Chapter 13

OPTICAL NMR AS A SHIELDING PHENOMENON

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Abstract

Optical NMR spectroscopy is a technique which has been demonstrated experimentally and theoretically. Laser irradiation of a sample in an NMR spectrometer produces chemical shifts which are different for each resonating nucleus in a molecule. It is shown that optical NMR and ESR spectroscopy can be interpreted generally as a shielding phenomenon, in which the applied laser generates an effective shielding constant $\sigma^{(ind)}$. The calculation of $\sigma^{(ind)}$ involves several different mechanisms of induction in general, and $\sigma^{(ind)}$ is a function of the molecular structure, varying from chromophore to chromophore or atom to atom. In principle therefore, ONMR spectroscopy provides a uniquely new fingerprint of any sample, because the laser induced shielding is unique to that sample.

1. Introduction

The theory of magnetization by electromagnetic radiation was first developed in terms of molecular property tensors by Piekara and Kielich [1–5] and was first demonstrated by Pershan et al. [6-8] and Shen [9] as the inverse Faraday effect. Recently, it has been proposed that circularly polarized electromagnetic radiation can shift NMR and ESR resonances [10], and several mechanisms have been developed theoretically [11–15]. Evidence for the effect has been provided by Warren et al. [16] in a variety of samples, for H and D nuclear magnetic resonances in one and two dimensions, using CW laser irradiation far from optical resonance. These shifts were found to be site specific, thus providing an individual fingerprint of the sample. Further evidence for the magnetizing ability of light has been provided recently as the optical Faraday effect [17, 18] in magnetic semiconductors. These effects have been rationalized by the present author [19–23] in terms of the photon's magnetostatic flux density operator $\hat{B}^{(3)}$, a fundamental property of the photon.

Section 2 of this paper shows that optical NMR spectroscopy can be thought of quite generally as a shielding phenomenon, whereby laser irradiation induces an electronic magnetic dipole moment which can be interpreted as a light induced shielding factor $\sigma^{(ind)}$. The effective permanent magnetic field seen by the resonating nucleus is therefore changed by the laser from the original $\boldsymbol{B_s}$ to $\boldsymbol{B_s}(1+\sigma^{(ind)})$. In general

$$\Delta\hat{H} = -\left(\hat{m}_{Ni} + \hat{m}_i^{(ind)}\right)B_{Si} + \ldots, \quad (1)$$

where $\hat{m}_i^{(ind)}$ is the light induced magnetic electronic dipole moment, and $\hat{m}_{Ni}$ is the nuclear magnetic dipole moment associated with the original NMR resonance line. The induced magnetic dipole moment $\hat{m}_i^{(ind)}$ is proportional in general to a number of molecular property tensors, such as magnetic electronic susceptibility and various electronic hyperpolarizabilities. It is well established [24–26] that these molecular property tensors depend on electronic distribution within the molecule, for example, there exist well documented molecular polarizability anisotropies. The susceptibility and hyperpolarizability for a given group or atom or chromophore within the molecule varies from site to site, and this variation can be computed ab initio [24]. The shielding factor $\sigma^{(ind)}$ is therefore different from site to electronic site and the applied field $\boldsymbol{B_s}$ is shielded differently by the laser for each resonating nucleus. The effect of the laser is site specific.

This result is generally valid, and agrees with the experimental findings of Warren et al. [16] that laser induced NMR shifts are different in nature for each resonating nucleus as a function of the applied laser intensity and frequency. ONMR spectra are therefore unique spectral signatures for a given sample, and the technique is therefore potentially generally useful.

Parts of the complete molecular tensor for the molecule are locally specific, i.e. the nature of the light perturbation of the nuclear resonance mediated by the electronic molecular property tensor depends on the immediately adjacent electronic topography of the resonating nucleus. In other words the light perturbation is also site specific as found experimentally [16]. In Sec. 3 the various mechanisms present in a chiral molecule such as that used by Warren et al. are examined in more detail using considerations of motion reversal ($\hat{T}$) and parity inversion ($\hat{P}$) symmetry to separate out the various contributions.

