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The Magnetic Fields and Rotation Generators of Free Space Electromagnetism

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The relation is developed between rotation generators of the Lorentz group and the magnetic fields of free-space electromagnetism. Using these classical relations, it is shown that in the quantum field theory there exists a longitudinal photomagneton, a quantized magnetic flux density operator which is directly proportional to the photon spin angular momentum. Commutation relations are given in the quantum field between the longitudinal photomagneton and the usual transverse magnetic components of quantized electromagnetism. The longitudinal component is phase free, but the transverse components are phase dependent. All three components can magnetize material in general, but only the transverse components contribute to Planck's law. The photon therefore has three, not two, relativistically invariant degrees of polarization, an axial, longitudinal, polarization, and the usual right and left circular transverse polarizations. Since the longitudinal polarization is axial, it is a phase-free magnetic field.

1. INTRODUCTION

In conventional, classical, electrodynamics it is customary to consider only the transverse, oscillating, phase-dependent components of a travelling plane wave in free space. Transverse magnetic components can be written as two orthogonal electromagnetic modes or components in a complex, circular basis,[1–4]

$$\mathbf{B}^{(1)}=\frac{B^{(0)}}{\sqrt{2}}(i\mathbf{i}+\mathbf{j})\,e^{i\phi},\qquad \mathbf{B}^{(2)}=\frac{B^{(0)}}{\sqrt{2}}(-i\mathbf{i}+\mathbf{j})\,e^{-i\phi} \tag{1}$$

where $\phi=\omega t-\boldsymbol{\kappa}\cdot\mathbf{r}$ is the phase. Component (1) is the complex conjugate of component (2). Here $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the X and Y axes of the

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laboratory frame (X, Y, Z), mutually orthogonal to the propagation axis, Z, of the plane wave. Here ω is the angular frequency in radians per second at an instant of time, t, and $\boldsymbol{\kappa}$ the wavevector in inverse meters at a position $\mathbf{r}$ in the laboratory frame. Modes (1) and (2) are both solutions of the free-space Maxwell equations. Correspondingly, there are oscillating, transverse, electric fields, usually written as

$$\mathbf{E}^{(1)} = \frac{E^{(0)}}{\sqrt{2}} (\mathbf{i} - i\mathbf{j})\, e^{i\phi}, \qquad \mathbf{E}^{(2)} = \frac{E^{(0)}}{\sqrt{2}} (\mathbf{i} + i\mathbf{j})\, e^{-i\phi} \tag{2}$$

in the same notation. In Eqs. (1) and (2), $B^{(0)}$ is the scalar magnetic flux density amplitude, and $E^{(0)}$ the scalar electric field strength amplitude.

These fields constitute the well-known classical Maxwellian description of electrodynamics in free space, a description in which there are only two degrees of polarization, i.e., in a circular (or Cartesian) basis, one (longitudinal) degree of polarization is missing. This is easily seen by considering the following circular representation of three-dimensional space,

$$\mathbf{e}^{(1)} = \frac{1}{\sqrt{2}} (\mathbf{i} - i\mathbf{j}), \qquad \mathbf{e}^{(2)} = \frac{1}{\sqrt{2}} (\mathbf{i} + i\mathbf{j}), \qquad \mathbf{e}^{(1)} \times \mathbf{e}^{(2)} = i\mathbf{e}^{(3)} = i\mathbf{k} \tag{3}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are cartesian unit vectors, defined by

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \tag{4}$$

Therefore if $\mathbf{i}$ and $\mathbf{j}$ are polar, $\mathbf{k}$ is axial; if $\mathbf{i}$ and $\mathbf{j}$ are axial, $\mathbf{k}$ is also axial. In the representation of transverse electric fields [Eqs. (2)], $\mathbf{i}$ and $\mathbf{j}$ are polar; for transverse magnetic fields [Eqs. (1)], $\mathbf{i}$ and $\mathbf{j}$ are axial. The unit vectors $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, and $\mathbf{e}^{(3)}$ in the circular basis form the cyclical algebra,

$$\begin{aligned} \mathbf{e}^{(1)} \times \mathbf{e}^{(2)} &= i\mathbf{e}^{(3)*} = i\mathbf{e}^{(3)} \\ \mathbf{e}^{(2)} \times \mathbf{e}^{(3)} &= i\mathbf{e}^{(1)*} = i\mathbf{e}^{(2)} \\ \mathbf{e}^{(3)} \times \mathbf{e}^{(1)} &= i\mathbf{e}^{(2)*} = i\mathbf{e}^{(1)} \end{aligned} \tag{5}$$

so that if any one is zero, the other two also vanish. The basic electrodynamical notion that there can be only two degrees of field polarization in three-dimensional space is therefore geometrical nonsense. This simple illustration translates into well-known[5] fundamental difficulties in the theory of electromagnetism. In the required language of special relativity, the electromagnetic four-potential A_μ loses manifest covariance, only two out of its four components are physically meaningful, and since A_μ is known to be physically meaningful through the Bohm–Aharonov effect,[6] this is obviously not a satisfactory description.

In this paper these difficulties are resolved by the use of the longitudinal magnetic field $\mathbf{B}^{(3)}$, which is phase free and which contributes nothing to the Planck radiation law because its associated frequency is zero. For this reason it is referred to as the "ghost field" (*Gespensterstrahlung*) of electromagnetism. In the quantum field theory it is represented by the longitudinal "photomagneton," which is the operator $\hat{B}^{(3)}$, directly proportional to the well-known spin angular momentum of the photon, whose eigenvalues are $\hbar$ and $-\hbar$ if the photon is considered to have no mass, and $\hbar$, 0, and $-\hbar$ otherwise. The $\hat{B}^{(3)}$ photomagneton is simply $B^{(0)}$ multiplied into the normalized photon angular momentum operator, $\hat{J}/\hbar$. In Section 2, this result is derived from simple geometrical considerations. Evidence for the existence of $\hat{B}^{(3)}$ is available in the inverse Faraday effect[7–12] (frequency independent magnetization by light). If $\mathbf{B}^{(3)}$ were zero, then both $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ would vanish. Since $\mathbf{B}^{(3)}$, the expectation value of $\hat{B}^{(3)}$, is directly proportional through $B^{(0)}$ to unit angular momentum, represented by the axial unit vector $\mathbf{k}$, it is relativistically invariant in free space, a requirement of the Maxwell equations. In the quantum theory this is interpreted through the fact that photon spin angular momentum, $\hbar$, is also frequency independent, i.e., does not depend on ν, the frequency of the light. (In contrast, photon linear momentum, $h\nu/c$, is proportional to ν, and photon energy, as originally proposed in Ref. 13, is the quantum of light energy $h\nu$.) It follows therefore that $\hat{B}^{(3)}$ is not absorbed at any frequency, because in the quantum field theory, absorption depends on the quantum of energy $h\nu$, and $\hat{B}^{(3)}$ has no energy. The photomagneton $\hat{B}^{(3)}$ is therefore far more difficult to detect experimentally than the usual transverse fields, and has no effect, as we have argued, on Planck's law of radiation. It can however, be detected because as a phase-free magnetic field, it participates in the magnetization of matter by light—the well-known inverse Faraday effect which has hitherto been interpreted in terms of nonlinear optics. This phenomenon occurs *without optical absorption* in general, meaning that it can occur without the absorption of a quantum of energy $h\nu$, in other words at frequencies where the sample is transparent to light.[7–12]

In Section 2, the rigorous geometrical basis of $\hat{B}^{(3)}$ is developed using rotation operators in $O(3)$ and in the Lorentz group of Minkowski spacetime. It is shown there that the neglect of $\hat{B}^{(3)}$ implies that one rotation generator is erroneously asserted to be zero. In the quantum field theory this translates into the conclusion that one angular momentum is missing, this being the longitudinal angular momentum, and this is of course diametrically inconsistent with the basic assumption that the photon (considered as massless) have an ineluctable spin angular momentum, whose components in the *longitudinal* axis are $\hbar$ and $-\hbar$. Therefore, photon spin immediately implies the existence of $\hat{B}^{(3)}$.

