbol{B}^{(3)}$ does not exist contradicts the Dirac equation as developed in Chaps. 1 and 2, and seems vanishingly improbable. In this

section a summary is provided of the conditions under which $\boldsymbol{B}^{(3)}$ interacts with a fermion in the weak and strong field limits. We hope that this summary will be useful in defining appropriate experimental conditions.

We have seen in earlier chapters that the interaction of a fermion with a circularly polarized electromagnetic field is described by the following Dirac equation after averaging over many field cycles,

$$\begin{aligned}\left(\left(En - mc^2 + ecA_0\right)\left(En + mc^2 + ecA_0\right) - c^2\left(\boldsymbol{p} + e\boldsymbol{A}^*\right)\cdot\left(\boldsymbol{p} + e\boldsymbol{A}\right)\right.\\ \left. - ie^2c^2\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*\right)\psi = 0.\end{aligned} \tag{178}$$

In the weak field limit it has been shown that this equation reduces to

$$W\psi = \left(En - mc^2\right)\psi \sim \left(\frac{e^2}{2m}\left(\boldsymbol{A}^*\cdot\boldsymbol{A} + i\boldsymbol{\sigma}\cdot\boldsymbol{A}^*\times\boldsymbol{A} - ecA_0\right)\right)\psi \tag{179}$$

and in the strong field limit it reduces to

$$W\psi = \left(ec\boldsymbol{\sigma}\cdot\frac{i}{\boldsymbol{A}^{(0)}}\boldsymbol{A}^*\times\boldsymbol{A}\right)\psi. \tag{180}$$

The Evans-Vigier field from Eq. (179) is defined by Eq. (2), and this definition makes the term in $\boldsymbol{\sigma}\cdot\boldsymbol{A}^*\times\boldsymbol{A}$ an ordinary Zeeman effect term, with fermion half integral spin eigenvalue $\pm\hbar/2$ as usual. The Evans-Vigier field from the strong field limit (180) is defined by

$$\boldsymbol{B}^{(3)*} := -i\frac{\kappa}{\boldsymbol{A}^{(0)}}\boldsymbol{A}\times\boldsymbol{A}^* = -\frac{i}{\boldsymbol{B}^{(0)}}\boldsymbol{B}\times\boldsymbol{B}^*, \tag{181}$$

and is fundamental because $\boldsymbol{B}^{(0)}$, $\boldsymbol{B}$ and $\boldsymbol{B}^*$ are fundamental. The concept of $\boldsymbol{B}^{(3)}$ is analogous in some ways to that of Poynting's vector [1—15], another fundamental vector, and another field cross product. The origin of $\boldsymbol{A}\times\boldsymbol{A}^*$ however is to be found in the founding axioms of special relativity itself — the Einstein equation of motion is *quadratic* in energy and momentum, and so therefore is Dirac's equation of motion. Assuming *only* that $\boldsymbol{A}$ is complex in general, so that it is not equal to $\boldsymbol{A}^*$, the (quadratic) term in $\boldsymbol{A}\times\boldsymbol{A}^*$ follows from either the Einstein or Dirac equation, and this term defines the Evans-Vigier field $\boldsymbol{B}^{(3)}$. The latter is therefore also a consequence of special relativity, and being a field, can be observed only by field-matter interaction.

Equation (2) is transformed into Eq. (181) in the limit,

$$\frac{\kappa}{A^{(0)}} \rightarrow \frac{e}{\hbar}, \tag{182}$$

and Eq. (181) applies only when $I \gg 7.72 \times 10^{-9}\,\omega^2$ for the electron in the field or $I \gg 0.026\,\omega^2$ for the proton in the field. In these limits the momentum magnitude $eA^{(0)}$ transferred from field to fermion is, F.A.P.P., the momentum magnitude, $\hbar\kappa$, of the free photon itself. Under these conditions the photon gives up essentially all its energy and momentum to the fermion. Its limits of application were first shown in Ref. 1, Eqs. (411) — (414), using the classical but relativistic Hamilton-Jacobi equation.

Rikken [74] based his claim of non-existent $B^{(3)}$ on an experiment in which the peak intensity was only $5.5 \times 10^{12}\ W\,m^{-2}$ at a Nd-YaG frequency of 10,640 cm^{-1} ($\omega = 1.77 \times 10^{16}\ rad\ s^{-1}$). Equations (156) and (157) both show that he would have been working well within the weak field limit, a limit in which $B^{(3)}$ must be defined, according to the Dirac equation, by Eq. (2). His sample (benzene) did not have free spin, however, so his conditions were entirely inappropriate for comparison with one fermion theory. Unfortunately, Rikken appears not to have been aware of Ref. 1 or of any recent reference to work on $B^{(3)}$, and interpreted the negative result in terms of Eq. (181) of the strong field limit as discussed earlier in this chapter. He appears then to have asserted that $B^{(3)}$ is zero under all conditions, whereas the precise opposite is indicated by the Dirac equation. Finally, the emergence of $B^{(3)}$ from the Dirac equation means that earlier criticisms [1] of $B^{(3)}$ based on symmetry and so forth are incorrect if the Dirac equation is accepted as a working hypothesis in relativistic quantum theory [22—25]. The Dirac equation is by no means perfection [22] but for our purposes it is perfectly adequate.

Chapter 4. Properties of $B^{(3)}$ in the Strong Field Limit

In the previous chapter it was shown that in the strong field limit and therefore in the vacuum, the $B^{(3)}$ field is defined through an equivalence principle $eA^{(0)} = \hbar\kappa$ that describes the complete transfer of photon momentum to fermion momentum in a photon-fermion collision. In this chapter this relation is developed systematically and a total of ten forms of $B^{(3)}$ are derived and tabulated (Table 3). It has been shown in earlier chapters that the $B^{(3)}$ field is the fundamental magnetizing field of relativistic magneto-optics: phenomena such as the inverse and optical Faraday effects the optical Zeeman effect, the OAB, RF-NMR and related technology, all depend on its existence in the vacuum. In the strong field limit these phenomena are experimentally proportional to the square root of beam intensity (I) and the ratio of κ to $A^{(0)}$ goes to the limiting constant $e/\hbar$. In this condition the fermion has been accelerated infinitesimally near the speed of light, c, so that its charge has become a field property. Based on these considerations, the fine structure constant can be interpreted classically, and the vector potential in vacuo, $\boldsymbol{A}$ becomes physically significant.

The existence of $B^{(3)}$ was first inferred [1—15] from the *experimentally* verified existence of the conjugate product, not in the form $\boldsymbol{A} \times \boldsymbol{A}^*$, or $\boldsymbol{B} \times \boldsymbol{B}^*$, but in the form $\boldsymbol{E} \times \boldsymbol{E}^*$ originally suggested phenomenologically by Pershan [41]. It has been clear for some time [16—21] that this cross product has magnetic symmetry, but the conventional viewpoint asserted that this symmetry does not imply the existence of a physical magnetic field. It is now understood that this view is insupportable because it would mean that the left and right hand sides of Eq. (2), for example, could not be equated; or that $B^{(3)}$ could not appear in the Dirac equation while $i\boldsymbol{A} \times \boldsymbol{A}^*/A^{(0)}$ *does* appear in the same Dirac equation (Eq. (180)). It is tantamount to asserting that a physically observable quantity on one side of an equation cannot be equated to a quantity on the other side of that equation with the same units and symmetry. When the quantity on the right hand side is a physical observable, that on the left hand side must be the same observable. The development [1,2] of cyclically symmetric relations based on the above conjugate products showed, therefore, that they produce in vacuo a physical and observable magnetic field, $B^{(3)}$. It was shortly afterwards realized [1—15] that these cyclical relations have the same O(3) symmetry [1,2] as rotation generators (or angular momenta) in three dimensional space, or their equivalents [1,2] in four dimensional space-time. This reinforced the initial inference that $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ are three physical fields in vacuo. The same symmetry arguments [1,2] show

that there is no physical $E^{(3)}$, the cyclical symmetry of the field relations demands, however, an imaginary $iE^{(3)}$ whose scalar square modulus $E^{(0)2} = -i^2E^{(3)} \cdot E^{(3)}$ is real and which can contribute to the total field energy. The fact that there is no real $E^{(3)}$ means that there is no free space Faraday induction [1—15], i.e., the time derivative of $B^{(3)}$ does not produce the curl of a real $E^{(3)}$ unless there is a sample present, as in the inverse Faraday effect. Experiments [87] to date confirm this expectation and the Dirac equation shows that the fermion spin interacts always with a magnetic field, there being no interaction term in the imaginary and unphysical electric field that emerges (Chap. 1) from the same analysis [26]. Finally, this conclusion can be shown to hold in the classical limit, when the Dirac equation becomes a Hamilton-Jacobi equation (Chap. 2). As described in earlier chapters, interesting new resonance effects are expected due to $B^{(3)}$. There can be no reasonable doubt, therefore, that $B^{(3)}$ is a novel and useful physical magnetic flux density arising from the fundamentals of special relativity, both in classical and quantum mechanics.

4.1 FERMION ACCELERATED INFINITESIMALLY CLOSE TO c PRODUCES $B^{(3)}$

The pre-$B^{(3)}$ approach to magneto-optical phenomena typified by the inverse Faraday effect therefore relied on the conjugate product in the form $E \times E^*$. This quantity multiplied by the vacuum permittivity ϵ_0 has, formally, the units of an electromagnetic torque density, Tq_V, or torque per unit volume carried by electromagnetic radiation in vacuo. The torque density is also given by $-iB \times B^*/\mu_0$, where μ_0 is the vacuum permeability, and, as shown in Chap. 7, can be expressed in terms of wavelength. The quantity $c|Tq_V|$ is the antisymmetric part of the tensor of light intensity, a tensor whose scalar part is

$$I = \frac{c}{\mu_0} B^{(0)2}, \tag{183}$$

an equation used in Chap. 1 to 3. Similarly, the energy density of the electromagnetic radiation in vacuo is given by the relation $B^{(1)} \cdot B^{(2)}/\mu_0$, and the intensity is c times the energy density.

The $B^{(3)}$ field emerges from Tq_V as follows:

$$B^{(3)} = \frac{\mu_0}{B^{(0)}} Tq_V, \tag{184}$$

and, like Tq_V, is real and physical. Its existence in vacuo implies that of the magnetic dipole moment whose origin is rotating charge e,

$$m^{(3)} = \frac{1}{\mu_0} B^{(3)}. \tag{185}$$

Similarly, Maxwell's rotating electric field strength $E^{(1)}(Vm^{-1})$ in electromagnetic radiation propagating through the vacuum can be expressed in terms of the same rotating electric charge in vacuo,

$$E^{(1)} = \frac{e}{\epsilon_0 V} r^{(1)}, \tag{186}$$

where dimensionality requires the rotating radius vector, $r^{(1)}$. The latter is complex in general because $E^{(1)}$ is complex. This analysis, based on fundamental S.I. units and definitions, introduces the charge e into electromagnetic radiation in the vacuum, and shows the origin of the $\hat{C}$ negative symmetry of the scalar amplitude $E^{(0)}$ [1]. It is therefore possible to describe electromagnetic radiation as a charge rotating around the origin. The length of $r^{(1)}$ can be shown as follows to be

$$r^{(0)} = |r^{(1)}| = \frac{1}{\kappa} = \frac{\lambda}{2\pi} = \left(\frac{V}{4\pi\alpha}\right)^{1/3}, \tag{187}$$

where λ is the wavelength of the radiation contained in a volume V, and where α is the fine structure constant. If V is the radiation volume, it is easily checked that Eq. (187) is dimensionally and physically consistent in the theory of electrodynamics [47] in S.I. units. The electromagnetic field propagates in the axis perpendicular to the plane of the rotating charge, which draws out a helical path at the speed of light in vacuo. In analogy with the solenoid, this movement produces the magnetic field $B^{(3)}$. If an electron is placed in the rotating $E^{(1)}$ of an electromagnetic field in the strong field limit, its classical trajectory would be exactly the same as that of our rotating charge and would be described by the Hamilton-Jacobi equation of Chap. 12 of Vol. 1 in this limit. This also leads to the $B^{(3)}$ field.

Under these conditions we have shown in previous chapters that

$$eA^{(0)} = \hbar\kappa, \tag{188}$$

where $\hbar$ is a constant which must have the units of angular momentum or action. It can be shown as follows that this definition *implies* that $\hbar$ must be the Dirac constant, because it *implies* the Planck-Einstein condition. The latter is therefore a direct consequence of accelerating an electron in the strong field limit.

From Eq. (186) the scalar magnitude of the rotating Maxwellian $E^{(1)}$ is

$$E^{(0)} = \frac{er^{(0)}}{\epsilon_0 V}. \tag{189}$$

The classical electromagnetic energy in the volume V is

$$En = \epsilon_0 E^{(0)2} V, \tag{190}$$

where the following definition is used of average volume

$$V := \int_0^V dV, \tag{191}$$

and so the energy in volume V is expressible in terms of the radius $r^{(0)}$,

$$En = eE^{(0)}r^{(0)} = \frac{e^2 r^{(0)2}}{\epsilon_0 V}. \tag{192}$$

Using the usual relation between $E^{(0)}$ and $A^{(0)}$,

$$A^{(0)} = \frac{E^{(0)}}{\omega}, \tag{193}$$

it is found that

$$En = \left(eA^{(0)}r^{(0)}\right)\omega, \tag{194}$$

i.e., *the electromagnetic energy is proportional to the electromagnetic angular frequency*, and this is the Planck-Einstein relation provided we make the identity

$$\hbar = eA^{(0)}r^{(0)}. \tag{195}$$

Therefore the first principle of quantum theory, the Light Quantum hypothesis, has emerged from classical theory in a new way. This is a consequence of thinking of an electron being accelerated in the strong field limit infinitesimally close to the speed of light c, and this point is vividly underlined if we identify $r^{(0)}$ with κ^{-1}, the inverse of the wavevector, which for radiation propagating at c is the quantity c/ω. The radius $r^{(0)}$ is, in this condition, the ratio of the forward and angular velocities of the radiation. The identification $r^{(0)} = \kappa^{-1}$ produces from Eq. (195) the equivalence condition (188), i.e.,

$$\hbar = \frac{eA^{(0)}}{\kappa}, \tag{196}$$

which in Vol. 2 [2] we also obtained from O(3) gauge theory of vacuum radiation. In Chap. 3 we saw that this condition is precisely that required to transform the weak field solution of the Dirac equation into the strong field solution, in which the fermion is accelerated infinitesimally near c. Equation (196) in Eq. (194) produces, self-consistently, the familiar

$$En = \hbar\omega, \tag{197}$$

of the famous Light Quantum hypothesis [88] first proposed heuristically by Planck and then by Einstein.

4.2 PHOTON RADIUS AND QUANTUM OF LIGHT ENERGY

Using Eq. (188), with the equation

$$A^{(0)} = \left(\frac{c}{\epsilon_0 \omega^2 V}\right)e, \tag{198}$$

the quantum of electromagnetic energy $\hbar\omega$ is defined as

$$\hbar\omega = \frac{e^2}{\epsilon_0 \kappa^2 V}, \tag{199}$$

and is proportional to e^2. This result can now be expressed in terms of the fine structure constant [89,90],

$$\alpha := \frac{e^2}{4\pi c \epsilon_0 \hbar}, \tag{200}$$

of quantum electrodynamics. From Eqs. (199) and (200),

$$\alpha = \frac{V}{4\pi r^{(0)3}}, \tag{201}$$

and so we arrive at Eq. (187) for the radius of the rotating charge, as we set out to show. The quantum of light energy is therefore

$$En = \hbar\omega = (4\pi\alpha V)^{3/2}\frac{e^2}{\epsilon_0}, \tag{202}$$

and becomes expressible in terms of $r^{(0)}$. The concept of photon radius $r^{(0)}$ in classical electrodynamics has linked the Planck-Einstein hypothesis (197) and the new equivalence condition (188), and has shown that the photon, whatever it is, can be produced by accelerating an electron in the strong field limit. Significantly, Eq. (187) defines the Thompson radius [91] of the photon, and so the latter, if particulate, has a finite radius c/ω that becomes longer at lower frequency.

This view is consistent with special relativity and Maxwell's vacuum equations and is derived by re-expressing the usual transverse fields $E^{(1)}$ and $B^{(1)}$ in terms of elementary charge e, a procedure equivalent to imagining an electron accelerated in the strong field limit infinitesimally near to c. This procedure results in $En = \hbar\omega$ if $\hbar = eA^{(0)}/\kappa$. The quantity $eA^{(0)}/\kappa$ is therefore a constant angular momentum, as required, but an *electronic* angular momentum of classical electrodynamics. In the quantum theory it is of course the universally constant angular momentum of one photon, the latter being the quantum of energy, $\hbar\omega$. *Provided* that we use the new equivalence condition $eA^{(0)} = \hbar\kappa$ the Light Quantum hypothesis is a logical outcome of classical electrodynamics.

Furthermore, the re-expression of $E^{(1)}$ as a rotating electric charge translating very near c results in the field $B^{(3)}$, which can be thought of as the outcome of helical

charge motion. Conversely the $B^{(3)}$ field drives an electron in a helical trajectory. This is consistent with the fact that $B^{(3)}$ from the Dirac equation in the strong field limit produces the equivalence condition (196), and consistent with the classical analysis given in Vol. 1 based on the Hamilton-Jacobi equation of e in the electromagnetic field.

The usual idea of a photon as being uncharged and therefore its *own anti-particle* is shown by Eq. (202) to have the narrowest of meanings. The photon can now be thought of as a *classical* amount of energy proportional to the square of e, and is not uncharged. The equivalence condition (196) shows that the origin of the Dirac constant $\hbar$ is the electronic charge e multiplied by $A^{(0)}/\kappa$, where $A^{(0)}$ is within a constant c the scalar potential of the classical wave in the strong field limit. Re-expressing $A^{(0)}$ through Eq. (199) leads to Eq. (201), which produces the fine structure constant of quantum electrodynamics in the form of a simple ratio of volumes, the ratio of V to $V^{(0)} = (4/3)\pi r^{(0)3}$, the Thompson sphere. The fine structure constant is

$$\alpha = \frac{1}{3}\frac{V}{V^{(0)}}. \tag{203}$$

Equation (187) shows that the magnitude of the classical Maxwellian wavevector is defined by the volume V through

$$\kappa = \left(\frac{4\pi\alpha}{V}\right)^{1/3}, \tag{204}$$

and because α is a universal constant [2,22—25], the de Broglie photon momentum becomes

$$p = \hbar\kappa = \left(\frac{4\pi\alpha\hbar^3}{V}\right)^{1/3} = \left(\frac{e^2\hbar^2}{\epsilon_0 c V}\right)^{1/3}. \tag{205}$$

The Planck-Einstein photon therefore becomes expressible as

$$En = \hbar\omega = \hbar\kappa c = \left(\frac{e^2\hbar^2 c^2}{\epsilon_0 V}\right)^{1/3}. \tag{206}$$

These equations show that the quantum of radiation energy $\hbar\omega$ and the quantum of radiation momentum, $\hbar\kappa$, are both defined in terms only of V and the universal fine structure constant α. This is consistent with the fact that they are fundamental quanta of energy and momentum, and with the fact that the magnitudes of these quanta vary

only with the volume V introduced in Eq. (186). On the most fundamental level in this Maxwellian theory, everything depends ultimately on e. In generalized gauge theory [2], from which the equivalence condition (188) emerges, e simultaneously plays the role of the elementary charge and a gauge scaling constant. Therefore e can be thought of as being topological in nature, so that an angular momentum such as $\hbar$ becomes understandable in terms of a geometrical entity, e, an elementary geometrical measure of the known universe. In electromagnetic radiation in the vacuum (or free space), e becomes subsumed into the definition of $E^{(1)}$ through the radius $r^{(1)}$ as in Eq. (186). The enigmatic photon, if it is a particle, must always have a finite radius.

The mass of this particle [1,2], if it exists, must be concentrated very near to the origin if the particle has a finite radius and is rotating about this origin. Otherwise it would rotate about a center of mass somewhere between the origin and the negative charge. What is known as photon mass may therefore be residual electron mass. Almost all of the latter's rest energy mc^2 must be transformed into other forms of energy as the electron is accelerated infinitesimally close to c in its helical trajectory, leaving a small residual mass, the photon mass, which we consider here to be elementary (Chap. 2) and irremovable in nature. If the electron mass is about 10^{-31} kgm and the photon mass no greater than about 10^{-44} kgm [92], a large fraction of the electron rest energy mc^2 is lost as other forms of energy; but not all. In the Maxwellian picture, radiation travels through a simple vacuum at c precisely, so there is no mass specified and no rest frame. The Maxwell equations in vacuo then show that the charge e rotates about an origin that is translating at c. The orbit of rotation is the Thompson radius, which becomes shorter at higher frequencies. The next section develops these ideas through the classical Hamilton-Jacobi equation of one electron in the strong field limit, and shows that the relativistic electron momentum p becomes $\hbar\kappa$ in this limit.

4.3 DERIVATION OF THE PLANCK-EINSTEIN CONDITION FROM THE RELATIVISTIC HAMILTON-JACOBI EQUATION

In this Section the equivalence condition (196) is used to derive the Planck-Einstein condition (197) from the classical Hamilton-Jacobi equation of a charge e in the electromagnetic field. In the limit where all the momentum of an incoming photon is transferred to an electron, the latter is accelerated to a state where it becomes indistinguishable from a photon, i.e., is accelerated infinitesimally near the speed of light, in which condition its concomitant electromagnetic fields become indistinguishable from those of Maxwell's vacuum theory, as discussed by Jackson [47].

By considering the classical, but relativistic, motion of an electron of mass m in the electromagnetic field, the Hamilton-Jacobi (HJ) equation can be used [1] to show

that the induced orbital angular momentum of the electron is given through $B^{(3)}$ as follows (in which spinors are missing)

$$J^{(3)} = \left(\frac{ec}{\omega}\right)^2\left(\frac{B^{(0)}}{(m^2\omega^2 + e^2B^{(0)2})^{1/2}}\right)B^{(3)}, \tag{207}$$

where ω is the angular frequency of the field and of the electron in equilibrium with the field. The relativistic factor in Eq. (207) is

$$\gamma := \frac{c}{\omega}(m^2\omega^2 + e^2B^{(0)2})^{1/2}, \tag{208}$$

and using $B^{(0)} = \omega A^{(0)}/c$ this becomes

$$En^2 = c^2\gamma^2 = m^2c^4 + c^2(e^2A^{(0)2}), \tag{209}$$

where En is total energy. The units of both sides in this equation are those of energy squared because γ has the units of linear momentum. Using the equivalence condition (196) is tantamount to assuming

$$ecA_0 \to ecA^{(0)} \gg mc^2, \tag{210}$$

because Eq. (209) is obtained in the strong field limit (Chap. 3). Therefore Eq. (209) becomes

$$En \to ecA^{(0)} = \hbar\kappa c = \hbar\omega, \tag{211}$$

which is the Planck-Einstein condition (197). If it is assumed that mass in equation (209) remains finite, Eq. (208) gives the de Broglie matter-wave equation

$$\omega^2 = \frac{m^2c^4}{\hbar^2} + c^2\kappa^2, \tag{212}$$

which is well known to be the Einstein equation after application of the quantum hypotheses,

$$En = \hbar\omega, \qquad p = \hbar\kappa. \tag{213}$$

If Eq. (212) is to be regarded as a matter wave equation for the electron of mass m then the electron must have acquired a linear momentum $\hbar\kappa$ from the quantized electromagnetic field. The classical momentum $eA^{(0)}$ introduced by the gauge has been identified with the quantized momentum $\hbar\kappa$, and in the limit $ecA^{(0)} \gg mc^2$ the relativistic factor of the classical relativistic Hamilton-Jacobi equation becomes the quantum relativistic Planck-Einstein condition. If we ignore the electron rest energy mc^2, it can be accelerated to such a degree as to become a photon, because a photon is *defined* by the Planck-Einstein condition. The electron, in order to be accelerated into a conventional photon, must lose all its rest energy in the form of radiation, but must retain its charge, e, fully intact.