Finally, in Sec. 4, the quantum mechanical nature of the problem is developed using Griffith coefficients to reinforce the conclusion in Sec. 2 that the laser induced shifts of the original NMR features are site specific.

2. The Light Induced Shielding Effect

Consider a laser induced magnetic electronic dipole moment:

$$\hat{m}_i^{(ind)} = \hat{\chi}'_{ij}B_{\Pi j} + .{}^{m}\beta^{ee}_{ijk}E_jE_k^* + .{}^{m}\beta^{me}_{ijk}B_jE_k^* + .{}^{m}\beta^{mm}_{ijk}B_jB_k^* + \ldots + \textit{any other mechanism}, \quad (2)$$

where we have included several of many possible mechanisms developed rigorously by the Poznań School [25, 26]. We have also allowed for the possibility of other mechanisms of light induction. In Eq. (2), there are various contributing molecular property tensors such as susceptibility χ'_{ij}, and the various hyperpolarizabilities $\boldsymbol{\beta}_{ijk}$. The latter premultiply the appropriate tensor products [25, 26] of the usual oscillating electric and magnetic field components of the electromagnetic plane wave. The superscript * denotes complex conjugation [25, 26] so that the phase is removed [11–15] in the tensor products. The real magnetic field $B_i^{(3)}$ is the classical equivalent of the photon's magnetostatic flux quantum [19–23] and is defined in vector notation by

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{E}\times\boldsymbol{E}^*}{2E_0ci}, \tag{3}$$

where E_0 is the electric field strength amplitude of the plane wave and c the speed of light. Note that each molecular property tensor appearing in Eq. (2) is complex in general [25], and each can be developed with time dependent perturbation theory, which shows that each is frequency dependent and shows optical resonances. The real and imaginary parts of each molecular property tensor has uniquely defined $\hat{P}$ and $\hat{T}$ symmetry.

There may also be present other mechanisms of induction of a magnetic dipole moment by the laser, as discussed by Warren *et al.* [16] for an H atom model. These are allowed for in the term "any other mechanism" in Eq. (2). The complete expression in Eq. (2) is meant therefore to encompass all possible mechanisms of induction of a magnetic electronic dipole moment by laser irradiation. In general each molecular property tensor is an operator in semi-classical theory [25], and this is noted in Eq. (2) by the symbol ^ over each tensor.

With these preliminaries the effect of the laser on the simplest possible NMR Hamiltonian operator is

$$\Delta\hat{H} = -\left(\hat{m}_{Ni} + \hat{m}_i^{(ind)}\right)B_{Si}, \tag{4}$$

where $\hat{m}_N$ is the nuclear magnetic dipole moment operator of the resonating nucleus and $\boldsymbol{B}_s$ the externally applied homogeneous and uniform magnetostatic field of the NMR instrument. Now, let

$$\sigma^{(ind)} \equiv \frac{\langle\hat{m}_i^{(ind)}\rangle}{\langle\hat{m}_{Ni}\rangle}, \tag{5}$$

where $\sigma^{(ind)}$ is the dimensionless light induced shielding factor. Equation (4)

becomes

$$\Delta\hat{H} = -\hat{m}_{Ni}\left((1+\sigma^{(ind)})B_{Si}\right), \tag{6}$$

which shows that the applied irradiation in ONMR can be treated generally as a shielding phenomenon, irrespective of the details of the individual mechanisms in Eq. (2) and of the various interactions that can occur between mechanisms, such as Landé coupling [11–15, 19–23]. Moreover, the order of magnitude of $\sigma^{(ind)}$ can be estimated simply, given the order of magnitude of the mediating molecular property tensor. For example, the order of magnitude of ${}^m\boldsymbol{\beta}_{ijk}^{ee}$ is $\sim 10^{-45}\,A\,m^4\,V^{-2}$. in small diamagnetic molecules, and can be estimated [27, 28] independently from the Faraday effect. The light intensity I_0 (watts per square meter) can be related to the square of E_0 through the free space equation