In Section 3, the experimental basis for $\hat{B}^{(3)}$ and $\mathbf{B}^{(3)}$ is examined through the inverse Faraday effect. The main conclusion of this section is that if there were no $\hat{B}^{(3)}$, there would be no inverse Faraday effect, contrary to experimental data.[7–12]

In Section 4, the nature of the longitudinal, concomitant electric field in free space is examined. It is clear from symmetry[14] and special relativity that there can be no real $\mathbf{E}^{(3)}$, because Fitzgerald–Lorentz contraction reduces any longitudinal *polar* axis to zero for any object travelling at the speed of light. For a *massless* photon, this axis remains zero in any frame of reference, because the Maxwell equations in free space do not vary under Lorentz transformations. For a *massive* photon,[15] it becomes possible that there be, in the observer frame, an additional, phase-dependent, longitudinal magnetic field $\mathbf{B}^{(3)}$ and electric field $\mathbf{E}^{(3)}$. It is shown that these equations can be resolved in an internally consistent manner by deducing that $\mathbf{B}^{(3)}$ is accompanied by a pure imaginary $i\mathbf{E}^{(3)}$. This implies that the *combined* contribution of $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ to the Poynting theorem in free space is zero, this being the classical statement equivalent to the fact that, in the quantum field, $\hat{B}^{(3)}$ has no Planck energy because it has no frequency. (In Planck's postulate of 1900 energy is directly proportional to frequency, so that $\hat{B}^{(3)}$ corresponds to an oscillator state of zero frequency.[16])

The use of $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ makes the theory of electromagnetism in free space fully consistent and manifestly covariant.

2. GEOMETRICAL BASIS FOR $\hat{B}^{(3)}$

The first indication of the existence of $\hat{B}^{(3)}$ in free space appeared in Ref. 17 through its relation to a quantity known in nonlinear optics as the conjugate product, the vector cross product $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$. This quantity is also referred to in nonlinear optics as the antisymmetric, or vectorial, part of the light intensity tensor $\varepsilon_0 c E_i E_j$,[18] and therefore has a well-defined physical meaning. Any quantity to which $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is algebraically equal also has a well-defined physical meaning by tautology. From Eq. (2), the conjugate product is the pure imaginary, *longitudinal* quantity

$$\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = iE^{(0)2}\mathbf{k} \tag{6}$$

where $\mathbf{k}$ is a unit axial vector. The product $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is therefore magnetic in nature, with positive parity inversion ($\hat{P}$) symmetry and negative motion reversal ($\hat{T}$) symmetry.[19] Using the fundamental free space result,

$$E^{(0)} = cB^{(0)} \tag{7}$$

immediately gives the field $\mathbf{B}^{(3)}$,

$$\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = c^2 \mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = ic^2 B^{(0)} \mathbf{B}^{(3)} \tag{8}$$

with

$$\mathbf{B}^{(3)} = B^{(0)} \mathbf{k} \tag{9}$$

It is elementary to show, with these relations, that,

$$\begin{aligned} \mathbf{B}^{(1)} \times \mathbf{B}^{(2)} &= iB^{(0)} \mathbf{B}^{(3)*} = iB^{(0)} \mathbf{B}^{(3)} \\ \mathbf{B}^{(2)} \times \mathbf{B}^{(3)} &= iB^{(0)} \mathbf{B}^{(1)*} = iB^{(0)} \mathbf{B}^{(2)} \\ \mathbf{B}^{(3)} \times \mathbf{B}^{(1)} &= iB^{(0)} \mathbf{B}^{(2)*} = iB^{(0)} \mathbf{B}^{(1)} \end{aligned} \tag{10}$$

where the * denotes complex conjugation. *It is seen that there is a symmetric, cyclical algebra between the three magnetic field components in free space. This structure is that of the complex basis vectors* $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, *and* $\mathbf{e}^{(3)}$, *[Eqs. (5)].*

This is precisely what is expected if there is a three-dimensional, geometrical, relation between the tranverse and longitudinal components of solutions of Maxwell's equations in free space. The longitudinal component, $\mathbf{B}^{(3)}$, must be phase free, because of the Maxwellian condition

$$\nabla \cdot \mathbf{B} = 0 \tag{11}$$

If it is accepted that $B^{(0)}$ is nonzero (otherwise there is no electromagnetism), then, Eqs. (10) show that if $\mathbf{B}^{(3)}$ is zero, both $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ *vanish*. Conversely, if $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ are nonzero, then so is $\mathbf{B}^{(3)}$. This result once more emphasizes the fundamental inconsistency in the conventional approach,[1–4] in which $\mathbf{B}^{(3)}$ bears no relation to the transverse $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$, and in most texts is not considered. Equation (10) *shows that there is* a well-defined geometrical relation, which shows that $\mathbf{B}^{(3)}$ is physically meaningful. The converse of this result is that if it is asserted that $\mathbf{B}^{(3)}$ is zero, then the conjugate prouct vanishes, in contradiction with experimental data on the inverse Faraday effect,[7–12] and in contradiction to the fundamental theoretical structure of non-linear optics.[12–18]

The question now arises of the fundamental properties of $\mathbf{B}^{(3)}$ in the classical and quantum theories of fields. In this section it is shown using elementary tensorial methods that it is defined in the quantum field theory by the operator

$$\hat{B}^{(3)} = B^{(0)} \frac{\hat{J}}{\hbar} \tag{12}$$

where $\hat{J}$ is the photon angular momentum operator. For one (massless) photon the eigenvalues of $\hat{J}$ are $\pm\hbar$, meaning that the projections in the *longitudinal* (Z) axis are $+\hbar$ or $-\hbar$. In the quantum field theory, therefore, $\hat{B}^{(3)}$ depends on the existence of $\hat{J}$, because $B^{(0)}/\hbar$ is a constant of the electromagnetism for a given intensity. Realizing this, any attempt to assert that $\hat{B}^{(3)}$ is zero becomes inconsistent with the fundamentals of quantum mechanics, because such an assertion would imply that the photon spin angular momentum is zero. It is well accepted that the photon spin is an intrinsic, irremovable property,[20] and therefore so is $\hat{B}^{(3)}$.

The interpretation of $\hat{B}^{(3)}$, the longitudinal "photomagneton" of electromagnetism, is therefore simple in the quantum field theory—it is an operator generated directly from photon spin. The latter has eigenvalues *$\pm\hbar$ independent of the frequency (ν) and phase (ϕ) of the electromagnetic field.* Any attempt to understand the meaning of $\hat{B}^{(3)}$ must therefore be based on the meaning of $\hat{J}$. Similarly, the interaction of $\hat{B}^{(3)}$ with matter must be understood in the same way as that of $\hat{J}$ with matter. In particuar, there must be conservation of angular momentum—the total angular momentum before and after the interaction must be the same. To uderstand this, one needs the theory of angular momentum coupling in quantum mechanics.[20] Therefore $\hat{B}^{(3)}$ is obviously not a static magnetic field in any conventional sense (e.g., a field generated from a bar magnet or a solenoid). It is a novel property of light.