Using the equivalence condition (196) the HJ equation (207) can be rewritten as

$$J^{(3)} \to \frac{\hbar\kappa}{(m^2c^2 + \hbar^2\kappa^2)^{1/2}} \frac{ec^2}{\omega^2} B^{(3)}, \tag{214}$$

in which the field $B^{(3)}$ drives the electron in an orbit at an angular frequency ω. All the other quantities in this equation are particulate in nature, describing an electron that has acquired a linear momentum $\hbar\kappa$ from the quantized field. In the strong field limit,

$$\hbar\kappa \gg mc, \tag{215}$$

and Eq. (214) reduces to

$$J^{(3)} \to \frac{ec^2}{\omega^2} B^{(3)}, \tag{216}$$

showing that the orbital angular momentum acquired by the electron is directly proportional to $B^{(3)}$. This produces a characteristic $I^{1/2}$ dependence in the strong field limit, as discussed in Chap. 3 and earlier volumes [1,2]. In this limit, the wavevector becomes indistinguishable from ω/c, and using $B^{(0)} = \kappa A^{(0)}$, we obtain

$$J^{(3)} \to \frac{ec^2 B^{(0)}}{\omega^2} e^{(3)} = \frac{eA^{(0)}}{\kappa} e^{(3)}. \tag{217}$$

The equivalence condition (196) finally reduces this equation to

$$J^{(3)} \to \hbar e^{(3)}, \tag{218}$$

in which $J^{(3)}$ is the angular momentum of the photon as we set out to show. The maximum orbital angular momentum that the electron can attain is $\hbar$, a universal constant, and this is so only if the rest energy mc^2 is regarded as negligible in comparison with the energy $ecA^{(0)}$ acquired from the quantized electromagnetic field. In this limit, $eA^{(0)}$ of the field becomes $\hbar\kappa$ of the electron, and if mc^2 is negligible, this becomes the total quantized linear momentum of the electron.

If the $B^{(3)}$ field were zero, no orbital angular momentum could be transferred from the photon to the electron, and this is something that contradicts conservation of angular momentum. There would be no magneto-optic effects, in contradiction with experience [16—21]. A similar analysis from the Dirac equation shows that if $B^{(3)}$ were zero, the intrinsic spin of the electron could not interact with the field through a Pauli spinor. In other words, there would be no optical anomalous Zeeman effect [1,2]. The field $B^{(3)}$ is the fundamental field responsible for magneto-optical effects and also for the interaction of fermions with the electromagnetic field.

The major unsolved problem, or so it seems at present, is to define more precisely the circumstances under which the rest energy of the electron can be rendered negligible. This is not a trivial question because if the electron is given a mass m of $\sim 10^{-31}$ kgm, its energy mc^2 is a constant of special relativity unless it is transmuted into radiation energy.

4.4 THE DERIVATION OF $B^{(3)}$ FROM A ROTATING CHARGE e IN VACUO

A magnetic field is due to a rotating charge, and in vacuo the field is the vacuum permeability multiplied by the magnetic dipole moment caused by the rotating charge. The S.I. unit of magnetic dipole moment is C m^{-1} s^{-1} and so it is possible to define a magnetic dipole moment in vacuo,

$$|m^{(3)}| = \frac{erv}{V}, \tag{219}$$

where e is the elementary charge rotating around the origin at the end of a radius r with tangential speed v in a volume V of electromagnetic radiation. It is straightforward to show, as follows, that the magnetic field $B^{(3)} = \mu_0 m^{(3)}$ from Eq. (219) is precisely equation (2). This derivation is given firstly, and the discussion of the validity of equation (219) secondly.

The magnetic field $B^{(3)}$ from Eq. (219) is

$$B^{(3)} = \frac{\mu_0 erv}{V} e^{(3)}, \tag{220}$$

and V can be defined as the average volume (191) without loss of generality. The radius r is the Thompson radius κ^{-1} as discussed already, and the tangential velocity v is by definition

$$v = \omega \times r, \tag{221}$$

where ω is the angular velocity. Therefore,

$$vj = \omega k \times \frac{c}{\omega} i, \tag{222}$$

where i, j, and k are Cartesian unit vectors, and so $v = c$. Therefore,

$$B^{(3)} = B^{(0)} e^{(3)} = \left(\frac{\mu_0 e c^2}{\omega V} \right) e^{(3)}, \tag{223}$$

is the magnetic field due to the rotating charge e used earlier in this chapter.

The validity of this result can be checked using the equivalence condition (188), which can be written as

$$\frac{c^2}{\omega} = \frac{\hbar\omega}{eB^{(0)}}. \tag{224}$$

Using this in Eq. (223) produces

$$\hbar\omega = \frac{1}{\mu_0} B^{(0)2} V, \tag{225}$$

which is the well known expression [47] for the photon in vacuo in terms of $B^{(0)}$ and the radiation volume V. The photon in equation (225) is the usual quantum of electromagnetic energy in vacuo.

Using Eq. (225) in Eq. (223) produces (*cf.* Eq. (4)),

$$B^{(3)} = \frac{e}{\hbar} \cdot \frac{c^2}{\omega^2} B^{(0)2} e^{(3)} = \frac{e}{\hbar} A^{(0)2} e^{(3)}, \tag{226}$$

which can be re-expressed as

$$B^{(3)} = -i \frac{e}{\hbar} A^{(1)} \times A^{(2)}, \tag{227}$$

and this is equation (2). Therefore Eq. (224) links Eq. (223) and Eq. (227), showing them to be *equivalent for radiation in vacuo*. This result was first derived in Vol. 1 and 2. The representation of electromagnetic radiation in vacuo by the rotating elementary charge e, propagating both tangentially and forward (along Z) at c, is therefore entirely self-consistent with the existence of the vacuum $B^{(3)}$. This result is consistent, furthermore, with O(3) gauge theory [2], which produces equation (2) using methods adapted from general relativity, and is also consistent with analysis based on rotation generators and developed elsewhere [1—15]. Thirdly, it is consistent with the equivalence equation (188), which was first shown in Vol. 2 [2], and referred to there as the *charge quantization condition* because it makes e proportional to $\kappa/A^{(0)}$ through $\hbar$. Fourthly, the result is consistent with Jackson's analysis [47] of a radiating electron accelerated infinitesimally close to c, whereupon the concomitant fields become vacuum electromagnetic fields.

If it asserted that $B^{(3)}$ is zero [62—65], then e must be zero, a reduction to absurdity.

Mass is absent from the above demonstration, which is relativistic in nature because we are describing radiation propagating in vacuo exactly at c. Obviously, the calculation is valid only in the vacuum, as soon as $B^{(3)}$ interacts with a fermion the Dirac equation shows that the two definitions of $B^{(3)}$ interlinked in vacuo by equation (188) remain valid, but become *inequivalent* because Eq. (188) is no longer valid. They must be used in the appropriate limits as discussed already, e.g. equation (2) is valid only in the weak field limit as shown by the Dirac equation. Equally obvious is the fact that $B^{(3)}$ can never be observed in vacuo, it can be observed only when it influences, or interacts with, a fermion. The same is true of any field [22—25].

4.5 ORIGIN OF THE ROTATING VACUUM CHARGE IN SIMPLE RADIATION THEORY

In this section we follow closely the classical (and classic) discussion by Jackson [47], of simple radiative processes. The origin of radiation is oscillating charge and current density, ρ and J, i.e., charge per unit volume and current per unit area. In the Maxwell equations, these are the famous source terms. In the vacuum Maxwell equations, the source terms are set to zero, because the *density* of charge and current is vanishingly small, the source of vacuum electromagnetic radiation can be thought of as being infinitely far away, but this is a mathematical ideal rather than anything that can be accepted as being physically meaningful. The important point is that charge conservation (the Noether theorem) demands that e cannot disappear from the analysis, so that even when the source terms are missing, charge conservation must be taken account of in the vacuum Maxwell equations, and therefore in the vacuum fields *themselves*. This means that the charge conjugation symmetry [1] ($\hat{C}$ symmetry), of the field amplitudes $E^{(0)}$ and $B^{(0)}$ (and also of $A^{(0)}$) must be negative. Since A is expressed [47] in terms of J, which is expressed in terms of e, the latter must appear in the vacuum fields themselves. This section demonstrates this conclusion mathematically using simple radiation theory. The classical picture therefore replaces the early nineteenth century concept of action at a distance (Coulomb interaction between two charges) by interaction via electromagnetic radiation propagating at c in vacuo. The d'Alembertian derivative [1,2] of the potential four-vector A_μ is *equated* to oscillating charge-current density J_μ (S.I. units),

$$\Box A_\mu = -\frac{J_\mu}{\epsilon_0}, \tag{228}$$

and since J_μ is defined in terms of e, so must A_μ, and since e is elementary and universal, it cannot disappear. Therefore electromagnetic fields in vacuo can be described through e as in the opening sections of this chapter. The charge e moves through the vacuum in a helical trajectory at c, depending on the radiation's wavelength, the propagation being made possible by the interaction of magnetic and electric fields as in Maxwell's vacuum equations. The source is an oscillating (or rotating) electron and the process of radiation can be thought of as transmutation of part of the electron energy to photon energy. This means that the photon carries with it the elementary charge e as in Eq. (202). In this process, as in any other, energy, momentum and charge must be conserved.

In S.I. units the general solution of Eq. (228) is given by Jackson [47],

$$A(r,t) = \frac{1}{4\pi\epsilon_0 c^2}\int d^3r' \int dt \frac{J(r',t')}{|r-r'|}\,\delta\left(t' + \frac{|r-r'|}{c} - t\right), \tag{229}$$

where the current density is defined through a Dirac delta function,

$$J = ev\,\delta(r - r'). \tag{230}$$

Here r is a coordinate vector. Equation (229) shows that *radiation* is a product of e with a space-time factor. For sinusoidally varying sources [47],

$$A(r) = e\left[\frac{1}{4\pi\epsilon_0 c^2}\int v \frac{\delta(r-r')}{|r-r'|} e^{i\kappa|r-r'|}\, d^3r'\right]. \tag{231}$$

Therefore the fields E and B of classical electromagnetic radiation are directly proportional to e, even though they propagate through the vacuum. This is precisely in accord with earlier sections of this chapter. This conclusion must carry through into relativistic quantum field theory [22—25] through the usual methods of canonical quantization, or alternative methods such as those proposed by Weinberg [93]. The photon as quantum of electromagnetic energy is proportional to e^2 (Eq. (202)) and can be thought of as *uncharged*, as in the literature on elementary particles [67], only with the utmost conceptual strain. Quantized fields, however, are creation and annihilation operators [1,2,22—25], and are proportional to e at first order. Consistently, electromagnetic energy (Eq. (225)) is again proportional to the square of field amplitude. An overemphasis on photon as *uncharged particle* leads to very great confusion, as evidenced in some recent theoretical papers [62—65] which attempt to assert that $B^{(3)}$ in vacuum is zero. As we have seen, this can only be so if e is zero, reductio ad absurdum.

In the radiation zone [47], the condition holds that $r \gg d$, where d is the dimension of the source. The oscillating electromagnetic field components become transverse to the direction of propagation. In much simpler language, and in a circularly polarized electromagnetic wave, the charge e describes a helix as it propagates in vacuo at c. The radius of the helix is Thompson's radius, κ^{-1}, and as discussed in the previous section, $B^{(3)}$ is formed by this motion, in analogy with a solenoid. In the latter, of course, the charge is carried by electrons, in the vacuum, Eq. (231) shows that electron mass is not present, but e clearly *is* present in free space. In the radiation zone [47], and in the standard dipole approximation,

$$A(r) = -e\left[\frac{1}{4\pi\epsilon_0 c^2}\int \nabla\cdot v\,\delta(r-r')e^{i\phi}\,d^3r'\right], \tag{232}$$

an equation that can be re-expressed as

$$A(r) := A_0 e^{i\phi}, \tag{233}$$

where ϕ is the phase of the electromagnetic radiation,

$$\phi = \omega t - \kappa\cdot r. \tag{234}$$

Therefore electromagnetic radiation in vacuo can be described entirely, and very simply, in terms of the motion of elementary charge, e. If e spirals through the vacuum with circularly polarized plane waves, we have

$$B = \frac{1}{c}k\times E, \tag{235}$$

and, as in Eq. (186), the transverse electric field component is

$$E^{(1)} = \frac{e}{\epsilon_0 V}r^{(1)}. \tag{236}$$

The transverse magnetic field component from Eq. (235) is

$$B^{(1)} = \frac{e}{\epsilon_0 cV}k\times r^{(1)}, \tag{237}$$

whose magnitude is $B^{(0)}$, given by

$$B^{(0)} = \frac{er}{\epsilon_0 cV} = \frac{e\mu_0 rc}{V}. \tag{238}$$

This is also the magnitude of $B^{(0)}$ from Eq. (220), a result which is consistent with the cyclical field relations,

$$B^{(1)}\times B^{(2)} = iB^{(0)}B^{(3)*}, \text{ et cyclicum,} \tag{239}$$

derived independently in Vol. 1 and 2. Again we find precise self-consistency in our theory. Experimentally, electromagnetic radiation is known to cause magnetization [16—21], and this magnetization can be pictured as circulating charge, motion caused in matter by an incoming electromagnetic field which is itself helical motion of e through the vacuum. A source electron at one end of the universe influences a matter electron at the other end through a helical journey of e through free space. This process conserves energy, charge and momentum, and the range of electromagnetic radiation is very great because there is very little mass associated with the radiation's e.

As we have seen in Chap. 1 and 2, $B^{(3)}$ can be detected only as it *interacts* with matter, and this interaction is controlled in the Dirac equation entirely by A and A^*, and not directly by the fields themselves. This lends support to the feeling that the four-potential A_μ is something *more fundamental* than the derivative fields, a line of thought that is supported by the Aharonov-Bohm effects [32—37,39] as described in Chap. 3. If this point is not appreciated, confusion will multiply, as in the experiment by Rikken [74]. This author's conclusion that $B^{(3)}$ is zero must mean that e is zero, and if there is no e, there is no radiation at all. Conversely therefore, $B^{(3)}$ is *non* zero whenever there is circularly polarized radiation present. Linearly polarized radiation is a superposition of left and right circular components, and $B^{(3)}$ is again present, but equal and opposite for these components. Similarly, $B^{(3)}$ is non-zero in elliptical polarization.

Finally in this section we emphasize a little known, but important, conclusion of special relativity, that two charges co-translating at c do not attract or repel. This point has been made clearly by French [70], and is used in a recent paper by Mac Gregor [69]. Therefore charged fields on neighboring photons (traditionally of zero mass and propagating at c) do not attract or repel in vacuo. As soon as field-matter interaction occurs, there is charge-charge interaction because the matter charge velocity is less than c and the incoming electromagnetic field influences an electron as demonstrated by Hertz in the nineteenth century. Photon-photon scattering appears to take place without Coulomb interaction because both beams propagate very near c.

4.6 BIOT-SAVART LAW FOR $B^{(3)}$ IN THE VACUUM

The origin of magnetic fields in moving charges was demonstrated [47] in the early nineteenth century by Biot and Savart, whose work was extended greatly by Ampère. Therefore if $B^{(3)}$ is a magnetic field, it must be described classically by the Biot-Savart-Ampère (BSA) law as usual. In this section this expectation is supported by the

following analysis in vacuo. If classical electromagnetism in vacuo is simply the spiralling motion of e at c, then our derivation follows clearly. The source of $B^{(3)}$ is a moving charge, e, but this is *not* the electron, because mass is not present in the Maxwellian point of view, it is in that view the elementary charge propagating without mass. If the photon has mass, then this is thought [1,2] to be at least thirteen or fourteen orders of magnitude less than that of the electron. The BSA law for $B^{(3)}$ follows from the fact that classical electromagnetism is the spiralling motion of e in vacuo, and the law can be expressed as

$$B^{(3)*} = -\frac{i}{c^2} v^{(1)} \times E^{(2)}, \tag{240}$$

in S.I. units. Here $v^{(1)}$ is the transverse velocity of the charge in vacuo, and as shown earlier in this chapter, the magnitude of this velocity must be c. Equation (240) is also a direct consequence of special relativity. In the circular basis ((1), (2), (3)) [1,2] the field $E^{(2)}$ is a rotating electric field, and so $v^{(1)}$ is also a rotating transverse velocity, that of the spiralling charge e in vacuo.

The self-consistency of Eq. (240) can be checked by expressing $E^{(2)}$ as

$$E^{(2)} = i\omega A^{(2)} = -\frac{\partial A^{(2)}}{\partial t}, \tag{241}$$

in terms of the rotating vector potential $A^{(2)}$. Similarly, the complex conjugate $E^{(1)}$ (whose physical, real, part is the same as $E^{(2)}$ by definition) can be expressed as

$$E^{(1)} = -i\omega A^{(1)} = -\frac{\partial A^{(1)}}{\partial t}. \tag{242}$$

The BSA law for $B^{(3)}$ therefore becomes expressible as

$$B^{(3)*} = \frac{\kappa}{c} v^{(1)} \times A^{(2)}. \tag{243}$$

The factor $\kappa v^{(1)}/c$ in Eq. (243) has the units of a wave-vector, and can be expressed in quantum mechanics as a momentum operator ($\hbar\kappa = ip$), giving the result

$$B^{(3)*} = -\frac{i}{\hbar} p^{(1)} \times A^{(2)}. \tag{244}$$

Re-expressing $p^{(1)}$, finally, as a del operator, $p^{(1)} = -i\hbar\nabla^{(1)}$, we recover the fundamental definition of a magnetic field in terms of the curl of a vector potential. As demonstrated in Vol. 1 and 2 [1,2], there is no physical $A^{(3)}$ or $E^{(3)}$, and so in the Dirac equation, the field and fermion interact through $A^{(1)}$ and $A^{(2)}$ only, these being the only physical components of the vector potential in vacuo. If classical electromagnetism is spiralling charge, then there is no longitudinal electric field in vacuo, in analogy with a solenoid. The latter produces a longitudinal $B^{(3)}$ but no longitudinal $E^{(3)}$. The only conceptual difference is that in a solenoid, the charge is carried by a massive electron in the windings of the solenoid, in the vacuum the charge is massless, F.A.P.P. There is no Faraday induction in vacuo of a putative $E^{(3)}$ from $B^{(3)}$ because for a given frequency ω, there is no experimental method available of changing the spiralling motion of e in vacuo. Its radius is fixed at c/ω and its forward and transverse velocities at c. In contrast, the ordinary Faraday induction law in vacuo is one of the Maxwell equations, and deals with the physical transverse electric and magnetic fields.

4.7 EQUIVALENT FORMS OF $B^{(3)}$ IN VACUO

The fundamental and widely known axioms of the quantum theory assert that

$$En = \hbar\omega, \quad p = \hbar\kappa, \quad p \to -i\hbar\nabla, \quad En \to i\hbar\frac{\partial}{\partial t}, \tag{245}$$

and so it is possible to write transverse momenta in the circular basis as wave-vectors and del operators. For example

$$p^{(1)} = \hbar\kappa^{(1)} = -i\hbar\nabla^{(1)}, \quad p^{(2)} = \hbar\kappa^{(2)} = -i\hbar\nabla^{(2)}. \tag{246}$$

The origin of these transverse momenta is, as we have seen, simply the spiralling motion of e at c. The famous axioms (245) identify the particulate and undulatory nature of radiation and matter, and make all particles waves and vice-versa. This is the idea originally proposed by de Broglie [94]. The classical forms of the Evans-Vigier field $B^{(3)}$ show that it is a physical observable, directly so in the strong field limit, and show that the group space of electromagnetism is that of O(3). The electromagnetic

sector of unified field theory is therefore O(3), perhaps with far reaching consequences. Some of these have been sketched in Vol. 2 [2]. In other words, vacuum electromagnetism is elementary and universal charge, e, spiralling through free space, and the often complicated mathematical machinery of field theory is subsidiary to, and dependent upon, this very simple view. Nature at the fundamental level appears to be inherently simple provided the source of understanding is found.

The most intricate derivation of $B^{(3)}$ must therefore depend on the simple, spiralling e, and all the expressions given for $B^{(3)}$ in this chapter must be equivalent. By using condition (188) in wave-vector form,

$$p^{(1)} = eA^{(1)} = \hbar\kappa^{(1)}, \tag{247}$$

two more equations for $B^{(3)}$ can be derived. In Eq. (247), $\kappa^{(1)} = \kappa^{(2)*}$ is a rotating, transverse, wave-vector in the circular basis [1,2], a vector whose magnitude is κ. Using Eq. (188) we obtain

$$B^{(3)*} = -i\frac{B^{(0)}}{\kappa^2}\kappa^{(1)} \times \kappa^{(2)} = -i\frac{B^{(0)}}{p^{(0)2}}p^{(1)} \times p^{(2)}. \tag{248}$$

The various forms of $B^{(3)}$ are collected in Table 3. They can all be interpreted physically to mean that e spirals forward at the speed of light, the radius of the spiral being c/ω. The elementary charge appears explicitly in the O(3) form, and implicitly in the nine others. All balance $\hat{C}$, $\hat{P}$, and $\hat{T}$ symmetry [1]. The BSA and curl A forms are standard, classical expressions for a magnetic flux density. Condition (188) produces a *Dirac form* directly from the O(3) gauge form. Therefore the equivalence condition (188) is confirmed as a fundamental part of vacuum electromagnetism. The same conclusion is obtained from the relativistic HJ equation of e in A_μ. The BSA form is obtainable from the double A form using it, and the BSA form is the earliest classical understanding of the source of a magnetic field. The source of $B^{(3)}$ is found in the propagating electromagnetic plane waves, which can be understood in circular polarization simply as spiralling elementary charge e. Each photon can be thought of as producing its own quantum of $B^{(3)}$, which we have alluded to as the photomagneton [4].