$$I_0 = \epsilon_0 c E_0^2, \tag{7}$$

where ϵ_0 is the free space permittivity in S.I. Therefore, for the mechanism ${}^m\boldsymbol{\beta}_{ijk}^{ee}E_jE_k^*$ of Eq. (2), the shielding factor $\sigma^{(ind)}$ for a laser of 100 watts per square centimeter intensity and for $|{}^m\boldsymbol{\beta}_{ijk}^{ee}|$ of $\sim 10^{-45}\,A\,m^4\,V^{-2}$ is about one part in 10^8, i.e. about 1.0 Hertz in 100 MHz. In other words the light shifts an original NMR resonance line at 100 MHz by the order of 1.0 Hz, and this is the order of magnitude observed experimentally [16], with intensities as low as 1.0 watt cm^{-2}.

More generally, ALL the various mechanisms in Eq. (2) contribute simultaneously to the light induced resonance shift, and the various terms may contribute additively or may cancel each other to some extent. The induced magnetic dipole moment is also dependent in general on the polarization of the light. Some mechanisms exist in circular polarization only [11–15] and vanish in linear polarization. In some circumstances, for example in a paramagnetic molecule with an odd number of electrons, there is a residual contribution in linear polarization [14], but the experimental results of Warren *et al.* [16] show clearly that the shifts are much larger in circular polarization than in linear polarization, and do not exhibit a simple dependence on circular polarization. Equation (2) allows for this by employing a combination of several different valid mechanisms. The biggest effect is expected to be from $\boldsymbol{B}^{(3)}$ [19], and this vanishes in linear polarization. However, $\boldsymbol{B}^{(3)}$ changes sign with circular polarization [19–23], and this is only partially observed experimentally [16], indicating the existence of other mechanisms. In general, the induced magnetic dipole moment is assumed to arise from all possible contributions.

It is also well established that in conventional NMR spectroscopy, in the absence of light irradiation, there exists a shielding factor σ such that the field at the nucleus is shifted from the original $\boldsymbol{B_s}$ to $\boldsymbol{B_s}(1-\sigma)$ [29]. This is the well known chemical shift, and in general σ is of the order parts per million, i.e. 10^{-6}. In the presence of the laser, the complete shielding effect is therefore

$$\Delta \hat{H} = -\hat{m}_{Ni}(1+\sigma^{(ind)})(1-\sigma)B_{Si} \doteq -\hat{m}_{Ni}B_{Si}(1-\sigma+\sigma^{(ind)}), \tag{8}$$

and it is clear without the need for further detailed calculation that the laser changes the site specificity, or chemical shift characteristics, of the original NMR spectrum, giving a unique ONMR fingerprint for each sample. ONMR spectroscopy is therefore a generally useful addendum to ONMR in N dimensions. This result is valid for all types of NMR, and also for all types of ESR.