It is also obvious that the source of $\hat{B}^{(3)}$ in free space is the source of $\hat{J}$—usually thought of as a charge-current system at infinity, i.e., matter infinitely removed from free space. This is the same as the source of the usual transverse, oscillating, fields, which in the quantum field theory are thought of in terms of creation and annihilation operators. Therefore, *the existence of $\hat{B}^{(3)}$ does not require a separate source.* In the same way, $\hat{J}$, or photon spin, does not require a source in any way distinct or different from that of electromagnetism.

We arrive at the inescapable conclusion that if $\hat{J}$ be accepted, as usual, then so must $\hat{B}^{(3)}$.

The classical interpretation of $\mathbf{B}^{(3)}$, the expectation value of $\hat{B}^{(3)}$, depends on the cross product $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ of negative and positive frequency transverse modes (1) and (2). This must also be reflected in the quantum field theory, so Eq. (1) is more fully written in the accepted notation[12, 18] as

$$\hat{B}^{(3)}(0; -\omega, \omega) = B^{(0)} \frac{\hat{J}}{\hbar}(0; -\omega, \omega) \tag{13}$$

This shows that $\hat{J}$ itself is generated from a phase-free cross product of negative and positive frequency waves, i.e., from a particular combination of creation and annihilation operators.[12] This is the same combination as that which defines[12] the Stokes operator $\hat{S}_3$. The latter is well known to be an angular momentum operator, and the commutator relations between Stokes operators are the same as those of angular momentum operators in quantum mechanics. In this section, we shall develop commutator relations for $\hat{B}^{(1)}$, $\hat{B}^{(2)}$, and $\hat{B}^{(3)}$. These are also angular momentum commutator relations.

Any interaction of $\hat{B}^{(3)}$ with matter must therefore reflect its fundamental character, i.e., account for the fact that it is defined as

$$\hat{B}^{(3)} \equiv \hat{B}^{(3)}(0; -\omega, \omega) \tag{14}$$

Similarly,

$$\hat{J} \equiv \hat{J}(0; -\omega, \omega) \tag{15}$$

$$\hat{S}_3 \equiv \hat{S}_3(0; -\omega, \omega)$$

Our earlier description[17] of $\hat{B}^{(3)}$ as "static" obviously refers to the fact that it has no *net* (i.e., explicit) functional dependence on phase, $\phi = \omega t - \boldsymbol{\kappa} \cdot \mathbf{r}$. In precisely the same way, $\hat{J}$ has none, i.e., its eigenvalues are $\pm\hbar$, which are frequency-independent quantities. Similarly, the Stokes operator $\hat{S}_3$ and parameter S_3 have no net phase dependence. For a given beam intensity in circular polarization, S_3 is a constant of magnitude $\pm E^{(0)2}$; + for left, − for right circular polarization.

We shall return to the question of how $\hat{B}^{(3)}$ $(0; -\omega, \omega)$ interacts with matter in Section 3, when dealing with the inverse Faraday effect and time-dependent perturbation theory. In the remainder of this section, Eq. (12) is derived from first principles.

The first step is to put the cyclic relations between $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$, and $\mathbf{B}^{(3)}$ into *classical* commutative form by using the result from elementary tensor analysis[12, 18] that an axial rank-one vector is equivalent to a polar rank-two antisymmetric tensor,

$$B_i = \tfrac{1}{2}\varepsilon_{ijk}\hat{B}_{jk} \tag{16}$$

where ε_{ijk} is the rank-three, totally antisymmetric, unit tensor (the Levi-Civita symbol). The rank-two tensor representation of the magnetic field, $\hat{B}_{jk}$, is entirely equivalent to the usual rank-one vector B_1, but has the key advantage of being accessible to commutator algebra. This allows a straightforward transition to the quantum field theory through a factor $\hbar$. Commutator algebra also provides a means of expressing $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$, and $\mathbf{B}^{(3)}$ in terms of $O(3)$ rotation generators.[31] In so doing, these magnetic fields are related directly to quantum mechanical angular momentum operators, and have the same commutator properties. This was originally deduced[17] using creation and annihilation operators, an independent method.

The classical fields $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$, and $\mathbf{B}^{(3)}$ in free space are all axial vectors by definition, and it follows that their unit vector components must also be axial. In matrix form, they are, using tensor analysis of the type illustrated in Eq. (16),

$$\hat{\imath} \equiv \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \hat{\jmath} \equiv \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{k} \equiv \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{17}$$

It follows that the matrix representation of the unit vectors in the circular basis is

$$\hat{e}_1 \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{bmatrix}, \quad \hat{e}_2 \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ i & -1 & 0 \end{bmatrix}, \quad \hat{e}_3 \equiv \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{18}$$

and that these form a commutator Lie algebra which is mathematically equivalent to the vectorial Lie algebra (5),

$$\begin{aligned} [\hat{e}_1, \hat{e}_2] &= -i\hat{e}_3^* = -i\hat{e}_3 \\ [\hat{e}_2, \hat{e}_3] &= -i\hat{e}_1^* = -i\hat{e}_2 \\ [\hat{e}_3, \hat{e}_1] &= -i\hat{e}_2^* = -i\hat{e}_1 \end{aligned} \tag{19}$$

If it is arbitrarily asserted that $\hat{e}_3$ is zero, then both $\hat{e}_1$ and $\hat{e}_2$ vanish, i.e., the assertion is fundamentally inconsistent with three-dimensional geometry, expressed in a circular basis (5) rather than a Cartesian basis. Nevertheless, this meaningless assertion is the root of the conventional approach to electrodynamics, an approach which considers only transverse components of the plane wave in free space, and does not recognize that the transverse field components are linked to the longitudinal field component. This conclusion becomes clear if the geometrical (19) is used to describe the plane wave in vacuo,

$$\begin{aligned}\hat{B}^{(1)} &= iB^{(0)}\hat{e}^{(1)}e^{i\phi}\\ \hat{B}^{(2)} &= -iB^{(0)}\hat{e}^{(2)}e^{-i\phi}\\ \hat{B}^{(3)} &= B^{(0)}\hat{e}^{(3)}\end{aligned} \tag{20}$$

from which emerges the classical commutator relations between the three magnetic field components.

$$\begin{aligned}[\hat{B}^{(1)}, \hat{B}^{(2)}] &= -iB^{(0)}\hat{B}^{(3)*} = -iB^{(0)}\hat{B}^{(3)}\\ [\hat{B}^{(2)}, \hat{B}^{(3)}] &= -iB^{(0)}\hat{B}^{(1)*} = -iB^{(0)}\hat{B}^{(2)}\\ [\hat{B}^{(3)}, \hat{B}^{(1)}] &= -iB^{(0)}\hat{B}^{(2)*} = -iB^{(0)}\hat{B}^{(1)}\end{aligned} \tag{21}$$