TABLE 3

Forms of the Evans-Vigier Field $B^{(3)}$

Form	$B^{(3)*}$
Double A	$B^{(3)*} = -i\frac{\kappa}{A^{(0)}}A^{(1)} \times A^{(2)}$
O(3) Gauge	$B^{(3)*} = -i\frac{e}{\hbar}A^{(1)} \times A^{(2)}$
Biot-Savart-Ampère	$B^{(3)*} = -\frac{1}{cp^{(0)}}p^{(1)} \times E^{(2)}$
AE	$B^{(3)*} = -\frac{1}{cA^{(0)}}A^{(1)} \times E^{(2)}$
Curl A	$B^{(3)*} = -\nabla^{(1)} \times A^{(2)}$
Dirac	$B^{(3)*} = -\frac{i}{\hbar}p^{(1)} \times A^{(2)}$
Double B	$B^{(3)*} = -\frac{i}{B^{(0)}}B^{(1)} \times B^{(2)}$
Double E	$B^{(3)*} = -\frac{i}{cE^{(0)}}E^{(1)} \times E^{(2)}$
Double p	$B^{(3)*} = -i\frac{B^{(0)}}{p^{(0)2}}p^{(1)} \times p^{(2)}$
Double κ	$B^{(3)*} = -i\frac{B^{(0)}}{\kappa^2}\kappa^{(1)} \times \kappa^{(2)}$

With the benefit of hindsight, the double E, B and A forms of Table 3 have already shown that $iB^{(0)}B^{(3)}$ has been observed [16—21] experimentally many times using visible frequency radiation, and once with 3 GHz microwave radiation [17c]. Examples include the inverse Faraday effect [17], optical Faraday effect [79,95], and light shifts

in atomic spectra [18]. Less clearly, but definitively, light induced NMR shifts have been reported [20] as described in Chap. 1 and 2. It is now clear from the work of Chap. 3 that these are phenomena of the weak field limit, and are proportional to intensity, I. The conditions of the strong field limit appear not to have been approached yet, as discussed in earlier volumes [1,2]. Electric and magnetic fields are space and time derivatives of vector and scalar potentials, and the latter determine the way in which electromagnetic fields interact with matter. This is the minimal prescription, a consequence of gauge theory and the conservation theorems. The interaction of $B^{(3)}$ with a fermion is therefore determined by the fermion's interaction with A and A^*, a consideration which shows that our first, heuristic, theory of $B^{(3)}$ [1—15] is the strong field limit of this chapter. In the weak field limit, Eq. (2) applies.

The existence in the vacuum of $B^{(3)}$ neatly demonstrates the acute weaknesses inherent in the view that electromagnetism is a flat U(1) sector of unified field theory. The $B^{(3)}$ field is both physical and perpendicular to the plane used to define the U(1) group. Furthermore $B^{(3)}$ is the immediate result of a spiralling e, which emerges from classical radiation theory [47]. The Euclidean little group E(2) [1,2,23] must be replaced by an O(3) little group. For this reason the photon as particle can no longer be thought of as massless, because in that view the little group is the unphysical [23] E(2) and fields in vacuo are transverse. Our overall conclusion is that there are no massless particles in nature.

4.8 CONSERVATION OF ENERGY-MOMENTUM IN VACUO

It is well known that the electrodynamical conservation laws are fundamental to any consideration of light in free space or the interaction of light with matter. The momentum and energy of a radiation pulse totally contained within a finite volume V [52] has the same Lorentz transformation properties as a material point particle, and so the laws of conservation of energy and momentum must be similar to those of a particle. The latter is conventionally asserted to have no mass, and is the *standard* photon. The previous section concluded, however, that there can be no massless particle in nature, so the standard photon is a flawed concept. In the received view [52] the energy-momentum conservation law of classical field theory can be expressed as

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} = 0, \tag{249}$$

where $T^{\mu\nu}$ is the energy momentum four-tensor and x^{μ} the metric. This implies that the quantity

$$G^{\mu}G_{\mu} = W^2 - c^2G^2 = 0, \tag{250}$$

is a Lorentz invariant, where W denotes classical electromagnetic energy density and where G denotes classical electromagnetic linear momentum density

$$\hbar\omega = \int W dV = c\int G dV. \tag{251}$$

These relations remain valid in the presence of the magnetic field $B^{(3)}$, because the same concept that produces (249) and (251) produces $B^{(3)}$. This concept is simply that of e spiralling in vacuo at c. In this section it is shown that the validity of the Rayleigh Jeans and Planck laws remain unaffected by $B^{(3)}$. This must be so because the source of $B^{(3)}$ (e spiralling through the vacuum) is the same as that of the conventionally accepted transverse fields $B^{(1)}$, $B^{(2)}$, $E^{(1)}$ and $E^{(2)}$.

For example, the value of the Planck constant remains the same, because $B^{(3)}$ simply causes the available electromagnetic energy density to be re-distributed among three space indices rather than two. The radiation oscillator must therefore be associated at each angular frequency ω by

$$U_{new} = \frac{1}{\mu_0}\left(B^{(1)}\cdot B^{(1)*} + B^{(2)}\cdot B^{(2)*} + B^{(3)}\cdot B^{(3)*}\right), \tag{252}$$

instead of the traditional

$$U = \frac{1}{\mu_0}\left(B^{(1)}\cdot B^{(1)*} + B^{(2)}\cdot B^{(2)*}\right), \tag{253}$$

When the sense of circular polarization (handedness) is switched from right (+) to left (-),

$$B_{+}^{(1)} = B_{+}^{(2)*} \rightarrow B_{-}^{(1)} = B_{-}^{(2)*}, \qquad B_{+}^{(3)} \rightarrow B_{-}^{(3)}, \tag{254}$$

and the total field is defined in both right and left circular polarization by three circular indices,

$$\begin{aligned} B_+ &= B_+^{(1)} + B_+^{(2)} + B_+^{(3)}, \\ &\Downarrow \\ B_- &= B_-^{(1)} + B_-^{(2)} + B_-^{(3)}, \end{aligned} \tag{255}$$

n terms of the electric components

$$U_{new} = \epsilon_0 \left(E^{(1)} \cdot E^{(1)*} + E^{(2)} \cdot E^{(2)*} + iE^{(3)} \cdot (iE^{(3)})^* \right), \tag{256}$$

eplaces the conventional

$$U = \epsilon_0 \left(E^{(1)} \cdot E^{(1)*} + E^{(2)} \cdot E^{(2)*} \right). \tag{257}$$

The premultiplier (factor of two) in the density of states calculated from the Rayleigh-eans law is therefore *two* in the presence of $B^{(3)}$, because there are still two senses of ircular polarization (+ and - subscripts).

The Rayleigh-Jeans law determines the number density of oscillators with requencies in the range ω to $\omega + d\omega$. For each oscillator there are two senses of ircular polarization, *each* of which is described by three circular indices. The two enses are physically distinct, and for each there exists the cyclically symmetric algebra 239). In left circular polarization the helical motion of e through free space at c is he mirror image of the motion in right circular polarization. These two physically listinct components determine the premultiplying value of two in the Rayleigh-Jeans law or the density of states,

$$dN = \frac{8\pi\nu^2}{c^3} d\nu, \tag{258}$$

where ν is the frequency of the wave. The existence of $B^{(3)}$ does not therefore affect Eq. (258) for the density of states, but $B^{(3)}$ does of course introduce the third space ndex to the analysis. When the sense of circular polarization of the beam is witched, $B^{(3)}$ changes its direction. In the strong field limit this effect might become xperimentally observable. In the weak field limit, the interaction is determined by $B^{(3)}$ at second order, through $A^{(1)} \times A^{(2)}$.

The Planck hypothesis of November 1900 [61] asserts that a radiation oscillator can possess only discrete energies, measured in quanta, $\hbar\omega$, of radiation energy. The quantum is the dictionary definition [61] of the photon. The effect of $B^{(3)}$ is to change

the mathematical expression of the Planck hypothesis for one photon from

$$\hbar\omega = \frac{1}{\mu_0} \int \left(B^{(1)} \cdot B^{(1)*} + B^{(2)} \cdot B^{(2)*} \right) dV, \tag{259}$$

to

$$\hbar\omega = \frac{1}{\mu_0} \int \left(B^{(1)} \cdot B^{(1)*} + B^{(2)} \cdot B^{(2)*} + B^{(3)} \cdot B^{(3)*} \right) dV_1 . \tag{260}$$

Thus, classical electromagnetic energy is expressed as a sum over three circular indices instead of two, but the total available energy density is the same. The numerical values of the right hand sides of equations (259) and (260) are the same, and the Planck constant $\hbar$ is unchanged. The photon also remains the same and there is no change in the Planck radiation law,

$$dU = \frac{8\pi\hbar\nu^3}{c^3} \left(\frac{e^{-\hbar\nu/kT}}{1 - e^{-\hbar\nu/kT}} \right) d\nu . \tag{261}$$

The total energy density of black body radiation is obtained by integrating dU over all frequencies [61]. Since $\hbar\omega$ is redistributed among three circular indices, so is black body radiation, but it is not possible to isolate the specific effect of $B^{(3)}$ from that of $B^{(1)} = B^{(2)*}$ simply through measurement of black body radiation with heat detectors. The specific effect of $B^{(3)}$ can be measured in the strong field limit, however, by isolating the expected $I^{1/2}$ profile in magneto-optic effects, as described in Vol. 1 and 2 [1,2].

The equilibrium between radiation and matter was shown by Einstein to be made up of several distinguishable processes, described by the well known Einstein coefficients [61]. The rate of absorption of electromagnetic radiation is described in terms of the Einstein B coefficient,

$$W_{f \leftarrow i} = B_{if} \rho , \tag{262}$$

from an initial quantum state $|i>$ to a final quantum state of higher energy $|f>$. Here ρ is the energy density of states at the absorption frequency ν_{fi}, and ρ must be evaluated at the transition frequency. For electric dipole transitions $W_{f \leftarrow i}$ is proportional to $En\,\hat{\rho}_N / V$ where En is the energy of the field at frequency ν and $\hat{\rho}_N$ is the frequency density of states for a given volume V occupied by the electromagnetic

radiation. The specific effect of $B^{(3)}$ in these calculations is to cause the electromagnetic energy to be redistributed among three space indices (1), (2) and (3). The rate of absorption (262) is therefore also unaffected.

We conclude that the absorption of a photon of energy $\hbar\omega$ by an atom at a frequency ν_{fi} defined by a transition from $|i>$ to $|f>$ is affected by $B^{(3)}$ only insofar that the definition of $\hbar\omega$ is modified by $B^{(3)}$ as in Eq. (260). The angular momentum conservation rules that apply in the absorption of a photon are not changed by the existence of $B^{(3)}$, which is generated by the angular momentum of the photon about the propagation axis. The specific $I^{1/2}$ effect of $B^{(3)}$ has evaded detection because the strong field limit has not been attained.

4.9 CONSERVATION OF CHARGE

Charge is always conserved experimentally, and this feature of the natural world has become understood in terms of a continuity equation,

$$\nabla\cdot\boldsymbol{J}+\frac{\partial\rho}{\partial t}=\nabla\cdot\boldsymbol{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}=0. \tag{263}$$

The scalar potential ϕ of electromagnetic radiation in vacuo due to a point charge $ee^{i\omega t}$ [96] is identical to the equivalent electrostatic potential except that $t - R/c$ is substituted for t where R is the source to observer distance. *The potential corresponds to a state of the charge at a previous time* $t - R/c$. Therefore the appearance of e in the vacuum equations in previous sections can be understood as corresponding to its state in the source volume at a time R/c earlier. If the charge e had been circling in the source at this time, then it would have formed a longitudinal magnetic field according to the Biot-Savart law. At a time R/c later this field appears in the vacuum and is the curl of a vector potential which can be defined in general through the retarded current, $[J]$,

$$\boldsymbol{A}=\frac{\mu}{4\pi}\int_{\infty}\frac{[\boldsymbol{J}]}{R}dV. \tag{264}$$

This is the well known mechanism of Liénard and Wiechert [96] and it is a mechanism which conserves e. The vacuum $B^{(3)}$ field corresponds to its state in the source a time R/c earlier, a source in which it was formed by circulating charge. This is also true of the transverse oscillating fields, and the circulating charge does not produce an axial electric field because the Biot-Savart law does not do so in the source. The phase

velocity of light in vacuo is c for the conventionally massless photon, and we see distant stars as they were, millions of years ago, due precisely to the Liénard-Wiechert mechanism. The complex nature of $A^{(1)}$ and $A^{(2)}$ allow therefore for phase differences in the source, as they were a time R/c ago [96].

Therefore all observable field components propagating in vacuo also correspond to a state of charge a time R/c earlier, and this is how e must be interpreted in vacuo. A circulating charge in the source transmits its influence, via the vacuum $B^{(3)}$, to produce a circulating movement of charge in matter, i.e., a magnetization known as the inverse Faraday effect. The machinery of interaction of field and matter is governed, however, through the Dirac equation using the potentials, and not directly through the fields themselves. This is a statement of the minimal prescription [22—25], which is itself a consequence of gauge invariance and the conservation of charge. This leads to the weak and strong field limits discussed in Chap. 3.

At a time t the field $B^{(3)}$ corresponds to a state of circulating charge a time R/c earlier; at time t' the field $B^{(3)}$ corresponds to a state of circulating charge a time R'/c earlier; and so on. So it is clear that $B^{(3)}$ propagates in vacuo at the phase velocity c. It propagates at the same phase velocity as the transverse components $B^{(1)}$ and $B^{(2)}$, the rotating transverse magnetic components that correspond also to a state of charge a time R/c earlier. Since $B^{(3)}$ is the curl of a vector potential by definition, its influence on matter is determined by the influence of that vector potential on matter, and this occurs through the use of the minimal prescription in the Dirac equation. The radiative corrections of quantum electrodynamics can now be incorporated in this classical picture, and it is clear that $B^{(3)}$ violates none of the principles of electromagnetic theory.

Chapter 5. The Classical Radiation Theory of $B^{(3)}$

There exists in free space a primordial $B^{(3)}$ field (Chap. 8). Its interaction with matter depends on the intensity and frequency of the electromagnetic field that transmits $B^{(3)}$ from source to sample. In this chapter we first prove this theorem using a simple development of the radiation theory summarized in the last chapter. The effects of the free space $B^{(3)}$ on radiation theory (boson ensembles) are then developed following the methods first proposed by Dirac for quantizing the electromagnetic field.

5.1 RADIATION OF $B^{(3)}$ FROM A CIRCLING ELECTRON

In this section it is shown that a circling source electron radiates $B^{(3)}$ to a sample electron, which is set in circular motion according to the reciprocity theorem [96]. The latter states that the current in a detector divided by the voltage applied at the source remains constant when source and detector are interchanged, as long as the frequency and impedances are left unchanged. This theorem applies to any electromagnetic field, and therefore applies to $B^{(3)}$. The energy emitted as radiation by the circling source electron is given up to the sample electron, which is thereby set in circular motion. This process can be understood entirely in terms of the field $B^{(3)}$, which is radiated through the vacuum by the Liénard-Wiechert potentials $A^{(1)}$ and $A^{(2)}$, which are transverse to the direction of propagation. The circular motion set up in the sample electron is magnetization, and this process is controlled by the Dirac equation as described in Chap. 1 and 2, using only the transverse potentials $A^{(1)}$ and $A^{(2)}$ through the minimal prescription. The overall process is simply the influence of one circling electron on another, initially static, sample electron.

It can be demonstrated qualitatively in the laboratory using a model consisting of an ideally massless charged object (A) on the end of a rotating rod. The amount and sign of charge and angular velocity can be varied experimentally. Another, initially stationary, ideally massless and charged object (B) is constrained to rotate in the same plane on the perimeter of a circular loop whose diameter is just larger than that of the rod carrying A. If A is highly charged and rotating very slowly, B is either attracted or repelled by A (depending on whether it is charged with the opposite or same sign respectively), and the angular velocity of B is approximately the same as that of A, and is the *same* in direction independently of the sign of charge. The charged object B is

either pushed (repelled) or pulled (attracted) in an orbit by the rotating A, clockwise if A is moving clockwise, or vice-versa. The motion of B is the result of the magnetic field set up by A. If this macroscopic laboratory model is replaced by a system A consisting of a source electron moving in a circle, and a system B consisting of a sample electron, the sample electron will move in a circle due to the magnetic field set up by the source electron. This is the field $\boldsymbol{B}^{(3)}$, and the source and sample electrons can be separated by millions of light years because electromagnetic radiation from the former reaches the latter through the *radiated* $\boldsymbol{B}^{(3)}$. In the strong field limit, the angular frequency of the radiation is low (source electron moving slowly) and its intensity is high. In this limit, as shown by the Dirac equation, the circling motion produced in the sample electron is proportional to $\boldsymbol{B}^{(3)}$ and to the square root of beam intensity.

In the weak field limit the massless object A is weakly charged (low field intensity) and is spinning very rapidly (high frequency). The effect on B will be indirect, it will take many revolutions of A before B starts moving, and it will be out of synchronization with A. In the weak electromagnetic field limit, the Dirac equation shows that the effect on the sample electron is second order in the magnitude of $\boldsymbol{B}^{(3)}$ and first order in field intensity. Intermediate cases will show a dependence on both I and square root I as discussed in Chap. 12 of Vol. 1 [1].

In both cases discussed above, the sample electron (charge) is set in motion in a circle by the source electron (charge), and this is magnetization. The magnetic field $\boldsymbol{B}^{(3)}$ is simply the rotation of charge. In the magneto-optics literature the magnetization is known as the inverse Faraday effect [16—21] if the circling motion is caused by a circularly polarized electromagnetic field. It becomes clear that the radiated $\boldsymbol{B}^{(3)}$ field is due to the rotation of the potentials $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, which according to the Liénard-Wiechert law refer to a state of charge [96] a time R/c earlier, as discussed in Chap. 4 (R is used in this section to distinguish the observer to source distance from the radius, r, of a charge circling in a source a time R/c earlier.) At a time R/c prior to that at which $\boldsymbol{B}^{(3)}$ is observed in the vacuum, the state of the charge was its state in the source of radiated $\boldsymbol{B}^{(3)}$. The radiated $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are transverse to the direction of propagation, but they represent a state of circling charge a time R/c earlier. This circling charge produces the $\boldsymbol{B}^{(3)}$ field, and the magnetization of the inverse Faraday effect. In order to observe the direct effect of $\boldsymbol{B}^{(3)}$ at first order, it is necessary to observe the square root intensity dependence of the magnetization [1,2]. The same rotating charge that produces $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ must inevitably produce $\boldsymbol{B}^{(3)}$, and since $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ exist in vacuo (being Liénard Wiechert potentials) so does $\boldsymbol{B}^{(3)}$. By causality, it is not possible to obtain electromagnetic radiation without a source of radiation having been present a time R/c earlier.

Conversely, if $\boldsymbol{B}^{(3)}$ is zero in vacuo, there are no potentials present, and no electromagnetic radiation. Any argument [62—65] that attempts to show that $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are non-zero while $\boldsymbol{B}^{(3)}$ is zero is therefore incorrect, because that argument violates the Biot-Savart law. This conclusion holds independently of the details of the argument because, as we show next, the rotating elementary charge, e, radiates $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$ and $\boldsymbol{B}^{(3)}$ simultaneously. In other words, the fact that e radiates the Liénard-Wiechert potentials means that it must also radiate $\boldsymbol{B}^{(3)}$, because without $\boldsymbol{B}^{(3)}$ there is no $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$.

5.2 RADIATION FROM THE ROTATING ELEMENTARY CHARGE e

Corson and Lorrain [96] have provided a clear summary of classical radiation theory, which produces electromagnetic fields in vacuo through the scalar and vector potentials, ϕ and $\boldsymbol{A}$ respectively. In S.I. units these can be written as a single potential four-vector,

$$A_\mu = (c\boldsymbol{A}, \phi). \tag{265}$$

The Aharonov-Bohm effects [32—39] show that the potentials are physically meaningful in classical electrodynamics and the interaction of radiation with matter takes place through the minimal prescription, i.e., through the four-vector A_μ and not through the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Electric and magnetic fields are therefore secondary concepts. In particular, a magnetic field is essentially a moving charge, and it is shown in this section that the circular motion of one elementary charge, e, radiates $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ simultaneously with $\boldsymbol{B}^{(3)}$, meaning that if $\boldsymbol{B}^{(3)}$ were zero there would be no source for the electromagnetic field, and by causality, no field. This conclusion is easily seen from the cyclic relations (154) by setting $\boldsymbol{B}^{(3)} = 0$. This results in $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)} = 0$ and the complete loss of electromagnetic radiation in vacuo.

The electromagnetic potentials satisfy the nonhomogeneous wave equations,

$$\nabla^2\phi - \epsilon\mu\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon}, \qquad \nabla^2\boldsymbol{A} - \epsilon\mu\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu\boldsymbol{J}, \tag{266}$$

where ρ and $\boldsymbol{J}$ are the source charge and current densities respectively. The solutions of these equations for ϕ and $\boldsymbol{A}$ at time t are expressible in terms of ρ and $\boldsymbol{J}$ defined at the earlier time $t - R/c$. The time R/c is that taken by an electromagnetic wave to travel the distance R between the element of volume where ρ and $\boldsymbol{J}$ are evaluated to the point of observation [96]. In the laboratory this interval of time is very short, and the interaction between charges appears to be instantaneous. On a cosmological scale

the interval can be millions of light years, but the ρ and J must still be defined as they were in the source. If there *was* no source there *is* no radiation. For electric dipole radiation for example [96], a transverse spherical wave radiates away from the dipole, and $E^{(0)}/B^{(0)}$ is the same as for a plane wave in the vacuum. Nearer the source, the radiation is more complicated in form. In essence, the transverse plane wave is a charge spiralling forward in the vacuum, and this single charge produces both $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ and $\boldsymbol{B}^{(3)}$. The latter are therefore descriptions derived from the same moving charge, and it is not possible to assert that $\boldsymbol{B}^{(3)}$ is zero for non-zero $\boldsymbol{A}^{(1)}$ because at an instant t they are descriptions of the same thing, a circling elementary e defined a time R/c earlier. Being different ways of describing the same thing, they must be equivalent and so one cannot be zero while the other is non-zero. These points are clarified by the following simple example.