Technically, the challenge is to maximize the laser intensity, and thus the light induced chemical shift, without damaging the sample, and this can be tackled in many different ways, for example by using a timing device which synchronizes a laser pulse with the usual radio frequency pulse (pi or pi/2 for example). This has the effect of maximizing $\sigma^{(ind)}$ at the same time, or approximately the same time, as resonance occurs in an r.f. pulse, i.e. as the nuclear spins are flipped by resonance with the rotating r.f. field pulse produced by a coil in the usual way [29] in Fourier transform NMR spectroscopy. Probably, the shortest available r.f. pulse (ca. microseconds) will have to be much longer than the longest effective laser pulse (ca. nanoseconds) in contemporary technology, but even so, if the laser pulse or pulse train is timed to occur in the center of the r.f pulse interval, the laser pulse will produce a change in resonance frequencies through $\sigma^{(ind)}$. This will produce laser pulse induced oscillations in the free induction decay function [29–31], measured over a much longer induction decay interval as usual, oscillations which upon Fourier transformation and suitable data reduction in a computer gives the ONMR spectrum. According to the mechanisms in Eq. (2), the more intense the laser, the greater the shielding effect $\sigma^{(ind)}$, and the more useful will be the ONMR spectrum. Pulsed lasers can deliver intensities of the order 10^9 those of CW lasers, and theoretically this can produce ONMR spectra of very high effective resolution provided that laser inhomogeneity is not found to be a problem. So far [16] laser inhomogeneity has NOT been found to affect the ONMR spectra, in which CW laser induced shifts have been clearly resolved.

3. Symmetry Considerations in the Molecular Property Tensors

In the simplest case the original semi-classical interaction Hamiltonian operator is described by $-\hat{m}\cdot\boldsymbol{B_s}$, where $\hat{m}_N$ is the nuclear magnetic dipole moment operator and $\boldsymbol{B_s}$ the permanent static magnetic field of the NMR instrument. Many ingenious and interesting developments have been made in contemporary NMR [29–31] but for our initial purposes it is sufficient to consider the perturbation caused

by light of this simplest type of nuclear Hamiltonian operator. Several perturbation mechanisms have been considered already and order of magnitude estimates made [11–15]. Considering the overall simplicity of the theory, (restricted to atoms) and the complexity of the chiral molecule chosen [16] for experimental investigation, fairly satisfactory agreement has been obtained between the order of magnitude estimates and limited light induced shift data available. In molecules with no net electronic angular momentum, there are perturbations of the type $-\hat{m}_N\cdot\hat{B}^{(3)}$, where both $\hat{m}$ *and* $\hat{B}^{(3)}$ are quantized operators in general, which in semi-classical approximation becomes $-\hat{m}_N\cdot\boldsymbol{B}^{(3)}$; of the type ${}^m\beta_{ijk}^{ee}B_{Si}\Pi_{jk}^{(A)}$, where $\Pi_{jk}^{(A)}$ is proportional to the antisymmetric part [21] of the light intensity tensor of circularly polarized radiation and ${}^m\beta_{ijk}^{ee}$ is a hyperpolarizability tensor of rank three, and Fermi contact mechanisms [12] in which light induced electronic magnetic dipole moments set up a magnetic field at the nucleus. Warren *et al*. have proposed a mechanism [16] in which the rotating electric field of the electromagnetic plane wave induces a polarization current which sets up an extra magnetic field at the nucleus. This is in satisfactory order of magnitude agreement with the observed shifts, but the calculation was carried out in atomic H, whereas the data were obtained for the H nuclear resonances of a fairly large chiral molecule, p methoxy phenylimino camphor. In atoms and molecules with net electronic angular momentum there are additional perturbation mechanisms mediated by the antisymmetric polarizability [11–15] α''_{ij} which can be reduced to an axial vector with the same symmetry as the magnetic electronic dipole moment. The tensor α''_{ij} forms an interaction Hamiltonian with the conjugate product $\boldsymbol{E}\times\boldsymbol{E}^*$, which is proportional to the novel vector $\boldsymbol{B}^{(3)}$, the classical equivalent of the photon operator $\hat{B}^{(3)}$.

Whatever the level of complexity of the detailed calculation of the light induced magnetic dipole moment, the basic mechanism of ONMR is always describable by the site specific shielding effect summarized very simply in the coefficient $\sigma^{(ind)}$. Chemical shifts of the conventional type, described by σ, are used widely for analysis by NMR, and in precise analogy, chemical shifts of the type described by $\sigma^{(ind)}$ potentially have the same utility, provided $\sigma^{(ind)}$ can be made large enough by technical ingenuity. Thus, ONMR has the clear potential advantage of extra information provided by a combination of two types of shielding, $\sigma^{(ind)}$ and σ compared with only one, described by σ, in conventional NMR. Clearly, this advantage is present potentially in all types of ONMR and OESR that technical ingenuity can devise, because the shielding factors σ and $\sigma^{(ind)}$ are always present.