This algebra can now be expressed in terms of the well-known[5, 21] rotation generators of $O(3)$ in three-dimensional space. These generators are complex matrices,[5, 21]

$$\begin{aligned}\hat{J}^{(1)} &= \frac{\hat{e}^{(1)}}{i} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & -i\\ -1 & i & 0\end{bmatrix}\\ \hat{J}^{(2)} &= \frac{-\hat{e}^{(2)}}{i} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & i\\ -1 & -i & 0\end{bmatrix}\\ \hat{J}^{(3)} &= \frac{\hat{e}^{(3)}}{i} = \begin{bmatrix} 0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\end{aligned} \tag{22}$$

providing the key link between the magnetic field matrices and rotation generators,

$$\begin{aligned}\hat{B}^{(1)} &= -B^{(0)}\hat{J}^{(1)}e^{i\phi}\\ \hat{B}^{(2)} &= -B^{(0)}\hat{J}^{(2)}e^{-i\phi}\\ \hat{B}^{(3)} &= iB^{(0)}\hat{J}^{(3)}\end{aligned} \tag{23}$$

This classical result shows that the commutator algebra of the magnetic fields (21) is part of the Lie algebra of the Lorentz group of Minkowski spacetime.[15–21] *It shows that the longitudinal component $\hat{B}^{(3)}$ is nonzero, because $\hat{J}^{(3)}$ is nonzero.* An assertion to the contrary means that if $\hat{B}^{(3)} = ?\,\hat{0}$, then $\hat{J}^{(3)} = ?\,\hat{0}$, which by reference to Eq. (22) is incorrect.

Furthermore, the generalization of the rotation generator from classical three-space [$O(3)$ group] to classical spacetime (Lorentz group) is well known[21] to be

$$
\begin{aligned}
\hat{J}^{(1)} = \hat{J}^{(2)*} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\hat{J}^{(2)} = \hat{J}^{(1)*} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\hat{J}^{(3)} = -\hat{J}^{(3)*} &= \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{24}
$$

It follows that $\hat{B}^{(1)}$, $\hat{B}^{(2)}$, and $\hat{B}^{(3)}$ can also be generalized in this way, and are also well-defined properties of spacetime in vacuo. This result is in turn consistent with the fact that all *three* magnetic components are well-defined solutions of Maxwell's equations, which in free space are invariant under Lorentz transformation.[1, 22] In this sense, $\hat{B}^{(1)}$, $\hat{B}^{(2)}$, and $\hat{B}^{(3)}$ are defined directly by the rotation generators of the Lorentz group, generators that form a Lie algebra in spacetime. By including $\hat{B}^{(3)}$ and not arbitrarily discarding it, as is the usual practice,[1–4] electrodynamics in spacetime actually becomes more fully self-consistent. For example, the Wigner little group[21] becomes O(3) instead of E(2): in other words three dimensional, not two dimensional. It is well known in field theory[21] that the Euclidean E(2) is physically meaningless, implying that classical electrodynamics is deeply flawed if the longitudinal field is not linked to the transverse fields as in this paper.

The fact that the approach that leads to E(2) is incorrect is seen through the fact that it leads to $\hat{J}^{(3)} = ?\,\hat{0}$. Uncritical acceptance of such an obviously incorrect result has become habitual because of the assumption that Maxwell's equations deal only with transverse field components in free space. Equation (10) shows that the longitudinal and transverse field components in free space are linked geometrically. This finding is tautological in nature, because Maxwell's equations are written in three, not two, space dimensions. The assertion $\hat{B}^{(3)} = ?\hat{0}$ not only makes nonsense out of Euclidean (and Minkowski) geometry, but also leads to difficulties throughout the gauge theory of electromagnetism, difficulties which are actually well known.[5] Many of these difficulties disappear when it is realized that $\hat{B}^{(3)}$ is a "spin field," which is phase free, and which therefore obeys the Maxwellian constraint $\nabla \cdot \mathbf{B} = 0$.

It is well known[5] that the rotation generators of $O(3)$ form a Lie algebra, part of the Lie algebra of the Lorentz group. In a circular basis (19), this becomes the following classical commutator algebra,

$$
\begin{aligned}
[\hat{J}^{(1)}, \hat{J}^{(2)}] &= -\hat{J}^{(3)*} = \hat{J}^{(3)} \\
[\hat{J}^{(2)}, \hat{J}^{(3)}] &= -\hat{J}^{(1)*} = -\hat{J}^{(2)} \\
[\hat{J}^{(3)}, \hat{J}^{(1)}] &= -\hat{J}^{(2)*} = -\hat{J}^{(1)}
\end{aligned}
\tag{25}
$$

which becomes

$$
\begin{aligned}
[\hat{J}_X, \hat{J}_Y] &= i\hat{J}_Z \\
[\hat{J}_Y, \hat{J}_Z] &= i\hat{J}_X \\
[\hat{J}_Z, \hat{J}_X] &= i\hat{J}_Y
\end{aligned}
\tag{26}
$$

in the cartesian basis, and which is, within a factor $\hbar$, *identical with the commutator algebra of angular momentum operators in quantum mechanics.* This result provides a simple rout to quantization of the magnetic fields of the plane wave in free space, giving the result

$$
\begin{aligned}
\hat{B}^{(1)} &= -B^{(0)} \frac{\hat{J}^{(1)}}{\hbar} e^{i\phi} \\
\hat{B}^{(2)} &= -B^{(0)} \frac{\hat{J}^{(2)}}{\hbar} e^{-i\phi} \\
\hat{B}^{(3)} &= iB^{(0)} \frac{\hat{J}^{(3)}}{\hbar}
\end{aligned}
\tag{27}
$$

where $\hat{B}^{(i)}$ are now field operators in quantum mechanics. In particular, the longitudinal operator $\hat{B}^{(3)}$ is the elementary quantum of longitudinal magnetic flux density, *the photomagneton* of electromagnetic radiation in free space. The photomagneton is the pilot wave of photon spin in the Einstein–de Broglie interpretation of the quantum theory of light. In the

Copenhagen interpretation, the quantized field operators $\hat{B}^{(1)}$, $\hat{B}^{(2)}$, and $\hat{B}^{(3)}$ form angular momentum commutators in free space,

$$
\begin{aligned}
[\hat{B}^{(1)}e^{-i\phi}, \hat{B}^{(2)}e^{i\phi}] &= -iB^{(0)}\hat{B}^{(3)} \\
[\hat{B}^{(2)}e^{i\phi}, \hat{B}^{(3)}] &= -iB^{(0)}\hat{B}^{(2)}e^{i\phi} \\
[\hat{B}^{(3)}, \hat{B}^{(1)}e^{-i\phi}] &= -iB^{(0)}\hat{B}^{(1)}e^{-i\phi}
\end{aligned}
\tag{28}
$$

The fields can now be thought of in terms of creation and annihilation operators[12, 17] as usual. In the Copenhagen interpretation, the three field components appearing in each commutator relation cannot be specified simultaneously.[20] If $\hat{B}^{(3)}$ is specified, then the transverse components remain unspecified, as in any angular momentum commutator relation in quantum mechanics. This is consistent with the fact that the (3) (i.e., Z) component of photon angular momentum is usually specified as eigenvalues, *longitudinal* projections $\hbar$ and $-\hbar$, and that for a massless photon travelling at the speed of light, the transverse angular momentum components are mathematically indeterminate. The longitudinal component of angular momentum in an object travelling at c is relativistically invariant. Therefore $\hat{B}^{(3)}$ in free space is also relativistically invariant. This must be so because it is a solution of Maxwell's equations, which are also relativistically invariant in free space. The specification of $\hat{B}^{(3)}$ in terms of photon spin is therefore fully consistent with relativistic quantum field theory.