Consider an idealization of a source of electromagnetic radiation, an idealization consisting of the elementary charge e describing a circle of radius r at an angular frequency ω. In the basis ((1), (2), (3)) this motion defines the dipole moments [96],

$$\boldsymbol{p}^{(1)} = ere^{i\omega t}\boldsymbol{e}^{(1)}, \qquad \boldsymbol{p}^{(2)} = ere^{-i\omega t}\boldsymbol{e}^{(2)}, \tag{267}$$

at the instant t. The real parts of these two equations are the same, and describe a charge e rotating clockwise in a plane. As shown in Chap. 4, this motion produces the magnetic flux density,

$$\boldsymbol{B}^{(3)} = \mu_0 \boldsymbol{m}^{(3)} = \frac{\mu_0 erv\boldsymbol{e}^{(3)}}{V}, \qquad \boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}. \tag{268}$$

Without loss of generality, the volume V can be defined as $(4/3)\pi r^3$, i.e., as a sphere of radius r. The magnitude $B^{(0)}$ of $\boldsymbol{B}^{(3)}$ is therefore

$$B^{(0)} = \frac{3\mu_0 ev}{4\pi r^2}, \tag{269}$$

which has the right units of J s C^{-1} m^{-2}, i.e., tesla or Wb m^{-2} i.e., V s m^{-2}. So $B^{(0)}$ is a magnetic flux $(3/4)\mu_0 ev$ in *Wb* divided by the area πr^2. This is another way (more complicated and more obscure) of saying that e circles with a radius r and angular velocity ω. The magnetic flux in *Wb* is proportional to the product ev. In terms of the dipoles,

$$ere^{(3)*} = -\frac{i}{|\boldsymbol{p}|}\boldsymbol{p}^{(1)} \times \boldsymbol{p}^{(2)}. \tag{270}$$

The transverse current (C rad s^{-1}) is $I = e\omega$. Therefore,

$$B^{(0)} = \frac{3\mu_0 I}{4\pi r}, \tag{271}$$

and the magnitude of the $\boldsymbol{B}^{(3)}$ field is proportional to the current divided by the radius. Putting these definitions together gives

$$\boldsymbol{B}^{(3)*} = -\frac{3\mu_0\omega}{4\pi r^3 e} i\boldsymbol{p}^{(1)} \times \boldsymbol{p}^{(2)}, \tag{272}$$

which shows that $\boldsymbol{B}^{(3)}$ is proportional in the source to the conjugate product of $\boldsymbol{p}^{(1)}$ with $\boldsymbol{p}^{(2)}$. This example clarifies what is meant physically in electrodynamics by any conjugate product, it represents, essentially, a circling motion of charge and therefore a magnetic flux density. The magnetic flux density $\boldsymbol{B}^{(3)}$ has been defined in the source in terms of circling charge, and in so doing the exponents $e^{i\omega t}$ and $e^{-i\omega t}$ have been used in a conjugate product.

This source radiates because the charge e is accelerating. The radiation is measurable at a point in space at an instant t if ωt is replaced in Eq. (272) by $\omega(t - R/c)$. The $\boldsymbol{B}^{(3)}$ in free space is no longer static and reappears as

$$\boldsymbol{B}^{(3)*} = -\frac{3\mu_0\omega}{4\pi r^3 e} i[\boldsymbol{p}^{(1)}] \times [\boldsymbol{p}^{(2)}], \tag{273}$$

where $[\boldsymbol{p}^{(1)}]$ and $[\boldsymbol{p}^{(2)}]$ signify [96] that the state of charge is defined at a time R/c earlier. This may be millions of light years or may be nanoseconds. As we have seen in earlier volumes [1,2] and chapters, Eq. (273) may be redefined as

$$\boldsymbol{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{274}$$

where $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ is the radiated transverse vector potential. Far enough away from the source, the only vector potentials present are the transverse potentials, as in classical radiation theory [96]. The transverse vector potential is defined in terms of the

transverse current I as

$$A^{(1)} = \frac{3\mu_0}{4\pi}[J^{(1)}] := \int \frac{\mu_0}{V_0}[J^{(1)}]dV, \tag{275a}$$

$$[J^{(1)}] = Ie^{i(\omega t - \kappa R)}e^{(1)}, \tag{275b}$$

where V_0 can be taken to be a sphere or any other volume. Similarly, its complex conjugate is

$$A^{(2)} = \frac{3\mu_0}{4\pi}[J^{(2)}] := \int \frac{\mu_0}{V_0}[J^{(2)}]dV = \frac{3\mu_0 I}{4\pi}e^{-i(\omega t - \kappa R)}e^{(2)}, \tag{276}$$

and this is the result of standard radiation theory [96] applied to a rotating and radiating e in a source. Gradually, the whole of the energy of the rotating e will be dissipated by radiation and it will stop radiating. The energy loss per unit time is given up as radiative power [96] and this can be expressed in terms of $B^{(3)}$. Finally, using the free space relation $B^{(0)} = \kappa A^{(0)}$, we obtain from Eq. (275)

$$B^{(3)} = \frac{\mu_0 ec^2}{\omega V_0}e^{(3)}, \tag{277}$$

which is equation (223). Therefore $B^{(3)}$ originates in the radiating e circling at radius r a time R/c prior to the instant, t, at which $B^{(3)}$ is detected in free space. The conjugate product $A^{(1)} \times A^{(2)}$ represents exactly the same thing, but in a different mathematical language, and since $A^{(1)}$ and $A^{(2)}$ are the ordinary radiated vector potentials, $B^{(3)}$ is also radiated. This is an inference which is missing from conventional radiation theory, but the fact that $B^{(3)}$ is radiated simply means that there was a charge e circling at r a time R/c ago in the source of that radiation. A circling charge does not produce an electric field in the axis perpendicular to the plane in which the charge is circling, and there is no radiated $E^{(3)}$. Finally, the interaction of $B^{(3)}$ with matter is controlled by A_μ interacting with matter, and is the reverse of the radiative process. The correct way to describe this interaction is through the Dirac equation, as described in Chap. 1 and 2. Qualitatively, the interaction can be understood in terms of a mechanical model which can be set up in the undergraduate laboratory and which has been described in this chapter.

In summary, the field $B^{(3)}$ in the source is static and is

$$B^{(3)*} = -\frac{\mu_0\omega}{eV_0}ip^{(1)} \times p^{(2)}, \tag{278}$$

where V_0 is a volume defined by the radius of the rotating elementary charge, e. The radiated $B^{(3)}$ in the vacuum is propagating and is

$$B^{(3)*} = -\frac{\mu_0\omega}{eV_R}i[p^{(1)}] \times [p^{(2)}], \tag{279}$$

where V_R is the volume of radiation being considered, e.g. the sphere defined by the Thompson radius $\lambda/2\pi$. Since $B^{(3)}$ is non-zero in the source, it is non-zero in the vacuum.

5.3 CONSERVATION OF ANGULAR MOMENTUM AND ENERGY

The quantum theory asserts that the angular momentum of the photon is $\hbar$. The classical angular momentum magnitude of the rotating charge in the radiating source is $eA^{(0)}r$, where $A^{(0)}$ is the amplitude of $A^{(1)}$ in the source. In the radiated field, $r = \kappa^{-1}$, and we have, from Eq. (182),

$$\hbar = e\frac{A^{(0)}}{\kappa} = e\frac{c^2}{\omega^2}B^{(0)}. \tag{280}$$

The angular momentum radiated by the circling charge e is therefore the field angular momentum,

$$J^{(3)} = e\frac{c^2}{\omega^2}B^{(0)}e^{(3)} = \frac{1}{\mu_0\omega}B^{(0)2}Ve^{(3)}, \tag{281}$$

from Chap. 4. This means that

$$B^{(0)} = \frac{\mu_0 ec^2}{\omega V}, \tag{282}$$

as in Eq. (277) or (223).

The magnetic dipole moment per unit volume of the radiated field is obtained

with $r = \kappa^{-1}$ and $v = c$ as

$$m^{(3)} = \frac{erv}{V} e^{(3)} \to \frac{ec^2}{\omega V} e^{(3)}. \tag{283}$$

Therefore the magnetic dipole moment of the field is

$$Vm^{(3)} = \frac{ec^2}{\omega} e^{(3)}. \tag{284}$$

Equations (281), (282) and (284) give the field's magnetogyric ratio

$$\gamma_f = \frac{\omega^2 V}{\mu_0 e c^2}, \tag{285}$$

such that

$$Vm^{(3)} = \gamma_f J^{(3)}. \tag{286}$$

It is easily verified that the units of γ_f are C kgm^{-1}, the units of the conventional magnetogyric ratio [45], $e/2m$, where m is mass. The equation

$$\gamma_f := \frac{e}{2m}, \tag{287}$$

gives

$$m = \frac{\mu_0 e^2 c^2}{2\omega^2 V}, \tag{288}$$

or,

$$\chi' = \frac{V}{\mu_0}, \tag{289}$$

where χ' is the susceptibility,

$$\chi' = \frac{e^2 c^2}{2m\omega^2}, \tag{290}$$

used in Chap. 12 of Vol. 1 [1] for the electron. Here it is being used for the electromagnetic field itself, and m is the equivalent mass of the field [52].

Therefore the circling elementary charge radiates, among other things: an angular momentum, $J^{(3)}$; a magnetic dipole moment, $m^{(3)}$; a magnetic field, $B^{(3)}$; a field mass, m; a field susceptibility, χ'; and a field energy, $\omega J^{(3)}$. All these quantities are present in free space at the instant t and refer to the state of circling charge [96] a time R/c earlier in the source, where R is the source to observer distance.

Introducing the fine structure constant [22—25],

$$\alpha = \frac{e^2}{4\pi\hbar\epsilon_0 c} = \frac{e^2 \mu_0 c}{4\pi\hbar}, \tag{291}$$

it follows from Eq. (288) that

$$mc^2 = \frac{3}{2}\alpha\hbar\omega. \tag{292}$$

This equation links the field energy mc^2 to the photon $\hbar\omega$, the quantum of electromagnetic energy. Since α is unitless, the equation is consistent, and should be compared with Eq. (1) of Vol. 1 [1], the de Broglie Guiding theorem.

$$m_0 c^2 = \hbar\omega, \tag{293}$$

where m_0 is the mass of the photon. Comparing equations (292) and (293),

$$m = \frac{3}{2}\alpha m_0. \tag{294}$$

If the photon mass were zero, as in much of the literature [23], then m would be zero from Eq. (294), because α is made up of the universal constants. Since m can be expressed as

$$m = \left(\frac{3}{8\pi}\mu_0 e^2 c\right)\omega, \tag{295}$$

it is zero if and only if ω is zero, in which case there is no classical radiation present. Therefore the photon mass is non-zero if $B^{(3)}$ is non-zero, because the gyromagnetic ratio from which m is calculated is defined in terms of the magnetic dipole moment

$$Vm^{(3)} = \frac{V}{\mu_0}B^{(3)} = \chi' B^{(3)}. \tag{296}$$

The appearance of the fine structure constant in this analysis is probably deeply significant [6], because α is the hallmark of quantum electrodynamics, or QED [22—25].

The conservation of angular momentum and energy can be demonstrated using Eq. (402) of Vol. 1 [1], derived from the relativistic Hamilton-Jacobi equation of the classical electron in the classical field. If we replace the electron by a sample consisting of the elementary charge, e, and assume that any mass associated with the charge is very small, we obtain from Eq. (402) of Vol. 1 [1],

$$J^{(3)} = \frac{e^2c^2}{\omega^2}\left(\frac{B^{(0)}}{\left(m_0^2\omega^2 + e^2B^{(0)2}\right)^{1/2}}\right)B^{(3)}, \tag{297}$$

$$\underset{m_0 \to 0}{\dashrightarrow} \frac{ec^2}{\omega^2}B^{(3)} = \hbar e^{(3)}. \tag{298}$$

This result is a clear demonstration of the transfer, under the right conditions, of all the field angular momentum ($\hbar$ per photon) to the sample. This process conserves angular momentum and the elementary charge e rotates on a radius c/ω with angular velocity ω. (If it had mass, the sample charge would be made to spiral, because the field has linear momentum.) Energy is similarly conserved through $\hbar\omega$, the photon energy, given up entirely to the sample charge e. The process of absorption can be thought of in these terms, because the light quantum hypothesis asserts that absorption takes place through the photon of energy $\hbar\omega$. This process is now seen to be controlled by $B^{(3)}$ through equation (297) or its quantized equivalent, the Dirac equation.

From Eqs. (297) and (298) it is clear that if $B^{(3)}$ were zero, there would be no field-charge interaction, contrary to experience. Equation (297) shows that the interaction is quadratic both in e and $B^{(0)}$, and the angular momentum given to the charge by the field is independent of the sign of either e or $B^{(0)}$, even in the limit $m_0 \to 0$, the strong field limit. A rotating charge pushes around a charge of the same sign (clockwise, for example), and pulls around a charge of opposite sign in the same sense, clockwise. The magnetization caused by $B^{(3)}$ (the inverse Faraday effect [16—21]) is proportional to $J^{(3)}$ and is independent of whether $B^{(3)}$ is radiated by a negative charge ($-|e|$) or positive charge ($|e|$). In experiments such as the inverse Faraday effect, m_0 in Eq. (297) is always non-zero, both in the weak and strong field limits, because a real material sample must be made up of massive particles. Therefore the observed magnetization does not change sign with the sign of $B^{(3)}$.

Unfortunately, there is a book review [97] in the literature that asserts without proof that $J^{(3)}$ changes sign with the sign of $B^{(0)}$, but the dependence of $J^{(3)}$ on $B^{(0)2}$ in Eq. (297) shows that this assertion is erroneous. The error is compounded by the assertion [97] that $B^{(3)}$ is zero because it is claimed that there is no experimental evidence for this assumed change of sign. Equation (297) shows *that no such change of sign occurs* in relativistic field theory. A related article [62] completely misapplies $\hat{C}$ symmetry [1] in an attempt to show that $B^{(3)}$ is zero. Any such conclusion violates the Biot-Savart law, that a magnetic field is rotating charge. Two related papers [63,64] appear to assert (obscurely) that $B^{(3)}$ is zero, but as argued self-consistently here and elsewhere [1,2] this would mean the disappearance of all radiation. These papers display a fundamental lack of understanding of electrodynamics and relativity. Similar papers [98] on topics other than $B^{(3)}$ but based on related symmetry arguments are probably incorrect, as argued elsewhere [4]. Their conclusions are at best fortuitously in accord with experience, at worst misleading.

In summary of this section it has been shown that a source consisting of a circling, elementary charge, e, with unspecified mass, radiates $B^{(3)}$, which induces circular motion in a sample consisting of the elementary charge e, again with unspecified mass. If the target mass is assumed to be very small, the angular momentum of the radiated field is given up entirely to the charge, and this is a process which conserves angular momentum and energy. The rotating charge e in the source produces a $B^{(3)}$ field through the Biot-Savart law, a field which induces rotation in the e of the sample. The whole process can be described in terms of the motion of the elementary charge e. In the source it circles at ω on a radius r, which is identified with the Thompson radius c/ω in free space radiation, which is the spiralling motion of e at the speed of light c. The charge e in the sample is put into motion when the spiralling e of the field interacts with it. The sample charge is pushed or pulled into circular motion by repulsion or attraction of the field charge. The language of fields, potentials and relativity which permeates these volumes is an attempt to describe this simple process.

5.4 MAXWELL'S CONCEPT OF THE ELECTROMAGNETIC FIELD

The electromagnetic field according to Maxwell's point of view is *an agent of interaction between two charges.* The existence of the $B^{(3)}$ field is a consequence of the Maxwellian picture, which leaves mass unspecified in the sense that mass does not appear directly in the Maxwell equations or the wave equations. The electromagnetic field is an example of the philosophical concept of fields in physics, a concept which has been clearly described by Barut [53], whose discussion we follow here.

External scalar or electromagnetic fields produce an *action at a distance* force on a sample particle or charge. More fundamentally, a field is a self-contained mechanical system in its own right, and can be described by equations of motion constructed in the appropriate mathematical limits [53]. The field carries energy, and has a continuously infinite number of degrees of freedom. It fills the whole of space like a fluid and has physical reality in relativistic field theory. The potential four-vector, A_μ, of the electromagnetic field is therefore directly proportional to the energy-momentum four-vector through the scaling constant e, which is also the elementary charge. Chapter three has shown that A_μ is physically meaningful in contemporary classical field theory because of the topology of the vacuum. In relativistic particle theory [53], the classical system is the mass point and the fields are auxiliary, phenomenological quantities. In relativistic field theory [53] the physical system is the field itself and physical laws are obeyed by the fields themselves. Interaction between fields and particles is described by an extra term in the appropriate Lagrangian [1,2], a term based on the minimal prescription as used in Chap. 1 and 2 for example. Therefore the field interacts with a particle through the vector and scalar potentials. This is why there occur weak and strong field limits.

The electromagnetic field is an energy carrying physical system in its own right [53], a system which produces interaction between charges. Therefore the field is an agent of interaction. Instantaneous action at a distance is replaced in special relativity by a signal velocity, or phase velocity, which is c in vacuo for an assumed massless photon. If there were no charges there would be no field, so the latter is described fundamentally in terms of the former, as we have shown in this and the preceding chapters. As described by Barut [53] the field is produced by a charged particle whose existence is assumed as the starting point of the analysis. The field is measured through the acceleration it produces when acting upon another charged particle. The field cannot be detected if this acceleration cannot be detected and all depends ultimately on the existence of the elementary e in the primordial universe. The equations which describe the motion of e are the Maxwell equations, or d'Alembert wave equations to which they are equivalent. The electromagnetic field propagates (moves forward) in vacuo because the relevant unit of time is $t - R/c$ as we have seen. Static electric and magnetic fields do not propagate and are equivalent to the limit $R = 0$, so the unit of time is t and there are no moving waves. The $B^{(3)}$ field propagates for precisely the same reason, it can be defined in terms of exponents in which appear the unit $t - R/c$.

The $B^{(3)}$ field becomes static, as for any other static field, when the exponents contain the time unit t and $R = 0$. This point has been illustrated in Eqs. (278) and (279). The fact that radiation of any kind occurs at all is due to the empirically supported equivalence of $\Box A_\mu$ and $-J_\mu/\epsilon_0$ in the d'Alembert equation. Therefore radiation is energy transfer from charge to charge and as we have seen in Chap. 4, the quantum of energy of the field, the photon, is proportional to the square of e. This is consistent with the fact that if there were no e there would be no photon. The interaction energy-momentum between field and charge is eA_μ, which is again proportional to the square of the elementary charge e. This gives the general rule that all equations of electrodynamics must conserve $\hat{C}$ symmetry, and that the $\hat{C}$ symmetry of fields and potentials is always negative.

These arguments are inherently relativistic, as is well known. For example Faraday induction is a relativistic phenomenon as pointed out clearly by Barut [53]. If we move a closed circuit in a magnetic field with velocity v_0, a charge in the loop will be acted on by a Lorentz force $ev_0 \times B$ with the same units as eE where E is an electric field. However, if v_0 is parallel to B, there is no force at all. Since $B^{(3)}$ is always parallel to the direction (Z) of propagation of electromagnetic radiation in vacuo, there is no electric force due to $v_0 \times B^{(3)}$ and no Faraday induction. Relativistically and equivalently, if we hold the circuit loop fixed and move the magnetic field, the same occurs, no Faraday induction. There is no Faraday induction due to $B^{(3)}$ and there is no $E^{(3)}$ [1,2]. This has been verified by available experiments [1,2,99] consisting of chopping a circularly polarized electromagnetic beam passed through a Faraday induction coil. These experiments can be repeated to any degree of precision and there will be no Faraday induction in free space due to $B^{(3)}$. (When $B^{(3)}$ encounters matter within an induction coil, there will occur the inverse Faraday effect [16—21].)

In the next section, these remarks are put into quantitative form by considering a radiating charge held in a circular orbit by a static magnetic field. This is a standard problem developed by Landau and Lifshitz [75] but shows with clarity why there is a radiated $B^{(3)}$.

5.5 RADIATION FROM A CIRCLING CHARGE HELD IN A STATIC MAGNETIC FIELD AT $R = 0$

Consider a particle with charge e and mass m constrained to move in a circle of radius r by a static magnetic field at $R = 0$. This means that the particle circles around the axis (Z) in which R is defined, i.e., if R were in Z, the charge circles in the XY plane. As ably demonstrated by Landau and Lifshitz [75], the transverse acceleration on the particle in its circular orbit is

$$w = \frac{e}{m}\left(1 - \frac{v^2}{c^2}\right)^{1/2} v \times B^{(3)}, \tag{299}$$

where v is its transverse velocity. This is the result of the Lorentz force equation. The relativistic Lorentz force defined by Eq. (299) is therefore

$$F = mw\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = ev \times B^{(3)}, \tag{300}$$

and can be expressed in the basis ((1), (2), (3)) [1,2] as

$$F^{(1)*} = -iev^{(2)} \times B^{(3)}. \tag{301}$$

This particle radiates because of the existence of a non-zero w, a centripetal acceleration of charge. If the static magnetic field were switched off, the charge would continue to radiate away energy until eventually only the rest energy, mc^2, would be left. This process is a well known part of standard classical radiation theory [75]. The $B^{(3)}$ field in Eq. (299) is static because it is calculated with $R = 0$, so that the unit of time being used is t. In the absence of radiation, this means that the charge is describing an orbit in a plane, and this orbit is a circle, not a helix, because there is no forward motion of charge along the Z direction.

As radiation occurs, however, F decreases in Eq. (300) due to a radiated $B^{(3)}$ field; the transverse acceleration w tends always to decrease due to radiation but is kept constant by the applied magnetic field. By considering the limit $|v| \to c$, it is easily demonstrated as follows that the radiation is the propagating $B^{(3)}$ field discussed earlier and in Chap. 4.

The angular frequency of the charged particle in the static magnetic field is given by [75]

$$\omega = \frac{e}{m}\left(1 - \frac{v^2}{c^2}\right)^{1/2} B^{(0)}, \tag{302}$$

where $B^{(0)}$ is the amplitude of the field in tesla. In the limit $|v| \to c$, the strong field limit, we can apply the equivalence principle, Eq. (280), in the form

$$e\frac{c^2}{\omega^2}B^{(0)} \xrightarrow{\quad} \hbar, \qquad |v| \to c \tag{303}$$

which shows that in this limit,

$$mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \xrightarrow{\quad} \hbar\omega. \qquad |v| \to c \tag{304}$$

Similarly, the angular acceleration in this limit becomes

$$|w| \xrightarrow[|v| \to c]{} \omega c, \tag{305}$$

and using $\omega = v/r$, the radius becomes c/ω. The rotating charge in this limit is described by

$$|r| \to \frac{c}{\omega}, \qquad |v| \to c, \qquad |w| \to \omega c, \tag{306}$$

and is radiating, because of the presence of non-zero transverse acceleration.

Equation (306) has all the properties of electromagnetic radiation except that we have not yet introduced forward motion along Z, we have considered the $|v| \to c$ limit of the charge circling in a plane. The property of propagation at c is now *introduced* through the familiar equation,

$$\kappa = \frac{\omega}{c}, \tag{307}$$

where κ is the wave vector. The photon forward momentum is thus identified as $eA^{(0)} \to \hbar\kappa$. The radiation process is now describable by replacing the unit of time t by the unit of time $t - R/c$. The introduction of κ, and its identification with ω/c, *is sufficient to describe the radiation process*. Here R is the source to observer distance introduced earlier in this chapter. The dynamics of the system are now to be evaluated [96] at the instant $t - R/c$, and in so doing, we refer to a state of circling charge at observer point R a time $t_1 = R/c$ earlier in the source, situated at $R = 0$. Since all is unaffected except t, the quantities r, v and w remain as they were originally, meaning that e at point R must be the charge described earlier at point $R = 0$. Similarly the charge at a point R_1 further along the Z axis is the same as that described at point R a

time $(R_1 - R)/c$ earlier and so on. The charge e spirals forward at c through the vacuum. The magnetic field $B^{(3)}$ is radiated from its position at $R = 0$ to its position at R a time R/c later, and the forward propagation has been introduced through the equation $\kappa = \omega/c$ linking the wavevector to the angular frequency. (If $B^{(3)}$ were not propagating there would be no wavenumber, no spiralling motion of e forward at c, and no radiation.)