It is also possible to by-pass complexity of calculation by a careful consideration of the symmetry and related properties of the molecular property tensors appearing on the right hand side of Eq. (2). Such a consideration reveals without further calculation the expected behavior of the spectrum as a function of the polarization, for example, of the incoming light, and the expected overall behavior of the ONMR spectrum with, for example, right and left enantiomers and their racemic mixture.

Table 1 summarizes the symmetry of the various molecular property tensors appearing in Eq. (2) under application of the motion reversal operator, $\hat{T}$, and the parity inversion operator, $\hat{P}$. Similarly, Table 2 summarizes the properties of the field variables of the incoming light beam as a function of circular polarization of the beam. (In linear polarization there is an equal amount of right and left circular polarization.)

It is clear from Tables 1 and 2 that the shift factor is the result of a combination of molecular property tensors, some of which are $\hat{P}$ negative and are non-zero only in enantiomers, changing sign between enantiomers and disappearing in the racemic mixture. It is this combination of terms that produces the individual ONMR fingerprint for each nuclear resonance.

With the use of symmetry it is possible to establish as follows which individual molecular property tensors MAY contribute in which polarization of the applied laser.

3.1. $\hat{T}$ Symmetry

For an atom or molecule containing an even number of electrons the expectation value of any $\hat{T}$ negative molecular property tensor operator vanishes [32]. Therefore all $\hat{T}$ negative operators in Table 1 vanish unless there is an odd number of electrons in the atom or molecule, i.e. unless there is a net electronic angular momentum. Thus, there is no permanent magnetic electronic dipole moment unless this condition is satisfied. Similarly, the real part of ${}^{m}\beta^{ee}_{ijk}$ for example vanishes in a molecule with an even number of electrons, but the imaginary part, being $\hat{T}$ positive, is non-zero in general. This rule is independent of the state of circular polarization of the applied laser radiation.

3.2 $\hat{P}$ Symmetry

If a property tensor is $\hat{P}$ negative, then it vanishes in an achiral molecule and in atoms. $\hat{P}$ positive property tensors are non-zero in general in all atoms and molecules if the rule on $\hat{T}$ symmetry is satisfied. In chiral molecules all the property tensors contribute to the shielding factor $\sigma^{(ind)}$ in general, and those properties that are $\hat{P}$ negative change sign between enantiomers. This change of sign does not occur however in the accompanying $\hat{P}$ positive property tensors, so that $\sigma^{(ind)}$ does not change to $-\sigma^{(ind)}$ between enantiomers because it is made up of a sum of terms, the majority of which (the $\hat{P}$ positive terms) remain the same from one enantiomer to the other. Qualitatively, this is what is observed experimentally [16], the

Table I. Some Property Tensor Symmetries

Molecular Property Tensor	$\hat{T}$	$\hat{P}$
$\hat{m}_i^{(ind)}$	-	+
$\hat{\chi}'_{ij}$	+	+
${}^{m}\beta^{ee}_{ijk}{}'$	-	+
${}^{m}\beta^{ee}_{ijk}{}'$	+	+
${}^{m}\beta^{me}_{ijk}{}'$	-	-
${}^{m}\beta^{me}_{ijk}{}'$	+	-
${}^{m}\beta^{mm}_{ijk}{}'$	-	+
${}^{m}\beta^{mm}_{ijk}{}'$	+	+