Therefore, $\hat{B}^{(3)}$ is a constant of motion,[12, 20] while $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ are governed by photon statistics and are subject to purely quantum effects such as light squeezing.[12] The field $\hat{B}^{(3)}$, being defined by photon spin, is not subject to light squeezing effects. In other words, photon spin itself is not affected by light squeezing, and its eigenvalues remain a constant $\hbar$ and $-\hbar$. If the photon is massive,[23] the eigenvalues become $\hbar$, 0, and $-\hbar$. The constancy of the field $\hat{B}^{(3)}$ is consistent with the fact that in quantum mechanics the general expression for the rate of change of an expectation value is[20]

$$\frac{d}{dt}\langle\hat{B}^{(3)}\rangle = \frac{i}{\hbar}\langle[\hat{H}, \hat{B}^{(3)}]\rangle \tag{29}$$

so that $\hat{B}^{(3)}$ commutes with the Hamiltonian $\hat{H}$. This is consistent with the fact that $\hat{B}^{(3)}$ has no Planck energy and does not contribute to classical electromagnetic energy density.[24] The expectation value of $\hat{B}^{(3)}$, being a constant of motion, is independent of time, and its eigenvalues are specified as the constant $\hbar$ and $-\hbar$. Similarly, the Stokes operator $\hat{S}_3$ is a constant of motion, so $\hat{B}^{(3)}$ is proportional to $\hat{S}_3$.[17] Therefore the photomagneton

$\hat{B}^{(3)}$ conserves angular momentum in free space, and this is a consequence of the isotropy of the Hamiltonian in free space,[20] and therefore a consequence of three-dimensional symmetry.[20]

This conclusion is simply a way of saying that the spin of the massless photon is $\pm\hbar$, and that the photomagneton $\hat{B}^{(3)}$ is a direct consequence of photon spin. The classical $\mathbf{B}^{(3)}$ is therefore a direct consequence of the fact that there exists right and left circular polarization in electromagnetic radiation. This is an expression of Eq. (10) in words.

The expectation values of $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$, on the other hand, are not constants of motion, and do not commute with the Hamiltonian. This is consistent with the fact that $\mathbf{B}^{(1)}$ or $\mathbf{B}^{(2)}$ contribute to classical electromagnetic energy density in free space. Similarly, the expectation values of $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$ are not constants of motion, and remain unspecified in the Copenhagen interpretation if $\hat{J}^{(3)}$ is specified. Such a result is consistent with special relativity, which deduces that the transverse classical angular momenta of an object travelling at c are indeterminate in the observer frame and that the longitudinal component is relativistically invariant. In other words, in special relativity,

$$J_Z = J'_Z, \qquad J_Y = \gamma J'_Y, \qquad J_X = \gamma J'_X \tag{30}$$

where $\gamma = (1 - v_Z^2/c^2)^{-1/2}$. We see that if the relative velocity of two frames, v_z, is c then J_Y and J_X (in the static, observer frame) become infinite unless $J_{X'} = J_{Y'} = 0$. In this condition, J_X and J_Y are mathematically indeterminate but $J_Z = J_{Z'}$ is well defined. This is what is indicated by the Copenhagen interpretation of Eqs. (28): the field $\hat{B}^{(3)}$ has specified, relativistically invariant, eigenvalues which are projections in the longitudinal axis of the propagating plane wave in free space. The transverse fields $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ are not constants of motion and are not specified if $\hat{B}^{(3)}$ is specified. It is well known that $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ are subject to quantum effects such as light squeezing, which are a consequence of the Heisenberg uncertainty principle applied to photons.

If we compare directly the classical and quantum equations,

$$\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = iB^{(0)}\mathbf{B}^{(3)} \tag{31a}$$

$$[\hat{B}^{(1)}, \hat{B}^{(2)}] = -iB^{(0)}\hat{B}^{(3)} \tag{31b}$$

it becomes immediately obvious that Eq. (31a) is *a relation between spins in the* Maxwellian interpretation. Each spin component (1), (2), and (3) is formed from a vector cross product of the other two, this being a requirement of Euclidean geometry. In order for this geometrical requirement to simultaneously satisfy Maxwell's equation $\nabla \cdot \mathbf{B} = 0$, the longitudinal component $\mathbf{B}^{(3)}$ *must be phase free*, otherwise its divergence is nonzero because the phase has a Z dependence. In order to satisfy this and the other three Maxwell equations, the transverse components $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ must be phase dependent. Equations (10) tie these considerations together in a circular basis, in the same way that rotation generators are tied together.

At the fundamental level, Eq. (13) shows that photon spin *itself* is nonlinear in nature, being an angular momentum. In the quantum field theory this has eigenvalues $\hbar$ and $-\hbar$. In Maxwellian electrodynamics the classical equivalent to photon spin in obtained from the *nonlinear* conjugate product, $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ (or $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$), *which removes the phase* ϕ. In this way the classical conjugate product and quantized photon spin are linked together in a relation which clarifies the physical meaning of both.

In the Copenhagen interpretation, the Heisenberg uncertainty principle applied to Eq. (31b) shows that

$$\delta \hat{B}^{(1)} \, \delta \hat{B}^{(2)} \geqslant \tfrac{1}{2} \, |B^{(0)} \hat{B}^{(3)}| \tag{32}$$

where $\delta \hat{B}^{(1)}$ and $\delta \hat{B}^{(2)}$ are root mean square deviations.[20] As usual,[20] the right-hand side is a rigorous lower bound on the product $\delta \hat{B}^{(1)} \, \delta \hat{B}^{(2)}$, *a lower bound which is therefore defined by the photomagneton* $\hat{B}^{(3)}$. If $\hat{B}^{(3)}$ were zero, $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ would commute, implying that $\delta \hat{B}^{(1)} = \hat{0}$ and $\delta \hat{B}^{(2)} = \hat{0}$ simultaneously. *The experimental observation of light squeezing*[12] *shows that this is inconsistent with data*; *therefore* $\hat{B}^{(3)} \neq \hat{0}$. In this sense, light squeezing indicates experimentally the existence of the photomagneton $\hat{B}^{(3)}$.

In the next section, other experimental indications of the existence of $\hat{B}^{(3)}$ are discussed.

3. DETECTION OF $\hat{B}^{(3)}$ IN THE INVERSE FARADAY EFFECT

In addressing the experimental effects of $\hat{B}^{(3)}$, the question arises of whether its interaction with matter (e.g., an electron) can be treated with time-dependent or time-independent perturbation theory. Magnetization by circularly polarized light, the inverse Faraday effect, can occur *without absorption*, as observed experimentally by van der Ziel *et al.*[7] However, it can also occur *with* absorption, as shown theoretically by Woźniak, Evans, and Wagnière.[25] It seems reasonable to assert that if there is no absorption, there is no transfer of photon energy, $h\nu$, and so the effect is frequency independent, meaning that in this limit, time-independent perturbation theory applies. In this limit there is transfer of angular momentum from the light to the sample, but no transfer of energy, so that the phenomenon of magnetization is in this sense "*elastic.*" Since $\hat{B}^{(3)} = B^{(0)}\hat{J}/\hbar$ and since $B^{(0)}$ is proportional to the square root of beam intensity I_0 (watts per unit area), it seems likely that such an effect is proportional to $I_0^{1/2}$.