To summarize therefore, the analysis has been based on the idea of a charge e attached to a particle of mass m circling at $R = 0$. The circling motion is caused by an applied static magnetic field. The circling motion of charge produces the static magnetic field

$$B^{(3)*} = -\frac{3\mu_0\omega}{4\pi r^3 e} i p^{(1)} \times p^{(2)}, \tag{308}$$

through the Biot-Savart law, as shown in Eq. (278). In this expression the time parameter is t. Conversely, therefore the field $B^{(3)}$ is the same as that needed to keep the charge in its circular orbit. In the limit $|v| \to c$, it has been demonstrated that the circling charge has all the characteristics of electromagnetic radiation except for forward motion (propagation at c in vacuo). The latter is *introduced* through the concept of wavevector $\kappa = \omega/c$ and this is sufficient to describe the phenomenon of radiation of $B^{(3)}$. The charge e spirals forward, and no longer remains in a plane. The radiated $B^{(3)}$ at position R a time R/c later is the same as in Eq. (308), except that

$$t \to t - \frac{R}{c}, \qquad |v| \to c. \tag{309}$$

The radiated $B^{(3)}$ is therefore

$$B^{(3)} = \left(\frac{3\mu_0\omega e}{4\pi r}\right) e^{(3)} := B^{(0)} e^{(3)}, \tag{310}$$

and is proportional to e. If there were no e there would be no radiation, in accord with the philosophy of the electromagnetic field summarized briefly in the previous section.

The radiated $B^{(3)}$ is accompanied by the radiated plane waves $B^{(1)}$ and $B^{(2)}$, through which $B^{(3)}$ is defined, and the radiated $B^{(1)}$ and $B^{(2)}$ are proportional to the radiated $E^{(1)}$ and $E^{(2)}$. As demonstrated in Chap. 4, all these fields can be thought of in terms of a spiralling charge e. In the limit $|v| \to c$, the range of the radiation is

effectively infinite, and the electromagnetic wave at the point R must be evaluated at the instant $t - R/c$. This remains true of $B^{(3)}$ because the latter is evaluated in terms of a product of conjugate phase exponents, each containing $t - R/c$. Therefore the circling e at $R = 0$ is the disturbance that initiates the travelling wave, and transmits the energy associated with the disturbance.

In the limit $|r| \to c/\omega$ Eq. (310) becomes

$$B^{(3)} \xrightarrow[|r| \to c/\omega]{} \frac{3\mu_0 e}{4\pi c}\omega^2 e^{(3)}, \tag{311}$$

showing that the intensity associated with the radiated $B^{(3)}$ is

$$I = \frac{c}{\mu_0} B^{(0)2} = \left(\frac{9\mu_0 e^2}{16\pi^4 c}\right)\omega^4, \tag{312}$$

and proportional to the fourth power of angular frequency. The intensity is inversely proportional to the fourth power of the wavelength

$$I = \frac{9\mu_0 e^2 c^3}{\lambda^4}. \tag{313}$$

It is interesting that the long wavelength limit of the Planck distribution,

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5}\left(e^{hc/(\lambda kT)} - 1\right)^{-1}, \tag{314}$$

is

$$I(\lambda, T) \xrightarrow[\lambda \to \infty]{} \frac{4\pi^2 \hbar ckT}{\lambda^4}, \tag{315}$$

and is also proportional to the inverse fourth power of wavelength. Therefore the result (313) is for one photon before thermal averaging in an ensemble of photons. In deriving Eq. (313) the volume occupied by the photon has been defined as a sphere of radius $(4/3)\pi r^3$, but the result can be generalized to any type of volume.

In essence, therefore, electromagnetic radiation is the replacement of time t by $t - R/c$, which is equivalent to the introduction of the relation (307) between wave-

number and angular frequency. In general, radiation is controlled by the d'Alembert wave equation, whose solutions in the plane wave approximation we have been considering in this section. In this context, the conventional literature asserts that radiation in the plane wave approximation is transverse to the direction of propagation, and this remains true in our analysis, the radiated potentials $A^{(1)}$ and $A^{(2)}$ are transverse, but their existence in vacuo means that there is a $B^{(3)}$ in vacuo. Under the right conditions, the characteristic square root intensity dependence of $B^{(3)}$ becomes observable experimentally [1,2].

5.6 THE LIÉNARD-WIECHERT POTENTIALS FOR $B^{(3)}$

The Liénard-Wiechert potentials are discussed in many excellent texts [7,52,53,96] to which we refer the interested reader. They describe radiation from a charge in an arbitrary trajectory. Radiation at observer point R is described at the instant $t - R/c$, and so the Liénard-Wiechert potentials are related to the retarded potentials. In general, and in S.I. units, the scalar and vector potentials radiated from the source are

$$\phi = \frac{ec}{4\pi\epsilon_0 R}\left(1 - \boldsymbol{k}\cdot\frac{[\boldsymbol{v}]}{c}\right)^{-1}, \tag{316}$$

and

$$\boldsymbol{A} = \frac{e\mu_0}{4\pi R}[\boldsymbol{v}]\left(1 - \boldsymbol{k}\cdot\frac{[\boldsymbol{v}]}{c}\right)^{-1}, \tag{317}$$

where $\boldsymbol{k}$ is the unit vector $\boldsymbol{R}/|\boldsymbol{R}|$, and where $[\boldsymbol{v}]$ is the velocity of the elementary charge e in the source at the instant $t - R/c$. The potentials are a direct outcome of special relativity [47,96]. Equations (316) and (317) are generally valid because they are manifestly covariant under Lorentz transformation, and therefore hold in any Lorentz frame. The theory of radiation is developed in many texts in terms of situations of interest and for various approximations to Eqs. (316) and (317). The fact that there is a source being considered means that the radiation emanates from that source, and this satisfactorily causal in character. If the charge in the source were not moving at $t - R/c$,

$$\phi = \frac{e}{4\pi R\epsilon_0}, \quad \boldsymbol{A} = \boldsymbol{0}, \tag{318}$$

which is Coulomb's law.

In this section we use the Liénard-Wiechert potentials to describe the radiation of $B^{(3)}$ from a charge e circling at radius r with angular frequency ω on a mass m. This can be a circling electron, for example. The tangential (or transverse) velocity of the charge can be represented in complex form $v^{(1)} = v^{(2)*}$ in the circular basis [1,2] ((1), (2), (3)). There is always present an inward seeking (centripetal) acceleration because the charge is constrained in a circular orbit, for example by a static magnetic field as in the previous section. The complex vector potentials radiated by the centripetal acceleration of the charge e are, in general, the complex Liénard-Wiechert potentials,

$$A^{(1)} = A^{(2)*} = \frac{e\mu_0 c}{4\pi R}[\beta^{(1)}](1 - \boldsymbol{k}\cdot[\beta^{(1)}])^{-1}, \tag{319}$$

where $\beta^{(1)} := [v^{(1)}]/c$. *It is important to note that mass does not appear specifically in these potentials*, and that the velocities are evaluated at $t - R/c$. There is energy flow associated with the radiation. The radiated $B^{(3)}$ field is therefore

$$B^{(3)*} = -i\frac{e}{\hbar}A^{(1)} \times A^{(2)}, \tag{320}$$

is described in terms of these potentials and is always non-zero in radiation from a circling electron. The radiated $B^{(3)}$ is described *without the mass of the sample being specified.* The $B^{(3)}$ field in the sample depends on the mass as in Eq. (299). We will return to this key point later in this section.

The usual transverse radiated fields are [53], in S.I. units,

$$B^{(1)} = B^{(2)*} = \frac{e\mu_0}{4\pi R}\frac{\left((\boldsymbol{k} - [\beta^{(1)}]) \times [\dot{\beta}^{(1)}] + \boldsymbol{k}\left(\boldsymbol{k}\cdot[\beta^{(1)}] \times [\dot{\beta}^{(1)}]\right)\right)}{(1 - \boldsymbol{k}\cdot[\beta^{(1)}])^3}, \tag{321}$$

and

$$E^{(1)} = E^{(2)*} = \frac{e\mu_0 c}{4\pi R}\frac{\left(k \times \left((k - [\beta^{(1)}]) \times [\dot{\beta}^{(1)}]\right)\right)}{(1 - k \cdot [\beta^{(1)}])^3}, \tag{322}$$

and these are generally valid equations without approximation. They illustrate the geometrical and dynamical intricacies associated with the many phenomena of radiation, e.g. from antennae [96], and at electron velocities close to c (synchrotron radiation).

When the source electron is circling slowly, then $\beta < 1$ [53], and in this limit the radiated vector potential at point R becomes directly proportional to the transverse velocity at time $t - R/c$,

$$A^{(1)} \underset{\beta < 1}{\dashrightarrow} \frac{e\mu_0}{4\pi R}[v^{(1)}]. \tag{323}$$

The radiated $B^{(3)}$ field therefore becomes

$$B^{(3)*} \underset{\beta < 1}{\dashrightarrow} -i\frac{e}{\hbar}\left(\frac{e\mu_0}{4\pi R}\right)^2 [v^{(1)} \times v^{(2)}], \tag{324}$$

where the conjugate product of velocities $[v^{(1)} \times v^{(2)}]$ must as usual be evaluated at the instant $t - R/c$. The radiated $B^{(3)}$ field is governed by $A^{(1)}$ and $A^{(2)}$ therefore, and so its interaction with matter is described by the interaction of $A^{(1)}$ and $A^{(2)}$ with matter, e.g. one electron. The equations of motion to be used to describe this interaction are the Dirac or Hamilton-Jacobi equations, based on the minimal prescription. This leads to the strong and weak field limits as described in Chap. 3. The radiated $B^{(3)}$ field adds nothing to the Poynting vector (energy flux) because it adds nothing to the linear momentum of radiation. Because it is radiated, it does however produce a novel rotational contribution to the vacuum energy density [1—15],

$$En = \frac{1}{\mu_0}\int B^{(3)} \cdot B^{(3)*} dV, \tag{325}$$

so that the radiation laws should be adjusted to take this into account, as described in Chap. 4. This simply means adjusting the effective volume of radiation V, leaving the value of $\hbar$ the same.

For low source electron velocities, the usual transverse radiated fields (321) and (322) become

$$B^{(1)} = B^{(2)*} \underset{\beta < 1}{\dashrightarrow} \frac{e\mu_0}{4\pi Rc} k \times [\dot{v}^{(1)}], \tag{326}$$

$$E^{(1)} = E^{(2)*} \underset{\beta < 1}{\dashrightarrow} \frac{e\mu_0}{4\pi R} k \times (k \times [\dot{v}^{(1)}]), \tag{327}$$

$$B^{(1)} \underset{\beta < 1}{\dashrightarrow} \frac{1}{c} k \times E^{(1)}, \tag{328}$$

and are radiated plane waves because Eq. (328) characterizes plane waves. There is a relation between the Liénard-Wiechert and retarded potentials which is satisfied by representing the source transverse velocity by

$$[v^{(1)}] = \frac{v_0}{\sqrt{2}}(ii + j)\exp\left(-i\omega\left(t - \frac{R}{c}\right)\right), \tag{329}$$

in which v_0 is the transverse speed of the source electron in its circular orbit. The transverse acceleration $[\dot{v}^{(1)}]$ is,

$$[\dot{v}^{(1)}] = -i\omega[v^{(1)}], \tag{330}$$

giving the center-seeking, or centripetal, acceleration of charge,

$$k \times [\dot{v}^{(1)}] = \omega[v^{(1)}], \tag{331}$$

which is responsible for the radiation of $B^{(3)}$. From these equations we find

$$B^{(1)} = \kappa A^{(1)} := \frac{\omega}{c} A^{(1)}, \tag{332}$$

which is Eq. (11) of Vol. 1, an equation which describes a plane wave in vacuo as expected, provided $\kappa := \omega/c$.

The amplitude of the radiated $B^{(3)}$ from this analysis is given in terms of v_0, ω and R,

$$B^{(0)} = |B^{(3)}| = e\frac{\mu_0 \omega v_0}{4\pi Rc}, \tag{333}$$

and is the same as the amplitude of the radiated $B^{(1)} = B^{(2)*}$. However, the radiation in vacuo must propagate at c, and must propagate as plane waves, as we see from Eq. (328), so from the analysis of Chap. 4 this means that in the vacuum,

$$v_0 \to c, \quad \frac{\omega}{c} \to \kappa, \quad r \to \frac{c}{\omega}. \tag{334}$$

From Eq. (220), however, we know that

$$B^{(0)} = e\mu_0 \frac{rv}{V}, \tag{335}$$

where V is formally, a volume of radiation. It is therefore possible to find an expression for V/R in the limit where the radiation becomes a plane wave propagating at c in vacuo. This volume in the present analysis can be thought of as being the volume of a cylinder [96] of length R and radius r, which in vacuo becomes the Thompson radius $1/\kappa$ as discussed in Chap. 4. The cylinder's volume is

$$V := \pi r^2 R, \tag{336}$$

and in the limit of plane waves propagating at c in vacuo we find

$$\frac{V}{R} \to \left(\frac{\lambda}{2}\right)^2. \tag{337}$$

This is an area, proportional to the square of the wavelength, through which passes the magnetic flux in weber associated with $B^{(3)}$, whose units are tesla, i.e., weber m^{-2}. The magnetic flux density at observer point R is $B^{(3)}$. Equation (337) shows that whatever the magnitude of V and R, the area defining the radiated flux density $B^{(3)}$ is always $\lambda^2/4$. This can be thought of as the area of a circularly polarized laser or microwave beam, or the area of starlight collected in a telescope. The source of the starlight may be many millions of light-years distant, so that the light being collected by the observer represents the state of the star many millions of years ago. More generally, the radiation from a source is radiated in a sphere, in which case spherical geometry must

be used to calculate the radiated energy density and flux density per unit solid angle in three dimensions [47,52,53,96].

In summary, therefore, the source of the radiated $B^{(3)}$ is the elementary charge e in a circular orbit of radius r. The angular frequency of this motion is ω and the transverse orbital speed is v_0; the motion is that of a particle *with mass*, m, such as an electron. The radiated $B^{(3)}$ is charge e in a spiralling orbit with Thompson radius $\lambda/(2\pi)$ and both forward and transverse speed c. *There is no mass specified with the charge e in this motion in free space.* If there is mass associated with this radiation, it is the mass of the photon [1,2], which is many orders of magnitude less than that of the electron. Therefore the radiative process (transfer of essentially massless energy) extends the range of e enormously, while leaving essentially all the mass of the radiating particle in the source. In this process, e itself is unchanged, or conserved [1,2, 22—25], but ceases to be localized in the source. This process is described through charge (ρ) and current (J) *densities* using the continuity equation [96],

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}. \tag{338}$$

Thus, a circling charge at one end of the universe influences another charge at the other end through the electromagnetic field. When the latter, carrying the radiated $B^{(3)}$, meets the target charge, (matter consisting of one fermion), the latter is made to circle according to the appropriate equations of motion. The radius and transverse speed of this circling motion depend on the *mass* of the fermion. The Hamilton-Jacobi equation, for example, shows that if the fermion is massless (a limit not encountered experimentally), it is made to circle at the Thompson radius with transverse speed c, and is therefore indistinguishable from a photon. This is the strong field limit. Otherwise the radius and transverse speed depend on the target fermion mass, and the circling of charge (magnetization) is proportional to a mixture of terms in I and $I^{1/2}$ where I is the field intensity.

The radiated $B^{(3)}$ field is therefore associated with a *very small amount of mass*. This distinguishes it from an ordinary induction field, such as the static magnetic field in a solenoid. The latter is caused by circling electrons and is not a radiated field. The $B^{(3)}$ field, being essentially massless, has, F.A.P.P., no rest frame, and can never be static in any frame of reference. It can be observed only through it interaction with matter, an interaction that must be described through $A^{(1)}$ and $A^{(2)}$ through which $B^{(3)}$ is defined.

5.7 THE VACUUM POYNTING THEOREM FOR $B^{(3)}$

We have seen in Eq. (202) that the quantum of electromagnetic energy in vacuo, the photon $\hbar\omega$, is proportional to e^2, and therefore the propagation of $\hbar\omega$ past the observer at point R is due to an acceleration of charge at the earlier instant $t - R/c$. The $B^{(3)}$ field in the source is due to a conjugate product involving exponents $i\omega t$ and $-i\omega t$; the $B^{(3)}$ field in vacuo is the same precisely, but with t replaced by $t - R/c$, the unit e being maintained constant. This section develops the Poynting theorem for $B^{(3)}$, the energy due to which is defined in Eq. (325). This is rotational energy, and the Poynting vector due to $B^{(3)}$ is proportional to an *angular* momentum density of radiation in vacuo. The vacuum $B^{(3)}$ does not contribute therefore to the well known Poynting vector N [1,2,47,52,53,96] which is a *linear* momentum density of radiation in vacuo. The novel Poynting vector due to $B^{(3)}$, and the energy density due to $B^{(3)}$ form a four-vector of special relativity, an angular momentum-angular energy four-vector.

In S.I. units the Poynting theorem in vacuo is

$$\nabla \cdot N = -\frac{\partial U}{\partial t}, \tag{339}$$

where N, Poynting's vector, is the electromagnetic power per unit area, i.e., the electromagnetic energy flux, and U is the electromagnetic energy per unit volume, the energy density. Since $B^{(3)}$ is a novel vacuum field, it generates the novel energy density [1,2],

$$U_J = \frac{1}{\mu_0} B^{(3)} \cdot B^{(3)*} \left(= \frac{1}{\mu_0} B^{(1)} \cdot B^{(1)*} \right), \tag{340}$$

which is rotational energy density. If U_J is constant in a given radiation volume V, then $\partial U_J / \partial t$ is zero, meaning that

$$\nabla \cdot \left(\frac{J^{(3)}}{V} \right) = -\frac{\partial U_J}{\partial t} = 0, \tag{341}$$

so that the Poynting vector associated with U_J is divergentless and independent of X, Y, and Z. Since there is no $E^{(3)}$, there is no contribution to N from a putative cross product of $B^{(3)}$ and $E^{(3)}$, as discussed on page 4 of Vol. 1. From Eq. (281) of this

volume we know, however, that there is an angular momentum, $J^{(3)}$, of vacuum radiation,

$$J^{(3)} = \frac{1}{\mu_0 \omega} V B^{(0)} B^{(3)}, \tag{342}$$

so that its angular momentum density is proportional to $B^{(3)}$,

$$\frac{J^{(3)}}{V} = \left(\frac{B^{(0)}}{\mu_0 \omega} \right) B^{(3)}. \tag{343}$$

The radiation's angular energy density is

$$U_J = \frac{En}{V} = \frac{1}{\mu_0} B^{(3)} \cdot B^{(3)*}, \tag{344}$$

and dividing Eq. (343) by Eq. (344) gives

$$En = \omega \left| J^{(3)} \right|. \tag{345}$$

We recover from this the definition of the photon if the angular momentum of radiation is the angular momentum, $\hbar$, of one photon, so Eq. (345) becomes the familiar Planck-Einstein hypothesis $En = \hbar\omega$.

The Poynting theorem for $B^{(3)}$ is therefore

$$\nabla \cdot J^{(3)} + \frac{\partial En}{\partial t} = 0, \tag{346}$$

and the four-vector defined by this theorem is

$$J_\mu := \left(J^{(3)}, \frac{En}{\omega} \right), \tag{347}$$

which is Lorentz covariant. Finally, the classical definition of $J^{(3)}$, as given, for example, by Jackson [47], is

$$J^{(3)} = \left(\frac{\epsilon_0}{\kappa} \int | E^{(1)} \times B^{(2)} | dV \right) e^{(3)}, \tag{348}$$

and is seen to be closely related to N integrated over V,

$$P^{(3)} = \epsilon_0 \int E^{(1)} \times B^{(2)} dV. \tag{349}$$

The ratio of Eqs. (349) and (348) is κ, which for one photon in the quantum field theory is the ratio of $\hbar\kappa$ to $\hbar$.

It is possible, therefore, to define Poynting's theorem entirely in terms of $B^{(3)}$.

5.8 Q.E.D., BOSON ENSEMBLES, AND $B^{(3)}$ IN VACUO

Dirac [26] was among the first to consider the application of the new quantum theory to radiation, initiating the subject of quantum electrodynamics [22—25]. It is convenient in this final section of this chapter on the irradiation of $B^{(3)}$ to sketch an introduction to the methods by which it can be quantized in free space, leading to the concept of the photomagneton operator [4], the phase independent magnetic flux density operator equivalent to the classical $B^{(3)}$. We first consider the question of the complex nature of the vector potential in the classical theory, a seemingly mundane property but one which is responsible for travelling waves from Maxwell's vacuum equations. In so doing we follow the clear discussion by Jackson [47] of plane wave solutions, in particular,

> "A basic feature of Maxwell's equations is the existence of *travelling wave solutions* which represent the transport of energy from one point to another. The simplest and most fundamental electromagnetic waves are transverse plane waves."

The travelling wave equation has the form [47]

$$\nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0, \tag{350}$$

where $v = c/(\mu\epsilon)^{1/2}$ is a constant, and where ϵ and μ are material permittivity and permeability respectively. The plane wave solutions of Eq. (350) are [47]

$$u = e^{i(k \cdot x - \omega t)}, \tag{351}$$

where $k = \omega/v$ and the fundamental solution is

$$u(x, t) = A e^{i(kx - \omega t)} + B e^{-i(kx + \omega t)}, \tag{352}$$

which represents waves travelling to the right, with complex exponent $x - vt$ and to the left with complex exponent $x + vt$. The phase angle $t - z/u$ is a constant, and the wave fronts are perpendicular to the axis of propagation. In the classical theory causality leads [96] to the neglect of the term in $x + vt$, leaving a fundamental solution of the form

$$u(x, t) = A e^{ik(x - vt)}. \tag{353}$$

Additionally [47], electromagnetic plane waves have a vector character and can be circularly polarized. This means that circularly polarized transverse electric and magnetic fields rotate as they advance.

The novel $B^{(3)}$ field satisfies the above criteria of electromagnetic plane waves, the reasons being as follows.