Table II. Polarization Properties of the Field Tensors of Light

Field Property	Right	Left	Linear
$B_{\Pi j}$	+	-	0
$\frac{1}{2}(E_jE_k^* + E_kE_j^*)$	+	+	+
$\frac{1}{2}(E_jE_k^* - E_kE_j^*)$	+	-	0
$\frac{1}{2}(B_jE_k^* + B_kE_j^*)$	+	-	0
$\frac{1}{2}(B_jE_k^* - B_kE_j^*)$	+	+	+
$\frac{1}{2}(B_jB_k^* + B_kB_j^*)$	+	+	+
$\frac{1}{2}(B_jB_k^* - B_kB_j^*)$	+	-	0

observed laser induced shifts change, but do not reverse, from one enantiomer to the other, and remain finite in the racemic mixture. In the latter the $\hat{P}$ negative property tensors sum to zero, but the $\hat{P}$ positive property tensors remain non-zero.

3.3 Polarization of the Laser

The shift factor $\sigma^{(ind)}$ is determined by products of the molecular property tensors with tensors of the laser field. In Table 2 the properties of the field tensors are given in terms of left and right circular polarization of the laser's plane wave. In general a field tensor can be split into symmetric and antisymmetric parts; for example,

$$E_iE_j^* = \frac{1}{2}\left(E_iE_j^* + E_jE_i^*\right) + \frac{1}{2}\left(E_iE_j^* - E_jE_i^*\right), \tag{9}$$

whose $\hat{T}$ symmetries are opposite [11–15]. The laser's magnetostatic field $\boldsymbol{B}^{(3)}$ [19–23] is defined by

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{E}\times\boldsymbol{E}^*}{(2E_0ci)}, \tag{10}$$

and the vector product $\boldsymbol{E}\times\boldsymbol{E}^*$ changes sign from left to right circular polarization. It is purely imaginary because $E_iE_j^*$ is hermitian [33], and is $\hat{T}$ negative. It vanishes in linear polarization. The vector $\boldsymbol{E}\times\boldsymbol{E}^*$ in tensor notation is formed from the product

$$\boldsymbol{E}\times\boldsymbol{E}^* \equiv \frac{1}{2}\epsilon_{ijk}E_jE_k^*, \tag{11}$$

where ϵ_{ijk} is the Levi Civita symbol and where $\boldsymbol{E}\times\boldsymbol{E}^*$ is the antisymmetric part of $E_iE_j^*$ defined in Table 2. The symmetric part of $E_iE_j^*$ is on the other hand $\hat{T}$ positive, and does not change sign from left to right circular polarization. The component $\boldsymbol{E}\cdot\boldsymbol{E}^*$ is finite, in consequence, in linearly polarized radiation. Similarly, the symmetric part of $B_iE_j^*$ is imaginary, $\hat{T}$ positive, and changes sign with circular polarization, vanishing in linear polarization; the antisymmetric part of $B_iE_j^*$ is real, $\hat{T}$ negative, and does not change sign with circular polarization, being finite, in consequence, in linear polarization.

Careful considerations of symmetry therefore lead to the conclusion that the state of polarization of the incoming laser beam has an effect on the factor

$\sigma^{(ind)}$ and therefore on the ONMR spectrum. This is again borne out qualitatively by the available experimental data [16]. It should be noted that reversing the polarization from right to left is NOT expected on the grounds of symmetry simply to reverse the sign of $\sigma^{(ind)}$, because such a reversal produces sign changes in some field tensors (Table 2) but not in others. Experimentally [16] the observed shifts are much larger in circular polarization than in linear, and do not simply reverse from right to left circular polarization. This can be interpreted again in terms of the fact that $\sigma^{(ind)}$ in general is made up of a large number of contributing mechanisms. It appears that the dominant mechanism is that in $\boldsymbol{B}^{(3)}$ [19–23], which vanishes in linear polarization as described already.