However, the fundamental angular momentum $\hat{J}(0; -\omega, \omega)$ needs two modes for its definition (one, (1), negative frequency; the other, (2), positive frequency) so it follows that the interaction of $\hat{B}^{(3)}$ with matter also requires the consideration of $-\omega$ and ω, even though $\hat{B}^{(3)}$ and $\hat{J}$ do not explicitly depend on frequency. (The eigenvalues of both depend on $\hbar$, which has no frequency dependence.) Therefore the interaction of $\hat{B}^{(3)}(0; -\omega, \omega)$ with matter requires, in general, time-dependent perturbation theory, based on the time-dependent Schrödinger equation. Since

$$\mathbf{B}^{(1)}(-\omega) \times \mathbf{B}^{(2)}(\omega) = iB^{(0)}\mathbf{B}^{(3)}(0; -\omega, \omega) \tag{33}$$

any effect due to $\mathbf{B}^{(3)}(0; -\omega, \omega)$ needs the simultaneous action of both modes (1) and (2). In time-dependent perturbation theory, this property must be accounted for in the molecular property tensor with which $\mathbf{B}^{(3)}(0; -\omega, \omega)$ interacts at any order in the field. For example, the inverse Faraday effect is described by Woźniak, Evans, and Wagnière[25] in terms of $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$. In a three-level atomic system the paramagnetic contribution to magnetism, $M(0)$, by circularly polarized light is given by

$$\mathbf{M}(0) = \frac{-iNc^2}{3\hbar kT}\left(\rho_1(\omega)A + \rho_2(\omega)B\right)\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} \tag{34}$$

which is Eq. (30) of Ref. 25 written in our notation. Here

$$A = \mathbf{m}_{aa} \cdot \mathrm{Im}(\boldsymbol{\mu}_{a1} \times \boldsymbol{\mu}_{1a}), \qquad B = \mathbf{m}_{aa} \cdot \mathrm{Im}(\boldsymbol{\mu}_{a2} \times \boldsymbol{\mu}_{2a}) \tag{35}$$

and

$$\rho_1(\omega) = \frac{\omega(\omega_1^2 - \omega^2 - \Gamma_1^2)}{(\omega_1^2 - \omega^2 + \Gamma_1^2)^2 + 4\omega^2\Gamma_1^2}, \qquad \rho_2(\omega) = \frac{\omega(\omega_2^2 - \omega^2 - \Gamma_2^2)}{(\omega_2^2 - \omega^2 + \Gamma_2^2)^2 + 4\omega^2\Gamma_2^2} \tag{36}$$

where ω_1 and ω_2 are resonance frequencies, $\mathbf{m}_{aa}$ is a ground state permanent magnetic dipole moment, and $\boldsymbol{\mu}_{a1}$, etc. are transition electric dipole moments. The transition to time-independent perturbation theory is given by setting the resonance frequencies ω_1 and ω_2 and damping factors Γ_1 and Γ_2 to zero, giving the result

$$\mathbf{M}(0) = -\frac{Nc^2}{3kT\hbar\omega}\mathbf{m}_{aa} \cdot \mathrm{Im}(\boldsymbol{\mu}_{a1} \times \boldsymbol{\mu}_{a2})\, B^{(0)}\mathbf{B}^{(3)} \tag{37}$$

Both equations (34) and (37) are to second order in the magnitude, $B^{(0)}$, of $\mathbf{B}^{(3)}$. However, Eq. (34) represents a transfer of energy (at the resonance frequencies ω_1 and ω_2) as well as a transfer of angular momentum from $\mathbf{B}^{(3)}$. Equation (37) represents a transfer of angular momentum only, because there is no resonance. Therefore Eq. (34) is an "inelastic" process, and Eq. (37) an "elastic" process. Both equations are semiclassical, in that the field is treated classically and the molecular property quantum mechanically.

Clearly, if $\mathbf{B}^{(3)}$ were zero, there would be no inverse Faraday effect of any kind.

The question now arises as to whether $\mathbf{B}^{(3)}$, having all the attributes of magnetic flux density, can act at first order, so that there is an inverse Faraday effect proportional to the square root of intensity in addition to process (34) or (37), which are both proportional to intensity. The time average of $\mathbf{B}^{(3)}$ is nonzero, because it has no phase dependence, *and this suggests that it can magnetize at first order.* If so, then there should be a component of the inverse Faraday effect proportional to the square root of intensity. This interesting possibility should be checked by further experimental work on the intensity dependence of the inverse Faraday effect, whose standard interpretation is at second order, in $B^{(0)}\mathbf{B}^{(3)}$, as we have seen. The oscillating components $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ also magnetize at first order, but the time-averaged magnetization vanishes. Any *energy* transfer process from the electromagnetic field to matter is second order, however, in the electric field strength or magnetic flux density of the field, but angular momentum transfer is first order in these quantities. For example, when a quantum of energy, $h\nu$ in the quantum theory, is transferred to an atom, there is a simultaneous transfer of angular momentum ($\hbar$) per photon, $h\nu$, a process which is governed by angular momentum selection rules.[20] Since $\hbar$, the magnitude of the photon's angular momentum, is energy divided by angular frequency, it has units proportional to $\mathbf{B}^{(3)}$ *at first order.* Therefore, it follows that $\mathbf{B}^{(3)}$ can produce pure, first-order, magnetization *only if there is no transfer of photon energy $h\nu$ to the sample.* (Otherwise the overall process is a mixture of first- and second-order effects.) In other words, any purely first-order process in $\mathbf{B}^{(3)}$ cannot be accompanied by absorption of light, because absorption is second order in $\mathbf{B}^{(3)}$. The molecular property tensor through which $\mathbf{B}^{(3)}$ produces magnetization must therefore be a susceptibility tensor calculated in the limit of time-independent perturbation theory, in which there are no optical resonances.

If further experimental work shows that there is a process in the inverse Faraday effect proportional to the square root of circularly polarized light intensity, then it would have been shown that $\mathbf{B}^{(3)}$ can act as a first-order magnetic field. If the data show no sign of such a process, however, *it would be incorrect to conclude therefrom that* $\mathbf{B}^{(3)} = ?\, 0$. As discussed in Section 2, such a possibility is excluded on several counts. The absence of a first-order process in $\mathbf{B}^{(3)}$ might mean that two modes (negative and positive ω) are needed to define the classical equivalent of photon spin angular momentum, $\mathbf{J}(0; -\omega, \omega)$, so that $\mathbf{J}$ *itself* is intrinsically nonlinear, and for this reason cannot act without the combined action of two electromagnetic modes, (1) and (2). The inverse Faraday effect has been observed *experimentally* in the absence of absorption, meaning that there is no transfer of $h\nu$ from radiation to sample, yet the sample is magnetized.[17] Therefore, the two modes making up $\hat{J}$ can act on a sample without transferring any photons $h\nu$, and the inverse Faraday effect is obviously not an absorption phenomenon. It is therefore confusing to allude to it as a "two-photon" process, because that would imply the absorption of two photons. Since $\hat{J}$ is nonzero and directly proportional to $\hat{B}^{(3)}$, the latter also depends on the simultaneous action of two modes, (1) and (2). The angular momentum, and $\hat{B}^{(3)}$, do not depend on frequency, however, and have no Planck energy. Any assertion that $\hat{B}^{(3)}$ is zero, however, is geometrically incorrect. The question is whether $\hat{B}^{(3)}$ can act at first order or not, and further experimental work is needed to clarify this point. In Ref. 17, the interpretation of the inverse Faraday effect is discussed in more detail. In diamagnetics, effects at first order in $\hat{B}^{(3)}$ are prohibited by the fact that the sample has no permanent magnetic dipole moment. In paramagnetics, such as the doped glasses used by van der Ziel *et. al.*,[7] effects at first order in $\hat{B}^{(3)}$ are allowed in principle, provided that the symmetry of the sample allows a net permanent magnetic dipole moment. Data are not available at present to test these matters further. A recent reinterpretation[26] of the results of Frey *et al.*[27] on the optical Faraday effect showed a *square root* intensity dependence of the light-induced Faraday rotation, which is a sign, albeit tenuous, that $\hat{B}^{(3)}$ is able to act at first order. It is tenuous because there were ony six data points available,[26] and these did not go through the origin. Also, the pump laser used by Frey *et al.*[27] was not circularly polarized before entering the intense magnetic field used in their experiment. However, it develops an excess of circular polarization through the ordinary Faraday effect when it passes through the magnetic field, producing a nonzero $\hat{B}^{(3)}$.