1) The $B^{(3)}$ field is a travelling and spinning solution of the vacuum Maxwell equations, because it is defined in terms of a conjugate product that includes the exponent product $e^{-i\phi} e^{i\phi}$, where $\phi := \omega(t - R/c)$ is characteristic of a travelling wave as we have just seen. Thus, $B^{(3)}$ is situated at a unique point R which propagates through the vacuum, and at no other point. At some later instant $B^{(3)}$ will be situated at another point in space, and thus propagates, or travels, with the wave front. Roughly speaking, it is carried along by the transverse plane waves. It is entirely wrong to think of $B^{(3)}$ as a uniform field stretching throughout the whole of space, at a given instant it is at a given locality and no other. Similarly, the photon is situated at a point in space, and as in the Compton effect, can act as a particle. The $B^{(3)}$ field is proportional to the angular momentum of this particle and is a pseudo vector.

2) There is a Poynting theorem for $B^{(3)}$ as described in the preceding section, and so $B^{(3)}$, as required, carries rotational energy and angular momentum through the vacuum. The field creates rotational energy density and angular momentum density. In contrast the transverse plane waves create translational energy density and linear momentum density.

3) The source of $B^{(3)}$ at observer point R is circling charge at the instant

$t - R/c$ earlier. This is also the source of the transverse plane waves.

4) The interaction of the classical $B^{(3)}$ with matter is described by the potentials through which it is defined. The minimal prescription, in which A_μ is complex in general, is the way in which this interaction is understood. This gives the weak and strong field limits and also describes the interaction of the transverse fields with matter. Since $B^{(3)}$ propagates, F.A.P.P., at c this interaction is relativistic in nature, requiring a relativistic factor in the Hamilton-Jacobi and Dirac equations.

When we come to apply Dirac's method of quantization [26] of the electromagnetic field it is necessary to find a way of translating these ideas into quantum mechanics, but there is no fundamental conceptual difficulty posed by the existence of the classical $B^{(3)}$. In Vol. 1 [1], for example, a simple scheme of quantization was adopted whereby $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ were recognized as angular momentum operators of quantum mechanics, which are within a factor $\hbar$ [23] infinitesimal rotation generators of O(3). This scheme is easy to develop and removes the anomaly of the unphysical E(2) little group [1], which is replaced by the physical O(3) little group. Dirac's method [26], the original methodology leading to Q.E.D. in the Heisenberg picture, relies however on a real A, and in Q.E.D. [58], fields are real in k (momentum) space. For example, Dirac [26] uses the Fourier expansion,

$$A = \int \left(A_k + \bar{A}_{-k}\right) e^{-ikx} d^3k, \tag{354}$$

which in our notation is

$$A = \int \left(A^{(1)}_{0k} e^{-ikx} + A^{(2)}_{0k} e^{ikx}\right) d^3k, \tag{355}$$

meaning that A is pure real. This is still classical, but quantization takes place directly from this Fourier expansion in k space by replacing the integral by a sum over a *dust* [26] of k values representing individual oscillators of the electromagnetic field. In order to get $B^{(3)}$ from this type of Fourier expansion, it is necessary to proceed on the basis of the Poynting theorem for $B^{(3)}$ by recognizing that

$$En = \frac{1}{\mu_0} \int B^{(3)} \cdot B^{(3)*} dV = \frac{1}{\mu_0} \int \left(\frac{\omega}{c}\right)^2 A^{(1)} \cdot A^{(2)} dV, \tag{356}$$

so that the quantized $B^{(3)}$ can be developed in terms of cross products of complex amplitudes $A^{(1)}_0$ and its complex conjugate $A^{(2)}_0$ for each k. For each photon,

therefore, there is a $B^{(3)}$ field, and in quantum mechanics it is proportional directly to an angular momentum operator. (In the classical theory, Eq. (342) shows that $B^{(3)}$ is proportional to the angular momentum, $J^{(3)}$, of radiation in a given volume V.)

With care, therefore, the quantization of $B^{(3)}$ using Dirac's methods [26] becomes straightforward, and there is no violation of the concepts of Q.E.D. This is intuitively acceptable because $B^{(3)}$ violates no classical concepts and exists in the classical theory of fields.

Chapter 6. Mass of the Photon if Particulate

The existence of the $\boldsymbol{B}^{(3)}$ field in vacuo has the fundamental implication that if the photon can be considered to be a particle, it must have mass in the theory of special relativity. Since $iB^{(0)}\boldsymbol{B}^{(3)*}$ is routinely observable in magneto-optical effects, then it is no longer possible to accept the idea of a massless, particulate photon. The Wigner little group becomes the physical O(3), and not the unphysical E(2). Empirical evidence for the expected $I^{1/2}$ dependence of magneto-optic effects due to $\boldsymbol{B}^{(3)}$ will strengthen this conclusion with further data, but it has already become clear that the observation of I dependent magneto-optical effects [16—21] means the existence of $iB^{(0)}\boldsymbol{B}^{(3)*}$ and therefore of $\boldsymbol{B}^{(3)}$. This conclusion means the introduction [1,2] of various mechanisms by which gauge invariance can be reconciled with non-zero photon mass. There are several inroads already available in the literature [1,2] to this interesting area of electromagnetic theory.

The prevailing orthodoxy relies on the *assertion* that the mass of the photon is zero, because it appears convenient to do so in the Lagrangian approach to field theory [1,2]. The assertion leads directly in special relativity to the conclusion [1,2,23] that the Wigner little group is the unphysical E(2). Therefore the theory of the massless photon is unphysical and internally inconsistent in special relativity, but is nevertheless routinely presented as physical and self-consistent. To the present authors this orthodoxy is unacceptable because it is fundamentally flawed. The recent emergence of $\boldsymbol{B}^{(3)}$ as an electromagnetic field, carrying energy in vacuo according to Eq. (346), shows that there are three degrees of freedom in electromagnetic fields propagating in vacuo; there exist $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ in the circular basis ((1), (2), (3)). Equivalently there are three degrees of freedom in the Cartesian frame (X, Y, Z).

6.1 SOME DIFFERENCES BETWEEN THE ORTHODOX AND $\boldsymbol{B}^{(3)}$ THEORIES OF LIGHT

It is convenient to list a selection of the major points of difference as follows.

1a) In the orthodox theory of light, the particulate photon is identically massless, and from special relativity must be associated with transverse plane waves propagating in vacuo. There can be no physical $\boldsymbol{B}^{(3)}$, it is asserted, because it is not a transverse plane wave.

1b) In the $B^{(3)}$ theory of light, the particulate photon must be three dimensional and therefore massive, because a massless particle must be two dimensional [56] from Wigner's paper of 1939.

2a) In orthodox classical electrodynamics the Poynting vector is N, which is proportional to linear momentum density, and translational electromagnetic energy in vacuo is defined in terms only of transverse plane waves.

2b) In the $B^{(3)}$ theory there is a new Poynting theorem, Eq. (346), and a new rotational contribution to electromagnetic energy density through $B^{(3)} \cdot B^{(3)*}$.

3a) In the orthodox theory propagating fields depend explicitly on the phase factor $\phi = \omega(t - R/c)$.

3b) The $B^{(3)}$ field propagates in vacuo because it is implicitly dependent on the phase through a cross product.

4a) In orthodox magneto-optics there occur conjugate products such as $B^{(1)} \times B^{(2)}$ but these are not accepted as indicating the existence of $B^{(3)}$.

4b) In the new understanding $iB^{(0)}B^{(3)*}$ is equal to $B^{(1)} \times B^{(2)}$, so $B^{(3)}$ is an everyday observable of magneto-optics. The orthodox thinking cannot accept this because $B^{(3)}$ is asserted to be zero. This is the point at which the old theory becomes internally inconsistent.

5a) In the orthodox view the mass of the photon is identically zero because otherwise gauge invariance is violated in the Lagrangian approach to field theory.

5b) In the new thinking there can be no photon whose mass is zero because of the existence of $B^{(3)}$. By making $A_\mu A_\mu$ infinitesimally small, gauge invariance and non-zero mass can be reconciled. There are probably several other ways of achieving the same aim [1,2].

6a) In the orthodox approach magneto-optical effects are always proportional to beam power density, I, and this is corroborated by available experimental results [16—21].

6b) In the new approach, there develops under the right conditions an $I^{1/2}$ dependence which has not been observed hitherto. This has been predicted clearly and independently by Chiang [100] and by Talin *et al.* [54], but the significance of these papers has not been fully realized. Both papers indirectly indicate the existence of $B^{(3)}$ through special relativity. The $I^{1/2}$ dependence is actually implicit in a standard text such as that of Landau and Lifshitz [75], as explained in Chap. 12 of Vol. 1 [1], but because this is an orthodox text in the classical theory of fields, $B^{(3)}$ is not recognized explicitly.

7a) In contemporary orthodoxy, the Faraday induction law is one of the vacuum Maxwell equations and always links transverse components of the electromagnetic field.

7b) In the new theory the field $B^{(3)}$ is dual [1,2] to the imaginary and unphysical $-iE^{(3)}/c$ through a relation which is formally equivalent to a Faraday induction law. There is no real $E^{(3)}$, however, and no Poynting theorem for it. In other words there is no $E^{(3)}$ to carry electromagnetic field energy through the vacuum, only a $B^{(3)}$.

8a) In the orthodox vacuum theory there are no cyclic relations such as

$$B^{(1)} \times B^{(2)} = iB^{(0)}B^{(3)*}, \text{ et cyclicum}, \tag{357}$$

because $B^{(3)}$ is asserted not to exist. At this point the old theory becomes internally inconsistent because setting $B^{(3)}$ to zero in Eqs. (357) results in $B^{(1)} = B^{(2)} = 0$, while it is known that this is not the case, and if so, the cross product of $B^{(1)}$ and $B^{(2)}$ leads back to $iB^{(0)}B^{(3)*}$, meaning that $B^{(3)}$ exists.

8b) In the new theory these cyclical relations indicate that $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ can be understood easily as O(3) rotational generators [1], or Lorentz group generators in space-time. The theory is easily quantized because these are also angular momentum operators in quantum mechanics to within a factor $\hbar$ [1,2].

9a) The assertion $B^{(3)} = 0$ in the old theory leads directly to the E(2) inconsistency, which is described for example by Weinberg [93] or by Ryder [23]. The E(2) little group is the planar Euclidean group and is unphysical.

9b) In the new thinking $B^{(3)}$ is accepted as a routine experimental observable and is, thereby, not zero. The Wigner little group [1,2,22—25] becomes O(3), the rotation group, and is physical.

10a) In the orthodox view the photon is asserted to be an uncharged particle, so that it is its own anti-particle.

10b) This view is shown to be untenable by Eq. (202), and by the experimentally observable $B^{(3)}$ field [16—21]. The energy of the photon is proportional to e^2, and there can be an anti-photon with a different sign of e but with the same mass [1] and energy but opposite $B^{(3)}$.

11a) In the orthodox view, canonical quantization is beset by inconsistencies caused [23] by the fact that A_μ need not be completely covariant.

11b) In the new view, A_μ is a fully covariant and physical four-vector, which in free space obeys a limiting form $A_\mu A_\mu \to 0$ of the Dirac condition.

The above selection of some key differences between the old and new theories leads to the expectation that there will eventually be found (for example in Q.E.D.) numerous examples of specific effects due to $B^{(3)}$ which can be interpreted as specific effects of finite photon mass. However these, although indicating the existence of such mass, may not be sufficient in themselves to determine it experimentally without some additional input [6]. However, $B^{(3)}$ *is* sufficient to invalidate the concept of the identically

massless photon.

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In the current orthodoxy, which is rapidly losing its
like E(2) is accepted uncritically because in the Lagrang
massless photon is needed to keep the term $m^2 A_\mu A_\mu$ inv
ion. Even in this context, however, the Higgs mecha
mass into the boson known as the photon using spontaneous symmetry breaking of the
vacuum. The mechanism is applied for a Lagrangian which is originally compatible
with gauge invariance. These points have been pursued in Vol. 2 [2]. Weinberg [93]
has shown that A_μ cannot be quantized for an identically massless photon because it
corresponds to a (1/2, 1/2) irreducible representation of the Poincaré group. Such a
representation is not allowed, however, for $m = 0$ because in this case the helicity of the
massless particle must be $\lambda = A - B$, where the irreducible representations are denoted
by (A, B). Thus, we encounter the familiar difficulties with canonical quantization of A_μ
for example, the Lorentz gauge: an indefinite metric, negative energies, unwel-
come c numbers, and so forth. In the Coulomb gauge [23], A_μ is not completely
covariant, thus compounding an already severe problem.

These difficulties are usually patched up in the orthodox theory using the Gupta-
Bleuler method [1,2,22—25], but Weinberg [93] chooses to avoid completely the use
of a Lagrangian formalism in favor of one based on an S-matrix. He concludes that all
field equations are simply relations between spin components of the vector field.
Therefore if spin is *defined* through a Wigner little group, the only physical particles are
those with mass, because E(2) is unphysical. This is a complicated way of recognizing
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mass is deeply embedded in general relativity theory and the bending of space-time
cannot be expected to vanish arbitrarily as in the older, orthodox theory of electrody-
namics. As we have just seen, such an idea is equivalent to throwing away a space
dimension, leading to the E(2) inconsistency.

The recognition of $\mathcal{B}^{(3)}$ has the further major advantage of rendering the little
group O(3) for the photon as particle. To know *all* the representations of the Lorentz
group for the photon with mass we need *only* know [23] the generators of O(3), which
are directly proportional to $\mathcal{A}^{(3)}$, $\mathcal{B}^{(3)}$ and $\mathcal{R}^{(3)}$. These are the three fields of the

and electric field becomes irrelevant [5b]. In chiral liquids the magnetization from IMB does not depend on the light polarization, and its direction is parallel to that of beam propagation, reversing sign for enantiomers. In crystals the effect depends on space group symmetry [157] and in general is about a hundred to a thousand times weaker than the IFE.

These volumes have shown that electromagnetic radiation includes the $\boldsymbol{B}^{(3)}$ field, now known to be a phase free component of a class of longitudinal solutions [158,159] in vacuo of the electromagnetic field equations. The $\boldsymbol{B}^{(3)}$ component propagates in vacuo and is a radiated field as described in earlier chapters. Its interaction with one electron is governed by the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, which can be used either in the Dirac or Hamilton-Jacobi equations of motion, leading to expressions such as Eq. (403) of Vol. 1 and in Appendix F of this volume. The I dependence at visible frequencies gives way to an $I^{1/2}$ half dependence at radio frequencies under the right conditions.

The IFE has been studied in plasmas [17c] and in the liquid and solid states [17—21]. In the original experiment of Deschamps *et al.* [17c] microsecond, megawatt pulses of microwave radiation at 3 GHz were used to ionize a low pressure gas. In the received view [17a] the electrons so produced are driven into circular orbits by the electric field of the circularly polarized radiation and in consequence produce an axial magnetic field. In the new $\boldsymbol{B}^{(3)}$ theory the effect is described in Appendix F. In both cases the induced magnetic field is detected through induction in a pick-up coil surrounding the ionized gas. The intensity of the induced field and its dependence on the polarization of the incoming radiation were measured using two polarizers [17c]. Changing the angle between them produced a cosinal intensity variation. Over the range of intensity investigated, the inverse Faraday induction was found to be linear in the intensity I of the electromagnetic radiation.

The theoretically expected square root intensity dependence due to $\boldsymbol{B}^{(3)}$ acting at first order will start to dominate under the conditions discussed in Chap. 12 of Vol. 1, and these can be achieved by reducing the frequency of the radiation to the MHz range, for example 0.3 MHz, at which frequency the linear dependence curves off. In such an experiment, care should be taken to eliminate any effects which could produce an artifactual non-linear I profile. Chiang [100] has discussed the fact that ions would be accelerated in the opposite direction in experiments of this type, and in consequence the sample must be an electron beam. In plasma, artifactual ionic effects interfere to the order of 10% for an incident field amplitude $\boldsymbol{E}$ of the order of 2.89×10^5 volts per meter. This corresponds to an intensity of about 2.5×10^8 W/m^2, of the order of that needed to see the real, artifact free, square root intensity dependence. If an electron beam is used these artifacts are eliminated and furthermore, for a given incident field and induction coil, the induced voltage will be increased since many more electrons will be contributing to it. For example, if an electron gun were used to deliver say one amp cm^{-2}, and if the beam cross sectional area were 1 cm^2, then the total number of electrons

exposed in a given pulse would be of the order 10^{13}, a hundred times greater than the electron density of the Deschamps experiment.

An alternative scheme would be to use continuous wave (cw) circularly polarized microwave or radio frequency radiation with a pulsed electron beam. Such an arrangement is commonly used in an undergraduate experiment to determine the charge to mass ratio of the electron. Helmholtz coils surround a glass bulb containing an electron gun and a low pressure gas. The electrons are accelerated perpendicular to the field produced by the Helmholtz coils and under the right conditions move in circular orbits of a few centimeters radius. Such apparatus can be used to detect and measure $\boldsymbol{B}^{(3)}$ by exposing it to circularly polarized cw radiation, pulsing the electron gun and using the Helmholtz coils as pick up coils to measure the induced voltage. The radio frequency radiation is circularly polarized using a log spiral or helical antenna. In this experiment, constant beam power is maintained and the frequency varied using a device such as that produced by Antenna Research. A vacuum system containing a high current electron gun is used (a few amps per square centimeter). The electron gun is pulsed and the pick up coil measures $\boldsymbol{B}^{(3)}$ induced in the sample as given in Appendix F. Theoretically, modest beam powers of the order of 40 W m^{-2} would be sufficient to see an induced voltage provided the electron density N were adequate.

9.1 CONCLUSIONS

A survey has been given of a variety of techniques for investigating the magnetizing effects of circularly polarized electromagnetic radiation, focussing on the experimental prospects for detecting and measuring $\boldsymbol{B}^{(3)}$ through its expected square root I profile.

Appendix A. Circular Basis for Pauli Spinors

In the text the vector potential, $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$, of the electromagnetic wave has been set up in the circular basis defined by Eq. (32). In this appendix, the Pauli spinors are defined in the same basis in order to introduce $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ consistently into the Dirac equation consistently. In the basis ((1), (2), (3)) the Pauli spinors, by analogy with Eq. (32), are

$$\boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma}^{(2)*} = \frac{1}{\sqrt{2}}\left(\boldsymbol{\sigma}_X - i\boldsymbol{\sigma}_Y\right), \qquad \boldsymbol{\sigma}^{(3)} = \boldsymbol{\sigma}_Z \,. \tag{A1}$$

In the usual Cartesian basis, the spinors are well known [1,2,22—25] to be

$$\boldsymbol{\sigma}_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A2}$$

and Eqs. (A1) and (A2) lead directly to Eq. (39) in the text. It can be checked by direct evaluation that the Pauli spinors in ((1), (2), (3)) have SU(2) symmetry, and are therefore representations of three dimensional space. They are real, and therefore obey the following rules for Hermitian transposition [2,22—25],

$$\begin{gathered} \boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma}^{(2)+} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)+} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}^{(3)} = \boldsymbol{\sigma}^{(3)+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{gathered} \tag{A3}$$

The $\boldsymbol{\sigma}^{(3)}$ spinor is the same as the $\boldsymbol{\sigma}_Z$ spinor in the usual Cartesian representation, which also has SU(2) symmetry [2,22—25]. The representations of the Pauli spinors in the Cartesian and circular bases are therefore equivalent, but the latter basis is convenient for the description of circular polarization [1,2].

Appendix B. Product Algebra of Spinors and Unit Vectors in the Circular Basis

In order to evaluate products such as $\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)}$ appearing in the Dirac equation it is necessary to define the product algebra of Pauli spinors and unit vectors in the circular basis ((1), (2), (3)).

The dot product in this basis is equivalent to the Cartesian dot product, because the two bases are equivalent representations of three dimensional space, and so,

$$\begin{aligned} \boldsymbol{\sigma} \cdot \boldsymbol{e} &= \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)} + \boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)} + \boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{e}^{(3)} = \boldsymbol{\sigma}_X \cdot \boldsymbol{i} \\ &+ \boldsymbol{\sigma}_Y \cdot \boldsymbol{j} + \boldsymbol{\sigma}_Z \cdot \boldsymbol{k} = \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}, \end{aligned} \tag{B1}$$

giving the sum

$$\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)} + \boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)} = \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}. \tag{B2}$$

The individual terms in this sum are

$$\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)} = \frac{1}{\sqrt{2}} \left(\boldsymbol{\sigma}_X - i\boldsymbol{\sigma}_Y \right) \cdot \frac{1}{\sqrt{2}} \left(\boldsymbol{i} + i\boldsymbol{j} \right) = \begin{pmatrix} 0 & 0 \\ 1+i & 0 \end{pmatrix}, \tag{B3}$$

and

$$\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)} = \begin{pmatrix} 0 & 1-i \\ 0 & 0 \end{pmatrix}, \tag{B4}$$

and are complex. The Hermitian transpose (the matrix transpose with simultaneous complex conjugation of each element) is, for each term,

$$\left(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)}\right)^{+} = \boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)}, \quad \left(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)}\right)^{+} = \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)}. \tag{B5}$$

The product of terms is

$$\left(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)}\right)\left(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}. \tag{B6}$$

Using these rules, it can be checked directly that the product is

$$\left(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)}\right)\left(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)}\right) = \boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} + i\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)}, \tag{B7}$$

which is Eq. (25a) of the text, and is the correct spinor algebra in the basis ((1), (2), (3)). It is of key importance because it introduces the conjugate product $\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)}$ multiplied by the spinor $\boldsymbol{\sigma}^{(3)}$. This term represents mathematically the interaction of the conjugate product of a circularly polarized electromagnetic field with the spinor $\boldsymbol{\sigma}^{(3)}$, and thus with the half integral spin of a fermion such as a proton or electron.

Finally in this appendix, it is shown that

$$\left(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)} v\right)^{+} = v^{+}\left(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)}\right)^{+} = v^{+}\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)}, \tag{B8}$$

a relation which must be used to construct the Hermitian transpose of Eq. (26a), i.e., to construct Eq. (26b). We write the two-spinor as the column vector,

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{B9}$$

whose components are in general complex. The Hermitian transpose of this column vector is the row vector,

$$v^{+} = \left(v_1^{*}\, v_2^{*}\right). \tag{B10}$$

Evaluating directly, it follows that

$$v^{+}\left(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)}\right)^{+} = \left(v_1^{*} v_2^{*}\right)\begin{pmatrix} 0 & 1-i \\ 0 & 0 \end{pmatrix} = \left(0\,(1-i)\,v_1^{*}\right), \tag{B11}$$

and

$$\left(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)} v\right)^{+} = \left(\begin{pmatrix} 0 & 0 \\ 1+i & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)^{+} = \begin{pmatrix} 0 \\ v_1(1+i) \end{pmatrix}^{+} = \left(0\,(1-i)\,v_1^{*}\right), \tag{B12}$$

and so we arrive at Eq. (B8). Hermitian transposition algebra of this kind is important in the Dirac equation [11,18] because it is used to show that the equation produces a rigorously non-negative probability, unlike the Klein-Gordon equation.