We emphasize that the question of whether $\hat{B}^{(3)}$ can act at first order is secondary to that of the existence of $\hat{B}^{(3)}$, which is proven unequivocally by the data of van der Ziel *et al.*[7] and by the arguments of Section 2 of this paper.

The simplest example of the inverse Faraday effect *without absorption* is when circularly polarized light interacts with one electron. This problem was first discussed by Talin *et al.*,[28] from which it is straightforward to show[29] that the magnetic dipole moment induced, without absorption, in one electron by circularly polarized light is

$$\mathbf{m} = \frac{-e^3c^2}{2m_0^2\omega^3} B^{(0)}\mathbf{B}^{(3)} \tag{38}$$

Here e and m_0 are the electronic charge and mass, and ω is the angular frequency of the light. It is therefore a simple matter to show that there is no elementary, one-electron, inverse Faraday effect if $\mathbf{B}^{(3)} = ?\,0$. Equation (38) is second order in the magnitude, $B^{(0)}$, of $\mathbf{B}^{(3)}$. If we assert[29] that $\mathbf{B}^{(3)}$ can act at first order, the effect becomes

$$\mathbf{m} = -\frac{e^2r_0^2}{2m_0}\mathbf{B}^{(3)} - \frac{e^3c^2}{2m_0^2\omega^3} B^{(0)}\mathbf{B}^{(3)} \tag{39}$$

where r_0 is an orbital electron radius. For ω of about 10^{15} rad sec^{-1}, and for a first-order electron radius of about 10 Å, the orders of magnitude for a beam intensity of about 10^{14} watt m^{-2} become

$$|\mathbf{m}| \sim -10^{-26}\,|\mathbf{B}^{(3)}| - 10^{-25}B^{(3)2} \tag{40}$$

and the second-order effect is roughly ten times larger. Under other conditions, the first-order effect may of course predominate. Again there are no data available to test these hypotheses. These data would require the careful measurement and analysis of the intensity dependence of the inverse Faraday effect in a suitable electron plasma.[8]

4. THE LONGITUDINAL ELECTRIC FIELD, $i\mathbf{E}^{(3)}$

The existence of a $\mathbf{B}^{(3)}$ appears to imply at first sight that there must be a concomitant longitudinal electric field from Maxwell's questions. However, such a field has never been detected experimentally, and no large, first-order, polarization effects of light have been observed to date. The only known polarizing effect of light is optical rectification,[30] a small, second-order process. There are several factors that point toward the fact that there is no real, (i.e., physical) electric field $\mathbf{E}^{(3)}$, but that there is an *imaginary* $i\mathbf{E}^{(3)}$.

1. In special relativity,[31] the square of the complex vector $c\mathbf{B}^{(3)} + i\mathbf{E}^{(3)}$,

$$(c\mathbf{B}^{(3)} + i\mathbf{E}^{(3)})^2 = c^2B^{(3)2} - E^{(3)2} + 2ci\mathbf{B}^{(3)}\cdot\mathbf{E}^{(3)} \tag{41}$$

is a Lorentz invariant. The real parts of the two independent invariants, $c^2B^{(3)2}-E^{(3)2}$ and $2ci\mathbf{B}^{(3)}\cdot\mathbf{E}^{(3)}$, are both zero. The first is the contribution of $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ to the free-space electromagnetic energy density, and the second is pure imaginary. They are well known[31] to be the two independent invariants of the electromagnetic four-tensor, $F_{\mu\nu}$ allows for the existence of longitudinal field components in free space. These are customarily asserted to be zero with the use of a suitable gauge.

2. The fact that

$$U^{(3)}=\frac{1}{\mu_0}\mathbf{B}^{(3)}\cdot\mathbf{B}^{(3)}+\varepsilon_0 i\mathbf{E}^{(3)}\cdot i\mathbf{E}^{(3)}=0 \tag{42}$$

in free space is consistent with the fact that $\hat{B}^{(3)}$ is directly proportional to the spin angular momentum of the photon, which has no Planck energy. It is not possible to assert that any real electric field be proportional to any kind of real angular momentum, because of $\hat{P}$ and $\hat{T}$ symmetries.[19] The imaginary $i\mathbf{E}^{(3)}$ therefore comes from fundamental special relativity, whose well-known dual transformation[31] converts a pure real magnetic field to a pure imaginary electric field without changing the Maxwell equations in free space.

3. The Maxwell equations in free space are satisfied by $\mathbf{B}^{(3)}$ accompanied by $i\mathbf{E}^{(3)}$ because both are phase free and therefore time independent and uniform in classical electrodynamics. If we write $\mathbf{B}^{(3)}$ in terms of a vector potential

$$\mathbf{B}^{(3)}=\nabla\times\mathbf{A}^{(3)} \tag{43}$$

and attempt to write a real $\mathbf{E}^{(3)}$ in terms of the scalar and vector potentials,

$$\mathbf{E}^{(3)}=?-\nabla\phi^{(3)}-\frac{\partial A^{(3)}}{\partial t} \tag{44}$$

it is found that this leads to $\mathbf{E}^{(3)}=\mathbf{0}$ in Maxwellian electrodynamics. Therefore if $\mathbf{B}^{(3)}$ is real, the real $\mathbf{E}^{(3)}$ vanishes.

4. The joint contribution of $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ to $U^{(3)}$ in free space is zero. The Poynting theorem then asserts that the vector

$$\mathbf{N}=\frac{1}{\mu_0}i\mathbf{E}^{(3)}\times\mathbf{B}^{(3)}=\mathbf{0} \tag{45}$$

is zero. This is consistent with the fact that $\mathbf{B}^{(3)}$ is parallel to $i\mathbf{E}^{(3)}$ in the propagation axis. It follows that,[26] if $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are asserted to be in general complex,

$$\begin{aligned} \mathbf{E}^{(1)} \times \mathbf{B}^{(3)} &= \mathbf{E}^{(3)} \times \mathbf{B}^{(1)} \\ \mathbf{E}^{(2)} \times \mathbf{B}^{(3)} &= \mathbf{E}^{(3)*} \times \mathbf{B}^{(2)} \end{aligned} \tag{46}$$

from which if $\mathbf{B}^{(3)}$ is real, then $i\mathbf{E}^{(3)}$ must be pure imaginary.