Appendix C. Hermitian Transposition of the Dirac Equation

We wish to prove that the Hermitian transpose of the Dirac equation,

$$\left(i\gamma^{\mu}\partial_{\mu} - e\gamma^{\mu}A_{\mu} - m\right)\psi = 0, \tag{C1}$$

is

$$\bar{\psi}\left(i\gamma^{\mu}\overleftarrow{\partial}_{\mu} - e\gamma^{\mu}A_{\mu}^{*} + m\right) = 0. \tag{C2}$$

In order to do so, it is necessary only to consider the term $\gamma^{\mu}A_{\mu}$ when A_{μ} is complex in general, because the Hermitian transposition of the other terms is entirely standard [22—25]. The relevant term is

$$\gamma^{\mu}A_{\mu} = \gamma^{o}A_{o} + \gamma^{i}A_{i}, \tag{C3}$$

and its Hermitian transposition implies the algebraic transposition of matrices elements with simultaneous complex conjugation of those elements. So Hermitian transposition of term (C3) results in

$$\gamma^{\mu+}A_{\mu}^{*} = \gamma^{o}A_{o}^{*} - \gamma^{i}A_{i}^{*}, \tag{C4}$$

where A_{μ}^{*} is the complex conjugate of A_{μ}. We have used [2,22—25],

$$\gamma^{o+} = \gamma^{o}, \qquad \gamma^{i+} = -\gamma^{i}, \tag{C5}$$

and multiplying by γ^{o}, using $\gamma^{i}\gamma^{o} = -\gamma^{o}\gamma^{i}$, results in

$$-\bar{\psi} e\gamma^{\mu} A_{\mu}^{*} := -e\psi^{+}\gamma^{o}\left(\gamma^{o} A_{o}^{*} + \gamma^{i} A_{i}^{*}\right), \qquad \text{(C6)}$$

as in the text. This means that the operator $-e\gamma^{\mu}A_{\mu}^{*}$ multiplies the spinor $\bar{\psi} = \psi^{+}\gamma^{o}$ from the right, the operation in this case being matrix multiplication of the row vector $\bar{\psi}$, with four elements. In the original Dirac equation (C1) the operator $-e\gamma^{\mu}A_{\mu}$ multiplies the four-spinor ψ, a column vector, from the left. In the textbooks [22—25] the use of a complex A_{μ} in the Dirac equation is not standard, because the phenomenon of the anomalous Zeeman effect is being described in a static magnetic field whose vector potential is real. The standard treatment of the Dirac equation of a fermion in a plane wave is almost always restricted to linear polarization, and the key resonance term is missed.

Appendix D. Theory of Electrodynamics

	Fundamental Concept	Standard Theory	New Theory
1	Structure of Theory	linear, Abelian	**non-linear, Non-Abelian**
2	Maxwell's Equations in Vacuo	$\frac{\partial F_{\mu\nu}}{\partial x_{\nu}} = 0,\ \boldsymbol{B}^{(3)} = \boldsymbol{0}$	$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = i\boldsymbol{B}^{(0)}\boldsymbol{B}^{(3)*}$, et cyclicum
3	Electromagnetic Sector Symmetry	$U(1) = O(2)$	$O(3)$
4	Poincaré Group Symmetry	$\hat{J}^{(3)}$ generator missing, unphysical little group	**magnetic fields are rotation generators**
5	Wigner Little Group	$E(2)$, unphysical planar, Euclidean	$O(3)$, **physical space rotation**
6	Source of $\boldsymbol{B}^{(3)}$ at Observer Point R and Instant t	not considered	**circling e at time $t - \frac{R}{c}$ earlier**
7	Propagation of $\boldsymbol{B}^{(3)}$	not considered	**through the Liénard-Wiechert potentials $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$**

	Fundamental Concept	Standard Theory	New Theory
8	Gauge Group Definition of $B^{(3)}$	not considered	$B^{(3)*} = -i\frac{e}{\hbar}A^{(1)} \times A^{(2)}$
9	Free Space Four-Tensor	$F_{\mu\nu}$, Abelian in space ((1), (2), (3))	$G_{\mu\nu}$, Non-Abelian in space ((1), (2), (3))
10	Photon Helicities	-1 and 1	-1, 0, 1
11	Translational Poynting Theorem	$\nabla \cdot N = -\frac{\partial U}{\partial t}$	same
12	Rotational Poynting Theorem	not considered	$\nabla \cdot J^{(3)} = -\frac{\partial U^{(3)}}{\partial t}$
13	Magnetic Fields	$B^{(1)} = B^{(2)*}$	$B^{(1)} = B^{(2)*}$, $B^{(3)}$
14	Electric Fields	$E^{(1)} = E^{(2)*}$	$E^{(1)} = E^{(2)*}$, $iE^{(3)}$
15	Vector Potentials	$A^{(1)} = A^{(2)*}$	$A^{(1)} = A^{(2)*}$, $iA^{(3)}$, A_0
16	Planck-Einstein Relation	$En = \hbar\omega$	$En = \hbar\omega = h_1\lambda$, $h_1 = ec\lvert B^{(3)}\rvert$
17	de Broglie Relation	$p = \hbar\kappa$	$p = \hbar\kappa = \frac{h_1\lambda}{c}$

	Fundamental Concept	Standard Theory	New Theory
18	Quantum of Energy	$\hbar\omega = \frac{1}{\mu_0}\int(B^{(1)} \cdot B^{(1)*} + B^{(2)} \cdot B^{(2)*})dV$	$\hbar\omega = \frac{1}{\mu_0}\int(B^{(1)} \cdot B^{(1)*} + B^{(2)} \cdot B^{(2)*} + B^{(3)} \cdot B^{(3)*})dV_1$
19	Quantum of Angular Momentum	$\hbar$	$\hbar = h_1\frac{\lambda}{\omega}$
20	Quantum of Torque	$\hbar\omega$	$\hbar\omega = h_1\lambda$
21	Momentum Equivalence Condition	not considered	cyclic relations imply in vacuo $eA^{(0)} = \hbar\kappa$, the quantum of linear momentum
22	Mass of Photon	identically zero	$m = \frac{e\lambda_0\lvert B^{(3)}\rvert}{c}$
23	Gauge Invariant Lagrangian Mass Term	not considered	$En = \frac{V}{\mu_0}\left(\frac{Mc^2}{\hbar\omega}\right)^2 B^{(3)} \cdot B^{(3)*}$ in $O(3)$ gauge group
24	Gauge Conditions on Four-Potential	1) transverse 2) Coulomb	1) not allowed 2) scalar, non-zero 3) $A_\mu A_\mu \to 0$

	Fundamental Concept	Standard Theory	New Theory
25	**Field Quantization**	canonical, beset with difficulties because A_μ is not covariant	direct quantization of cyclic field relations
26	**de Broglie Theorem**	not considered	$\hbar\omega_0 = h_1\lambda_0 = mc^2$
27	**Interaction with Fermion**	via $A^{(1)} = A^{(2)*}$ in minimal prescription, $A_0 = \phi = 0$	same as Standard Theory but finite scalar potential $A_0 = \phi \neq 0$
28	**Magneto-optics**	I dependence from $A_0 = 0$	$I^{1/2}$ dependence observable under the right conditions, $A_0 \neq 0$
29	**Q.E.D.**	no mass term	finite mass term
30	**Observation of $B^{(3)}$ Field**	not considered	routinely observable through $iB^{(0)}B^{(3)*}$ in magneto-optics

Notes:

N := Poynting vector; U = energy density;

$J^{(3)}$ = radiation angular momentum; $U^{(3)} = \frac{1}{\mu_0} B^{(3)} \cdot B^{(3)*}$;

λ := wavelength; λ_0 = rest wavelength;

V = volume of radiation; M = mass of radiation;

$\phi = A_0$ = scalar potential.

Appendix E. Dynamical Analogies of the Maxwell Equations

The existence of the cyclical field equations,

$$B^{(1)} \times B^{(2)} = iB^{(0)}B^{(3)*}, \text{ et cyclicum}, \tag{E1}$$

implies that there is a precise dynamical analogy to the Maxwell equations in free space. Equations (E1) can be used to derive the Maxwell equations because the former are fundamental relations between fields in free space, relations to which there is a precise dynamical analogy,

$$J^{(1)} \times J^{(2)} = iJ^{(0)}J^{(3)*}, \text{ et cyclicum}, \tag{E2}$$

where J denotes angular momentum. The close similarity between Eqs. (E1) and (E2) stems from the fact that both equations describe O(3) rotation generators, suggesting a profound analogy between the structure of space-time and electromagnetism. This can be extended to general relativity. In particular, the rotation generators obey a cyclically symmetric, compact, group algebra, that of O(3), whereas linear momentum generators do not. In consequence, there is a $B^{(3)}$ field, corresponding to $J^{(3)}$, but no $E^{(3)}$, because the polar $p^{(3)}$ cannot be generated from the axial product $p^{(1)} \times p^{(2)}$ of transverse momentum generators. This is true in any system of coordinates, both in special and in general relativity. At the most fundamental level, electromagnetism in free space, like gravity, is a consequence of the structure of space-time itself, be this Euclidean or Riemannian. Furthermore, the existence of cyclic relations between space rotation generators implies that $B^{(1)}$ and $J^{(1)}$ must be complex in nature, and since they are rotation generators, time dependent in some way. The Maxwell equations are invariant under the duality transformation [1,2] (Minkowski notation),

$$\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) \longrightarrow \epsilon_{\mu\nu\rho\sigma}\left(\frac{\partial A_\sigma}{\partial x_\rho} - \frac{\partial A_\rho}{\partial x_\sigma}\right), \tag{E3}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the four-dimensional Levi-Civita symbol, and from the analogy between Eqs. (E1) and (E2), it is expected that there is a direct dynamical analogy to this result,

$$\left(\frac{\partial p_\nu}{\partial x_\mu} - \frac{\partial p_\mu}{\partial x_\nu}\right) \longrightarrow \epsilon_{\mu\nu\rho\sigma}\left(\frac{\partial p_\sigma}{\partial x_\rho} - \frac{\partial p_\rho}{\partial x_\sigma}\right), \tag{E4}$$

where p_μ is the energy-momentum four-vector. If $F_{\mu\nu}$ is the four-tensor of electric and magnetic fields [1,2], and $J_{\mu\nu}$ that of rotation and boost generators, then the above-mentioned analogy between electromagnetism and dynamics can be summarized through the statement that $F_{\mu\nu}$ is proportional to $J_{\mu\nu}$. Thus [1,2], magnetic fields are rotation generators, electric fields are boost generators, and the free space Maxwell equations become those of classical dynamics.

In free space, the Maxwell equations in S.I. are,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \cdot \boldsymbol{B} = 0, \tag{E5a}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \qquad \nabla \cdot \boldsymbol{E} = 0. \tag{E5b}$$

The pair of equations (E5a) and (E5b) can be written respectively as

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 0, \qquad \frac{\partial}{\partial x_\nu}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} = 0, \tag{E6}$$

where $F^D_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual tensor [1,2] of $F_{\mu\nu}$. The duality transform (E3) can therefore be expressed through the fact that Maxwell's free space equations are invariant under

$$F_{\mu\nu} \longrightarrow F^D_{\mu\nu}. \tag{E7}$$

The duality transform offers considerable insight to the nature of electromagnetism and space-time at the most fundamental level. It can be expressed in terms of space-time as

$$\nabla\times \longrightarrow -\frac{i}{c}\frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t} \longrightarrow ic\nabla\times, \tag{E8}$$

and in terms of electromagnetism as

$$\boldsymbol{B}^{(0)} \longrightarrow i\frac{\boldsymbol{E}^{(0)}}{c}, \qquad \boldsymbol{E}^{(0)} \longrightarrow -ic\boldsymbol{B}^{(0)}. \tag{E9}$$

It is easily checked that both (E8) and (E9) leave the Maxwell equations (E5) unchanged. Therefore Eqs. (E5) can be written as

$$\boldsymbol{B} \longrightarrow i\frac{\boldsymbol{E}}{c}, \qquad \boldsymbol{E} \longrightarrow -ic\boldsymbol{B}. \tag{E10}$$

On the most fundamental level, therefore, Maxwell's equations indicate that the duality transform

$$\sqrt{+1}\,\boldsymbol{B} \longrightarrow \sqrt{-1}\frac{\boldsymbol{E}}{c}, \tag{E11}$$

is equivalent to the duality transform

$$(\nabla\times)\boldsymbol{B} \longrightarrow \left(-\frac{i}{c}\frac{\partial}{\partial t}\right)\frac{i\boldsymbol{E}}{c}. \tag{E12}$$

This can be seen clearly through the vector potential $\boldsymbol{A}$, because

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \longrightarrow -\frac{i}{c}\frac{\partial}{\partial t}\boldsymbol{A} = \frac{i\boldsymbol{E}}{c}, \tag{E13}$$

thus combining Eqs. (E8) and (E9). Ultimately, therefore, the generation of an electric from a magnetic field is a matter of replacing the operator $\nabla\times$ by the operator $-\frac{i}{c}\frac{\partial}{\partial t}$, both acting on $\boldsymbol{A}$, and this is a consequence of the structure of space-time itself. In the minimal prescription, the vector potential $\boldsymbol{A}$ is $\boldsymbol{p}/e$, where $\boldsymbol{p}$ is linear momentum, and from the Lorentz force equation, $\boldsymbol{F}$ is expressible dimensionally as $e\boldsymbol{E}$. Relations such as these provide an opportunity, therefore, of writing the Maxwell equations as equations of pure dynamics. Using $\boldsymbol{E} = -1/e(\partial\boldsymbol{p}/\partial t) = -\boldsymbol{F}/e$, it is easily checked that

$$\boxed{\nabla\times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}} \quad \longleftrightarrow \quad \boxed{\nabla\times \boldsymbol{F} = -\frac{\partial \boldsymbol{J}_A}{\partial t}} \tag{E14}$$

where $\boldsymbol{J}_A$ is angular momentum per unit area. So in classical dynamics, the curl of force is the negative of torque per unit area, $-\partial \boldsymbol{J}_A/\partial t$. This is a precise analogy to the free space Maxwell equation on the left hand side of Eq. (E14), $\boldsymbol{F}$ being analogous to $\boldsymbol{E}$, $\boldsymbol{J}_A$ to $\boldsymbol{B}$. Similarly,

$$\boxed{\nabla\times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}} \quad \longleftrightarrow \quad \boxed{\nabla\times \boldsymbol{J}_A = \frac{1}{c^2}\frac{\partial \boldsymbol{F}}{\partial t}} \tag{E15}$$

It follows, also, that

$$\boxed{\begin{array}{c}\nabla\cdot \boldsymbol{B} = 0 \\ \nabla\cdot \boldsymbol{E} = 0\end{array}} \quad \longleftrightarrow \quad \boxed{\begin{array}{c}\nabla\cdot \boldsymbol{J}_A = 0 \\ \nabla\cdot \boldsymbol{F} = 0\end{array}} \tag{E16}$$

showing that if the rotational motion is taking place around the Z axis, there is no phase dependence of the type $\exp(i(\omega t - \kappa Z))$ in the Z components of $\boldsymbol{J}_A$ and $\boldsymbol{F}$.

In order to demonstrate more clearly that electric fields are boost generators and magnetic fields rotation generators [1,2] it is necessary to introduce linear momentum, $\boldsymbol{p}$, into the above relations, rather than force $\boldsymbol{F}$, whereupon

$$\boxed{\nabla\times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}} \quad \longleftrightarrow \quad \boxed{\frac{v^2}{\omega}\nabla\times \boldsymbol{p} = -\frac{\partial \boldsymbol{J}}{\partial t}} \tag{E17}$$

with, as in classical dynamics,

$$\boldsymbol{v} = \boldsymbol{\omega}\times \boldsymbol{r}, \qquad \boldsymbol{J} = \boldsymbol{r}\times \boldsymbol{p}. \tag{E18}$$

Here $\boldsymbol{\omega}$ and $\boldsymbol{J}$ denote angular velocity and momentum respectively, and $\boldsymbol{r}$ is the radius vector of the angular motion [1,2]. The free space Maxwell equation in analogy (E17) is therefore equivalent to the fundamental relation between linear and angular momentum in classical dynamics.

From Eq. (E14),

$$\nabla\times \boldsymbol{F} = \frac{1}{\pi r^2}\frac{\partial \boldsymbol{J}}{\partial t}, \tag{E19}$$

where, in general, and in analogy with the general relation between $\boldsymbol{E}$ and $\boldsymbol{A}$,

$$\boxed{\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi} \quad \longleftrightarrow \quad \boxed{\boldsymbol{F} = -\frac{\partial \boldsymbol{p}}{\partial t} - \nabla U} \tag{E20}$$

where U is potential energy. The curl of $\boldsymbol{F}$ is independent of U, and the curl of $\boldsymbol{E}$ is independent of ϕ. Continuing the analogy,

$$\boxed{\boldsymbol{B} = \nabla\times \boldsymbol{A}} \quad \longleftrightarrow \quad \boxed{\boldsymbol{J} = \pi r^2 \nabla\times \boldsymbol{p}} \tag{E21}$$

showing that $\boldsymbol{J}$ is independent of U just as $\boldsymbol{B}$ is independent of ϕ. Comparing Eqs. (E18) and (E21) shows that $\boldsymbol{r}\times = \pi r^2 \nabla\times$. The right hand side of Eq. (E21) can be obtained by direct integration of Eq. (E19),

$$\boldsymbol{J} = -\pi r^2 \int \nabla\times \boldsymbol{F}\, dt = -\pi r^2 \nabla\times \int \boldsymbol{F}\, dt = \pi r^2 \nabla\times \boldsymbol{p}, \tag{E22}$$

using Eq. (E20). Therefore, the fundamentally analogous quantities are

$$\boldsymbol{B} \longleftrightarrow \boldsymbol{J}, \quad \boldsymbol{A} \longleftrightarrow \pi r^2 \boldsymbol{p}, \quad \boldsymbol{E} \longleftrightarrow \frac{v^2}{\omega}\boldsymbol{p}, \tag{E23}$$

showing that $\boldsymbol{B}$, $\boldsymbol{E}$, and $\boldsymbol{A}$ in electrodynamics are as fundamental as $\boldsymbol{J}$, $\boldsymbol{F}$ and $\boldsymbol{p}$ in dynamics. The dynamical analogy shows clearly that both fields and potentials are physically meaningful.

The duality transformation in dynamics can be developed in its clearest form by accepting the validity of the de Broglie guiding theorem,

$$En = mc^2 = \hbar\omega_0 . \tag{E24}$$

If v is identified with c in Eq. (E17), then,

$$\frac{c^2}{\omega_0} = \frac{\hbar}{m}, \tag{E25}$$

so that in the photon rest frame,

$$\nabla \times \boldsymbol{p} = -\frac{m}{\hbar}\frac{\partial \boldsymbol{J}}{\partial t}, \tag{E26}$$

where m is the mass of the photon. The duality transform therefore becomes

$$\boxed{\sqrt{+1}\boldsymbol{B} \longrightarrow \sqrt{-1}\frac{\boldsymbol{E}}{c}} \quad \longleftrightarrow \quad \boxed{\sqrt{+1}\boldsymbol{J} \longrightarrow \sqrt{-1}\frac{\hbar}{m}\boldsymbol{p}} \tag{E27}$$

which, with $\boldsymbol{v} = \boldsymbol{p}/m$, is equivalent to

$$\boldsymbol{v} \longrightarrow -\frac{i\boldsymbol{J}}{\hbar}, \quad \frac{\boldsymbol{J}}{\hbar} \longrightarrow i\boldsymbol{v}. \tag{E28}$$

Equation (E28) means that $\boldsymbol{J}/\hbar$ plays the role of $\boldsymbol{B}$ in the duality transform, and $\boldsymbol{v}$ that of $\boldsymbol{E}/c$. Thus, the magnetic field is a rotation generator and the electric field a boost generator [1,2], which is the momentum-energy within a factor. If there is photon

mass, $\boldsymbol{B}^{(3)}$ and the $\boldsymbol{J}^{(3)}$ are missing, or at best, ill-defined.

Equations of motion can be constructed directly from relations with group symmetry O(3) because this is the group of infinitesimal rotation generators and angular momentum operators [1]. Space itself is described by a set of relations between complex unit vectors [1,2], $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, and $\boldsymbol{e}^{(3)}$ in the basis ((1), (2), (3)),

$$\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)*}, \text{ et cyclicum}. \tag{E29}$$

In order to convert this into an equation of motion, it is written as

$$\left(\boldsymbol{e}^{(1)}e^{i\phi}\right) \times \left(\boldsymbol{e}^{(2)}e^{-i\phi}\right) = i\boldsymbol{e}^{(3)*}, \text{ et cyclicum}, \tag{E30}$$

where $e^{i\phi}$, $\phi = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}$, is the de Broglie wave function. This is a generator of wave motion with relativistically invariant phase. Equations (E29) represent the framework of space, while Eqs. (E30) represent wave propagation with simultaneous rotational motion. If the propagation velocity of the wave is v, then $\boldsymbol{e}^{(3)}$ also propagates at this velocity, because at instant t, the frame ((1), (2), (3)) is centered on an origin at point $\boldsymbol{R}$. The complete frame must move forward to point $\boldsymbol{R}'$ at instant t' because if $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ propagate, so does $\boldsymbol{e}^{(3)}$, being always defined by $\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)}$.

Equation (E30) is therefore an equation which describes the characteristics of wave motion superimposed on the structure of space. It is a frame of reference spiralling forward at the propagation velocity of the wave. Equations (E30) become Eqs. (E1) provided

$$\boldsymbol{B}^{(1)} = i\boldsymbol{B}^{(0)}\boldsymbol{e}^{(1)}e^{i\phi}, \quad \boldsymbol{B}^{(2)} = -i\boldsymbol{B}^{(0)}\boldsymbol{e}^{(2)}e^{-i\phi}, \quad \boldsymbol{B}^{(3)} = \boldsymbol{B}^{(0)}\boldsymbol{e}^{(3)}, \tag{E31}$$

showing that $\boldsymbol{B}^{(3)}$ propagates at the same speed as $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$. Since Eq. (E1) is analogous with Eq. (E2), cyclically symmetric relations between angular momenta, and since angular and linear momentum and angular momentum and force are related fundamentally through equations identical in structure to the free space Maxwell equations, the cyclic equations (E1) imply the existence of the Maxwell equations themselves. *Additionally*, they imply the existence of $\boldsymbol{B}^{(3)}$ in vacuo and in matter, and are therefore field equations, relations between fields, which are more fundamental than the Maxwell equations. Equations (E1) show that all types of angular momentum theory can be applied directly to field theory, both in classical and quantum mechanics, in special and in general relativity.

It *is* possible to modify the Maxwell equations to account for $\boldsymbol{B}^{(3)}$ from equations (E1), but this procedure is self-consistent only in the O(3) gauge group, which is non-

Abelian [2]. Equations (E1) show that Maxwell's equations must be generalized, and are therefore more fundamental as we have argued. In the O(3) gauge group, covariant derivatives are used, borrowing techniques of gauge theory taken from general relativity [2,23]. This leads to the possibility of developing $\boldsymbol{B}^{(3)}$ in Riemannian geometry, with an eye to proving that space-time becomes curved in the presence of electromagnetism as well as gravitation. This would be a step towards a unified description of electroweak and gravitational fields. A glance at equations (E1) shows them to be non-Abelian and non-linear. The much older Maxwellian point of view is linear and Abelian, making unification with gravitational theory difficult.