5. An attempt to construct for an assumed *real* $\mathbf{E}^{(3)}$ a cyclic algebra akin to (10) results in $\hat{T}$ violation.[32] This is consistent with the fact that a real longitudinal electric field would have to be a polar vector, which cannot, on basic geometrical grounds, be constructred from the cross product of two mutually orthogonal polar or axial vectors. In other words, any cross product such as $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ or $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ must produce an *axial* vector. Cross products such as $\mathbf{E}^{(1)} \times \mathbf{B}^{(2)}$ (those defining the Poynting vector) produce a polar vector which is $\hat{T}$ negative. Any real, physical electric field must be $\hat{T}$ positive, and therefore cannot be produced from Poynting-type cross products.

For these reasons $\mathbf{B}^{(3)}$ is accompanied, for consistency in classical Maxwellian electrodynamics, by $i\mathbf{E}^{(3)}$, a conclusion with emerges from special relativity. Dual transformation produces $i\mathbf{E}^{(3)}$ from $\mathbf{B}^{(3)}$ and vice versa, as required, and these two components (one magnetic, physical, and real, the other electric, unphysical, and imaginary) take their place in the electromagnetic four-tensor $F_{\mu\nu}$. Classical electrodynamics is therefore rendered more fully self-consistent by their inclusion. $\mathbf{B}^{(3)}$ produces physical effects, $i\mathbf{E}^{(3)}$ produces no physical effects.

5. CONCLUSIONS

There is experimental evidence[7–12] for the fact that the product $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ is not zero. Expressing this product as $iB^{(3)}\mathbf{B}^{(3)}$ shows that the real, longitudinal, phase-free $\mathbf{B}^{(3)}$ is not zero in free space, a deduction which is supported on geometrical grounds in Section 2. Using these methods, it has been shown that in the quantum field theory, $\hat{B}^{(3)}$ is proportional to the photon spin angular momentum operator, $\hat{J}$. The source of $\hat{B}^{(3)}$ is therefore the same as that of photon spin, and $\hat{B}^{(3)}$ is able to propagate in free space with photon spin. The operator $\hat{B}^{(3)}$ has no Planck energy because the eigenvalues of $\hat{J}$ for one photon are $\hbar$ and $-\hbar$, which are independent of frequency. Classically, the joint contribution of $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ to free-space electromagnetic energy density is zero. Therefore $\mathbf{B}^{(3)}$ (and $\hat{B}^{(3)}$) is not absorbed in field–matter interaction, and does not contribute to the Planck radiation law. We reach the fundamental conclusion that the photon has *three* degrees of polarization, and that this does not conflict with the Planck law. The same conclusion is reached from the hypothesis[23, 32] that the photon have mass, however tiny in magnitude. The Maxwellian $\mathbf{B}^{(3)}$ can therefore be regarded as the "zero mass limiting" form of a more general theory, in which the photon is non-zero. For several reasons, it is concluded that $\mathbf{B}^{(3)}$ is accompanied in free space by an imaginary $i\mathbf{E}^{(3)}$, which produces no polarization. If $\mathbf{B}^{(3)}$ is asserted to be zero, the inverse Faraday effect disappears, in conflict with experimental results.[7–12] It appears plausible that $\mathbf{B}^{(3)}$ act at first order as well as at second order in the inverse Faraday effect, and other related magneto-optic effects.

APPENDIX. CYCLICAL ALGEBRA INVOLVING ELECTRIC FIELDS

There are symmetric cyclical relations of type (10) which involve electric fields and which can also related to rotation and boost generators of the Lorentz group. In three dimensions there is, for example, the algebra

$$\begin{aligned}\mathbf{E}^{(1)}\times\mathbf{E}^{(2)}&=-E^{(0)}(ic\mathbf{B}^{(3)})^*\\ \mathbf{E}^{(2)}\times(ic\mathbf{B}^{(3)})&=-E^{(0)}\mathbf{E}^{(1)*}\\ (ic\mathbf{B}^{(3)})\times\mathbf{E}^{(1)}&=-E^{(0)}\mathbf{E}^{(2)*}\end{aligned}\tag{A1}$$

which becomes a relation between boost and rotation generators when we come to consider the four dimensions of the Lorentz group. The electric fields are proportional to the boost generators, and the magnetic fields to the rotation generators.

In order to derive a perfectly cyclical algebra involving electric fields only, we first not that the existence in special relativity of the complex vector $c\mathbf{B}^{(3)}+i\mathbf{E}^{(3)}$ means that the symmetry of the imaginary $i\mathbf{E}^{(3)}$ can be regarded as magnetic, i.e., regarded as the same as that of real $c\mathbf{B}^{(3)}$. This of course means that $i\mathbf{E}^{(3)}$ is not a real electric field. The square of the complex vector $c\mathbf{B}^{(3)}+i\mathbf{E}^{(3)}$ gives Lorentz invariants as in the text. With this realization, the following cyclical algebra can be written,

$$\begin{aligned}\mathbf{E}^{(1)}\times\mathbf{E}^{(2)}&=-E^{(0)}(i\mathbf{E}^{(3)})^*\\ \mathbf{E}^{(2)}\times(i\mathbf{E}^{(3)})&=-E^{(0)}\mathbf{E}^{(1)*}\\ (i\mathbf{E}^{(3)})\times\mathbf{E}^{(1)}&=-E^{(0)}\mathbf{E}^{(2)*}\end{aligned}\tag{A2}$$

which in spacetime is an algebra involving boost generators of the Lorentz group,

$$[\hat{K}^{(1)}, \hat{K}^{(2)}] = -\hat{J}^{(3)*}$$
$$[\hat{K}^{(2)}, \hat{J}^{(3)}] = -\hat{K}^{(1)*} \qquad \text{(A3)}$$
$$[\hat{J}^{(3)}, \hat{K}^{(1)}] = -\hat{K}^{(2)*}$$

Here, the boost generators in the circular basis are

$$\hat{K}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \end{bmatrix}, \qquad \hat{K}^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 \end{bmatrix},$$

$$\hat{K}^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \text{(A4)}$$

The boost and rotation generators are related to the electric and magnetic fields of the Lorentz group by

$$\hat{E}^{(1)} = E^{(0)}\hat{K}^{(1)}e^{i\phi}, \quad \hat{E}^{(2)} = E^{(0)}\hat{K}^{(2)}e^{-i\phi}, \quad \hat{E}^{(3)} = E^{(0)}\hat{K}^{(3)}$$
$$\hat{B}^{(1)} = -B^{(0)}\hat{J}^{(1)}e^{i\phi}, \quad \hat{B}^{(2)} = -B^{(0)}\hat{J}^{(2)}e^{-i\phi}, \quad \hat{B}^{(3)} = iB^{(0)}\hat{J}^{(3)} \qquad \text{(A5)}$$

Finally, the cyclically symmetric algebra of the Lorentz group is completed by the relations

$$[\hat{K}^{(1)}, \hat{J}^{(1)}] = 0$$
$$[\hat{K}^{(2)}, \hat{J}^{(2)}] = 0 \qquad \text{(A6)}$$
$$[\hat{K}^{(3)}, \hat{J}^{(3)}] = 0$$

Therefore, in the space part of the Lorentz group, the complete set of fields are $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$, and $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(1)}$, $\mathbf{E}^{(2)}$, and $i\mathbf{E}^{(3)}$. These components, expressed in the circular basis (1), (2), and (3), take their place in the antisymmetric four-matrix $F_{\mu\nu}$, the four-curl of A_μ in spacetime. The fields $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ are therefore generators of the Lorentz group in free space, and also in the presence of matter.

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