The dynamical operation,

$$\boldsymbol{e}^{(1)} \longrightarrow \boldsymbol{e}^{(1)} \exp\left(i\omega\left(t - \frac{r}{c}\right)\right), \tag{E32}$$

is a boost of $\boldsymbol{e}^{(1)}$ [1], and is alone sufficient to produce a generalized version of the Maxwell equations in free space. It is an operation on $\boldsymbol{e}^{(1)}$ with the well known exponent [21-25] $\exp(iS/\hbar)$ where S is the electromagnetic action $\hbar(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r})$. We have introduced $\hbar$ into the analysis, which has therefore become quantum mechanical in nature. In space-time the action in Minkowski notation is $S = \hbar\kappa_\mu x_\mu$. Therefore the generalized Maxwell equations are produced by the boost,

$$\boldsymbol{e}^{(1)} \longrightarrow \exp\left(i\hbar\kappa_\mu x_\mu\right)\boldsymbol{e}^{(1)} = \exp\left(ieA_\mu x_\mu\right)\boldsymbol{e}^{(1)} \tag{E33}$$

which is a rotation in space-time in special relativity. Defining the operator by the symbol $\overset{\Delta}{\mathcal{Q}} = e^{ieA_\mu x_\mu}$, transition to fields is achieved, finally, through

$$\frac{\boldsymbol{B}^{(1)}}{\boldsymbol{B}^{(0)}} = i\overset{\Delta}{\mathcal{Q}}^{(1)}\boldsymbol{e}^{(1)}, \qquad \frac{\boldsymbol{B}^{(2)}}{\boldsymbol{B}^{(0)}} = -i\overset{\Delta}{\mathcal{Q}}^{(2)}\boldsymbol{e}^{(2)}, \qquad \frac{\boldsymbol{B}^{(3)}}{\boldsymbol{B}^{(0)}} = \boldsymbol{e}^{(3)}, \tag{E34}$$

and the duality transformation,

$$\sqrt{+1}\,\boldsymbol{B}^{(0)} \longrightarrow \sqrt{-1}\,\frac{\boldsymbol{E}^{(0)}}{c}, \tag{E35}$$

takes Eqs. (E1) to their equivalents for electric fields

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = -\boldsymbol{E}^{(0)}(i\boldsymbol{E}^{(3)})^* \text{ et cyclicum}, \tag{E36}$$

where $i\boldsymbol{E}^{(3)}$ is pure imaginary and unphysical [1,2]. The electric field equations (E36) are as fundamental as the magnetic field equations (E1), being transformed into (E1) by fundamental tensorial duality.

Appendix F. $B^{(3)}$ in an Electron Gas

For purposes of experimental investigation, we list in this Appendix equations relating $B^{(3)}$ in free space to its value in a sample made up of N electrons in a volume V. In general, using the text's notation,

$$B^{(3)}_{\text{in sample}} = \frac{N}{V} \cdot \frac{\mu_0 e^3 c^2}{2m\omega^2} \left(\frac{B^{(0)}}{(m^2\omega^2 + e^2 B^{(0)2})^{1/2}} \right) B^{(3)}_{\text{free space}} . \tag{F1}$$

In *the low field (visible frequency) limit*, $m\omega \gg eB^{(0)}$, Eq. (F1) reduces to

$$B^{(3)}_{\text{in sample}} \longrightarrow \frac{N}{V} \left(\frac{\mu_0 e^3 c^2 B^{(0)}}{2m^2\omega^3} \right) B^{(3)}_{\text{free space}} , \tag{F2}$$

and in *the high field (radio frequency) limit*, $m\omega \ll eB^{(0)}$, Eq. (F1) becomes

$$B^{(3)}_{\text{in sample}} \longrightarrow \frac{N}{V} \left(\frac{\mu_0 e^2 c^2}{2m\omega^2} \right) B^{(3)}_{\text{free space}} . \tag{F3}$$

The *free space* value of $B^{(3)}$ is [1,2],

$$B^{(3)}_{\text{free space}} = \left(\frac{\mu_0}{c} I \right)^{1/2} e^{(3)} = \left(\frac{I}{\epsilon_0 c^3} \right)^{1/2} e^{(3)} . \tag{F4}$$

In terms of beam intensity, the low field limit is

$$\boldsymbol{B}^{(3)}_{\text{in sample}} \longrightarrow \frac{N}{V}\left(\frac{\mu_0^2 e^3 c}{2m^2}\right)\frac{I}{\omega^3}\boldsymbol{e}^{(3)}, \tag{F5}$$

and under the conditions used by Rikken [74], i.e., $I = 5.5 \times 10^{12}$ Wm^{-2}, $\omega = 1.77 \times 10^{16}$ rad s^{-1}, we obtain from Eq. (F5),

$$\boldsymbol{B}^{(3)}_{\text{in sample}} = 1.06 \times 10^{-35}\frac{N}{V}\boldsymbol{e}^{(3)} \sim 10^{-9} \text{ tesla}, \tag{F6}$$

$$\text{or } \sim 10^{-5} \text{ gauss},$$

which, for $N/V = 10^{26} m^{-3}$, is about the same order of magnitude as reported experimentally by van der Ziel *et al.* [17] in the first inverse Faraday effect experiment. *This magnitude is about four or five orders below the limit of sensitivity reported by Rikken* [74]. It is important to note that the interaction of $\boldsymbol{B}^{(3)}$ with matter is relativistic, because $\boldsymbol{B}^{(3)}$ propagates, F.A.P.P., at c in vacuo. It is important not to confuse the free space magnitude of $\boldsymbol{B}^{(3)}$, i.e., Eq. (F4), with its magnitude within a sample. Finally, it is important to test $\boldsymbol{B}^{(3)}$ under conditions appropriate to the theory, i.e., ideally, in a free electron gas or electron beam. Liquid benzene, used by Rikken, has no free fermion spin, and is inappropriate for comparison with Eq. (F1), for N *electrons* in a sample V.

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Index

$A^{(1)}$ and $A^{(2)}$
  Liénard-Wiechert potentials 89
$A^{(1)} \times A^{(2)}$
  conjugate product 51
  gauge transformed 51
$A^* \times A$
  conjugate cross product 8
Abelian electrodynamics 125
Acceleration, transverse 103
Action at a distance 58, 100, 150
Aharonov-Bohm effects 2, 41, 42, 56, 77
Angular momentum
  conservation of 71
  field 95, 98
  four-vector 112
  intrinsic 8
  matter-wave 53
  orbital, of the fermion 32
  radiated 95
Angular-momentum
  four-tensor 125
Anti-fermions 13
Anti-particle 121
$A \times A^*$
  conjugate cross product 42
  origin of 59
  physical cross product 41
A_μ
  completely covariant 14
  Einstein equation for 135
  four-potential 137
  potential four-vector 100
  timelike component 25

$B^{(1)} \times B^{(2)}$
  observability 130
$B^{(3)}$
  and energy density 112
  and Lagrangian mass term 129
  and Liénard-Wiechert potential 106
  and magnetization 49, 99
  and microwave background 177
  and non-zero mass 136
  and non-zero photon mass 98, 122
  and Poynting vector 112
  and propagation 100, 102
  and radiation 90, 94, 95, 101, 104, 105, 109, 111
  and relativistic cosmology 161
  and Young experiment 149
  as a pilot field 149, 152
  as a pilot wave 146
  classical radiation theory of 89
  derivation of 71
  Evans-Vigier field 9
  existence-observability 41, 126, 131
  field 25, 67
  forms of 61, 80
  heuristic theory of 82
  $I^{1/2}$ effect of 86
  in vacuo 106
  inverse square frequency dependence of 3
  magnetic field 90
  observability 58
  propagation in vacuo 87
  radiated field 107, 108
  radiation from an electron 89
  relativistic explanation of 123
  theory of light 119
Beam 2, 35, 151
  square root intensity in 90
Beth effect 159
Bianchi identity 128
Biot-Savart law 77, 99
Biot-Savart-Ampère law 77

$\hat{C}$ 28, 101
$\hat{C}$, $\hat{P}$, $\hat{T}$, symmetry 80
Canonical
  energy-momentum tensor 125
  quantization 15, 28
Casimir invariants 122

harge e
a helix 75
and radiated angular momentum 95
in the primordial universe 100
rotating 71, 73
rotating, and radiation 91
spiralling 104
harge motion, helical 67
harge
agent of interaction 100
and acceleration 107
and $B^{(3)}$ and $A^{(2)*}$ 92
and radiation as energy transfer 101
centripetal acceleration of 102, 109
circular motion of 76, 93, 111
conservation 74
propagating without mass 77
quantization condition 73
rotating electric 63
rotation of 90
target 111
vacuum, rotating 74
harged fields on neighboring photons 77
ristoffel-like asymmetric connection 163
rcular basis 10, 119
rcular indices 83
llision, elastic 27
mpton effect 44
nductivity
coefficient 166
coefficient, and torsion 172
vacuum, and magnetic field 175
njugate product 7, 8, 42, 51, 93, 120
nnection coefficient 128, 165
nservation
current 19
energy-momentum-charge 137
general physical law of 138
laws, electrodynamical 82
of $\hat{C}$ symmetry 27
of charge 43
of energy-momentum 22
nstant, fine structure 63, 97
ntinuity equation 86, 111
penhagen interpretation 151
smogony 176
smology, relativistic 161
ulomb
gauge 9
law 107
variant derivative 127
Covariant differentiation 165
Current
retarded 86
transverse 93
Cyclical relations 61, 76, 121, 132

d'Alembert equations 100, 135
d'Alembertian 17, 74
de Broglie
and the photon 14, 142
guiding theorem 97, 135, 146
matter-wave equation 69
photon momentum 67
theorem 139
Dipole
approximation 75
moments, oscillating and magnetic 147
Dirac
condition 12
equation 29, 60, 94, 164
equation, limiting forms of 23
four-spinor 6
matrix 4, 10, 161
wave equation 30
Dynamo 177

E(2) group 122, 133
Euclidean 82
inconsistency 121
planar, Euclidean 132
unphysical 117, 119
Einstein
Cartan theory 164
coefficients 85
equation 16, 69, 128, 135
quantized equation 140
Elastic collision 15
Electric field
rotating 45
Electrodynamics
Abelian 125
equations of 101
Electromagnetic
coupling, minimal 166
energy 154
energy, per unit volume 113
field 100
field tensor 165
phase 11
plane waves 116
power, per unit area 113
sector 56, 79
theory, and photon mass 135

torque density 147
Electromagnetism
and gravitation, unified 126, 127
group of 42
Electron
acceleration, & field limit 64
anomalous magnetic moment of 131
beams, interfering 57
circling source 89
classical 27
fringe pattern 57
induced orbital angular momentum 69
matter wave 41
momentum 45
photon collision 44
positron annihilation 28
residual mass 68
resonance 4, 31
source, circling slowly 108
theory of Dirac 5
Empty wave 142, 146, 152
Energy
and wavelength 155, 157
density 62, 112
density, radiated 111
density, vacuum 108
expectation values 13
rest 29
term, kinetic 31
Equivalence 16, 61, 70
ESR 1, 37
ultra-high resolution 21
Evans-Vigier field 14, 49
Existence arguments 49

Faraday
induction 101, 120
inverse effect 4, 33, 34, 61, 81, 90
optical effect 61, 81
Fermion 8, 27
accelerated 44
resonance 1
slow 11
spin 1, 5, 52
total energy 25
Feynman slash notation 16, 18
Field and matter, interaction 87
Field limit
high 43
low 42
strong 47, 59, 70, 82, 87, 90, 99, 100, 102, 111
strong, and electron acceleration 64
weak 47, 59, 60, 82, 87, 90, 100
Field
and the magnetic dipole moment 96
angular momentum 98
equivalent mass of 97
heplon 166
induction 111
magnetogyric ratio 96
pump electromagnetic 36
radiated, transverse 107
vierbein, and metric tensor 161
Fock-Ivanenko coefficients 163
Four-tensor
energy momentum 82
Four-vector
complex potential 7
momentum 6
potential 6
Fourier transform infra-red spectrometer 37
Frequency 157
Friction, vacuum 140

Galaxy, young disc 177
Gauge invariance 120, 127
in O(3) 135
Lagrangian 129
Lagrangian density 133
local 19
mass term 132
of the second kind 50
type two 13, 23
Gauge
non-Abelian theories 133
theories, spontaneously broken 49
transformation 51, 53, 54
General relativity 48, 123
Gravitation 126
and electromagnetism, unified 126, 127
non-Abelian theory 127
Group space 43, 48, 49
Gupta-Bleuler method 122

$\hbar_1$
definition of 158
Hamilton-Jacobi equation 22, 46, 68, 98, 150
Harmony of phases theorem 155
Hermitian transpose 11
Homotopy group 49
Hubble
constant 167
universe and spin density 172

2
dependence 70
effect of $B^{(3)}$ 86
profile 126
$^{(0)}B^{(3)}$
observed 81
$^{(0)}B^{(3)*}$
and magneto-optics 126
existence 119
observability 132
aginary $-iE^{(3)}$ 62
ra-red 3
erferogram 149
verse Faraday effect 4, 33, 34, 61, 81, 90

), classical definition of 114
piter, and the Pioneer 10 flyby 173

ein Gordon equation 17, 30

grangian 124, 129, 133
mb shift 131
ndau-Ginzburg Lagrangian 50
algebra 123, 127
énard-Wiechert potential 89, 106
ght
aberration 143
in vacuo, phase velocity 86
induced NMR shifts 82
intensity, tensor of 62
magnetic effect of 54
orthodox theories of 119
shifts 81
rentz
and particle mass 136
force 45, 102, 154
frame 13
matrix and spacetime 162
matrix and spacetime spinor 162
scalar, mass 24
transformation 28
w frequency, high intensity limit 23

agnetic dipole moment 96
agnetic field
and cosmogonic implications 176
and vacuum conductivity 175
$B^{(3)}$ 90
cosmic 161
fundamental 61
primordial 161, 176
primordial and non-zero 172
rotating 45
static 8, 104, 111
Magnetic
effect 5
Magnetization 4, 76
Magnetization 99
and intensity dependence 90
Magneto-optics 4
and $iB^{(0)}B^{(3)*}$ 126
effects 58, 71, 120
effects, $I^{1/2}$ profile 85
field responsible for 71
Magnetogyric ratio, of the field 96
Mass
a Lorentz scalar 24
invariant 24
unit of 29
Maxwell
Boltzmann exponential 38
equations 170
equations, homogeneous 128
equations, vacuum 14
wavevector 67
Microwave radiation, and $B^{(3)}$ 177
Microwaves, circularly polarized 22
Minimal prescription 2, 15, 18, 43, 55, 87, 91
Motion (of e)
helical through the vacuum 77
spiralling in vacuo 78

Naive interpretation 151
NMR 1, 37, 38
ultra-high resolution 21
Noether's theorem 19, 55, 125
Non-Abelian
Lie algebra 127
theory 43, 127, 133
Non-locality 51, 150
Non-relativistic
approximation 29
limit 11
Nuclear resonance 31
Nucleon resonance 4
Nucleosynthesis 177

O(3) group 49, 125
gauge theory 14, 48, 65, 73
generators 123
infinitesimal generators 124
irreducible representations 124

non-Abelian gauge theory 22
physical 117
symmetry 10
vacuum topology doubly connected 50
Optical-effects
Aharonov-Bohm (OAB) 41, 54, 56
conjugate product 1
equivalent of NMR and ESR 34
Faraday 61, 81
vacuum topology for OAB 42
Zeeman 61
Oscillating charge-current density 74

Particle mass
and Lorentz group 136
implications if zero 139
zero 28
Particles, physical, with mass 122
Pauli
equation 22, 29
matrices 4
spinor $\sigma^{(3)}$ 3, 7, 11
spinors, $\sigma^{(3)}$ 51
wave equation 30
Phase
angle 116
factor 120
shift, geometric 57
Photomagneton 27, 80, 115
Photon mass 29, 68, 135
and $B^{(3)}$ 122
and primordial magnetic field 172
and spacetime defect 166
finite 23, 48, 133, 142
in electromagnetic theory 135
non-zero 13, 98, 119, 172
Photon
absorption of 86
as particle 28, 119
coupled to torsion 165
electron collision 44
electron interaction 5
free 15
mass formula 167
massless, hypothetically 123
momentum, de Broglie 67
radius 66
rest frame 139, 141
spin 149
Thompson radius 66
Pilot field
and $B^{(3)}$ 146, 149, 151, 152
Planck 149
Pioneer 10 flyby of Jupiter 173
Planck
Einstein relation 64
hypothesis 84
pilot field 149
radiation law 85
Planck-Einstein condition 68
Poincaré group 24, 122, 124
Polarization
circular 83
elliptical 77
longitudinal state 24
Potential four-vector 100
Power density 2
Poynting
theorem for $B^{(3)}$ 112
vector 108, 112, 120
Principle of least action 3
Probe field 4
Proca equation 13, 138

Q.E.D 131
Quanta, of energy & momentum 67
Quantization, canonical 121, 122, 132
Quantum
electrodynamics (QED) 5
hypothesis 55
of energy 46
prescription 8
theory, axioms 79

Radiated $B^{(3)}$ 90, 94, 95, 101, 104, 105, 107-109, 111
Radiated potentials 106
Radiation 75
absence of $E^{(3)}$ 94
and angular momentum 95
and charge e 91
and dipole moments 147
angular energy density 114
angular momentum density of 112
applied to e in a source 94
as energy transfer 101
electric dipole 92
energy 71
energy density of 85
gauge 9, 15, 24
linear momentum density of 112

linearly polarized 77
microwave background, and $B^{(3)}$ 177
of $B^{(3)}$ 89
origin 73
oscillator 83
process 103
process of 74
range of 132
theory 74
theory, classical 89, 91
thermal, background 176
vacuum, angular momentum of 114
volume (V) 72
zone 75
adiative corrections 5
adiative power 94
adio frequency
field, circularly polarized 58
NMR 42, 54
adio-frequency
generator 5
NMR 38
or microwave generator 4
ayleigh Jeans and Planck laws 83
ayleigh-Jeans law 84
elativistic
cosmology 161
factor 69
field theory 100
Hamilton-Jacobi equation 46, 68
Lorentz force 102
particle theory 100
quantum theory 42
quantum wave equation 5
elict magnetic field, primordial 161
esolution enhancement 3
esonance
^{1}H (proton) 36
condition 35
equations 34
fermion 1
frequency 58
frequency, probe 36
nucleon 4
proton 3
spectroscopies 34
term 20
theory 21, 34
est frame
and standing wave 145
approximation 15
photon 139, 141
Riemann-Christoffel curvature tensor 128
Rotation group
symmetry 122
Rotation
generators 61, 73
group 10
Roy-Evans condition 25

S-matrix 122
Scalar potentials, radiated 106
Schrodinger wave equation 30
Solitons 49
Solution, fundamental 116
Solutions, stringlike 49
Space-time
and photon mass defect 166
and the Lorentz matrix 162
curved 160
flat 160
spinor, and the Lorentz matrix 162
Spin 122
density and Hubble's constant 172
density, background 172
intrinsic 25
intrinsic fermion 39
intrinsic, and torsion 161
population 38
Spinors 11
algebra 13
curvature 164
Pauli $\sigma^{(3)}$ 3, 7, 11, 51
space-time, and Lorentz matrix 162
term 33
two-spinors 10
Symmetry breaking, spontaneous 122

Tensor
and vierbein field 161
curvature 164
electromagnetic field 165
metric 170
torsion 164
Thompson
photon radius 66
radius 75, 110
scattering 176
Torque
density 62
density, electromagnetic 147
electromagnetic 158, 159
per unit length 158
quantum, $\hbar\lambda$ 159

Torsion 161
and conductivity coefficient 172
and defect density 164
and geometrical meaning 164
background, and curved space 164
non-zero, in a vacuum 161
potential 172
tensor 164
with photons 165
Two frequency problem 144, 154

U(1) group
flat theory 50
gauge theory 48
theory and four-potential A_μ 132
theory, inconsistencies of 132

Vacuum topology
non-trivial 54
Vacuum
with non-zero torsion 161
Vector potential
curl of 79
in vacuo, components 79
radiated 106
radiated, transverse 93
Vector
potential $\boldsymbol{A}$, in vacuo 61
row 11
Velocity, signal 100
Velocity, tangential 72
Vierbein field, and metric tensor 161
Volume, average 72
Vorticity
primordial 177
stable 49

Wave
d'Alembert equations 100
electromagnetic, plane 116
empty 142, 146
hypothesis 144
interfering fermion matter 52
nonhomogeneous equations 91
pilot 146
radiated plane 109
standing, in the rest frame 145
travelling solution 115
travelling, equation 115
travelling, initiating disturbance 105
Wave-particle
and three space dimensions 139
Wavelength 157
and energy 154, 155, 157
limit, long 105
Wien filter 56, 57
Wigner little group 119

Young experiment, and $B^{(3)}$ 149

Zeeman effect 32, 59, 61

Contents, Volume 1

PREFACE ix

1. WAVE AND PARTICLE 1
   - 1.1 The Enigma of Wave and Particle: Planck, Einstein, De Broglie 1
   - 1.2 Symmetry in Classical Electrodynamics, Wavenumber and Linear Momentum 6
   - 1.3 The Effect of $\boldsymbol{B}^{(3)}$ on the Fundamentals of the Old Quantum Theory 17

2. FUNDAMENTAL SYMMETRIES 21
   - 2.1 The Seven Discrete Symmetries of Nature 21
   - 2.2 The $\hat{C}\hat{P}\hat{T}$ Theorem 24
     - 2.2.1 $\hat{P}$ Conservation 25
     - 2.2.2 $\hat{T}$ Conservation 26
   - 2.3 The Concept of Photon and Anti-Photon 26
   - 2.4 Parity of the Photon and Anti-Photon 28
   - 2.5 Motion Reversal Symmetry of Photon and Anti-Photon 29
   - 2.6 The Photon's Charge Conjugation Symmetry, $\hat{C}$ 30
   - 2.7 Scheme for the Photon's Fundamental Symmetries, $\hat{C}$, $\hat{P}$ and $\hat{T}$ 30
   - 2.8 Symmetry of the Pure Imaginary Field $-i\boldsymbol{E}^{(3)}/c$ 33
   - 2.9 Experimental Demonstration of the Existence of the Anti-Photon 35

3. THE ORIGINS OF WAVE MECHANICS 37
   - 3.1 The Phase as Wave Function 39
   - 3.2 The Wave Mechanics of a Single Photon in Free Space 40
   - 3.3 Stationary States of One Photon in Free Space 42
   - 3.4 Heisenberg Uncertainty and the Single Photon 46