In summary there is a combination of mechanisms, ONMR shifts are allowed by Eq. (2) in all atoms and molecules in general, and the factor $\sigma^{(ind)}$ is not expected to simply change sign from right to left enantiomer or right to left circular polarization. In general, the results for right enantiomer, right circular polarization, right-left; left-left; and left-right; are expected to be all different in general, and again, this is in qualitative agreement with the experimental data [16].

4. Some Mechanistic Details

The expectation value of the Hamiltonian operator (2) must be evaluated in general between eigenstates which combine nuclear and electronic angular momentum states, using the theory of angular momentum coupling [19–23]. This is because the laser causes a perturbation in the electronic structure of a molecule or atom, a perturbation which in ONMR is picked up through nuclear resonances. Some specific mechanistic details have been evaluated recently [19–23] for simple model systems such as atoms. The development uses the standard algebraic methods of the quantum mechanics of multiple angular momentum systems, using symbolic machinery such as the n-j symbols and the theory of irreducible tensorial sets. In molecular point groups the n-j symbols are replaced by the Gruffydd V, W, and X coefficients [34], and there is an algebra of angular momentum coupling in molecular point groups based on the properties of these coefficients. However, this is formal and symbolic, and it appears that there is no easily obtainable code for the reduction of the V, W, and X coefficients to numbers, as is the case for the 3-j, 6-j, and 9-j symbols in atoms [35]. The V, W, and X coefficients were worked out originally for a very limited number of point groups of high symmetry. It is clear therefore that computation is the only practical way of rigorously evaluating the shielding factor $\sigma^{(ind)}$ for anything bigger than an atom. The preferred method would be *ab initio* computation with high quality basis sets [25], but more approximate methods could be used initially. The shielding factor σ of conventional NMR also is not accessible in general with algebraic methods. Clearly, this does not prevent conventional NMR from being a very useful analytical technique, and the same conclusion applies for the novel shielding factor $\sigma^{(ind)}$ introduced in this paper. In other words, the analytical laboratory uses spectra usually on a semi-empirical basis, rarely is the full NMR spectrum computed rigorously *ab initio*, the experimental spectrum is obviously used in a direct way to identify the sample. The same applies, clearly, to ONMR, which has the advantage of providing fingerprints of the shielding factor $\sigma^{(ind)}$ as a function of extra experimental variables such as laser intensity, polarization state, and frequency, *at least three new dimensions* for the data bank.

As examples of the way in which $\Delta\hat{H}$ is developed theoretically the following mechanisms are considered in this section in an atom with net electronic angular momentum.

Consider the well known hyperfine coupling Hamiltonian operator in an atom with an odd number of electrons,

$$\Delta\hat{H} = -(\gamma_e\hat{S} + \gamma_N\hat{I})B_Z + \lambda\hat{S}\cdot\hat{I}, \tag{12}$$

which is among the simplest ways of dealing with the combined effect of electronic and nuclear angular momenta. It is well known [36, 37] that the quantum theory of angular momentum coupling produces, in the coupled approximation, the following expectation value for this operator,

$$\Delta H = \langle SIFM_F|\Delta\hat{H}|SIFM_F\rangle = -\frac{\hbar}{2}\gamma_e M_F B_Z \times\left(\frac{F(F+1)+S(S+1)-I(I+1)}{F(F+1)} + \frac{\gamma_N}{\gamma_e}\left(\frac{F(F+1)+I(I+1)-S(S+1)}{F(F+1)}\right)\right) + \lambda\langle SIFM_F|\hat{S}\cdot\hat{I}|SIFM_F\rangle . \tag{13}$$

In this expression there appears the quantum number F, which takes the values

$$F = S+I, \ldots, |S-I|, \tag{14}$$

with the azimuthal components defined by

$$M_F = M_S + M_I . \tag{15}$$

Therefore, there are several expectation values of the interaction Hamiltonian operator, the spectrum is split into components by angular momentum coupling between the electronic and nuclear spin angular momenta $\hat{S}$ and $\hat{I}$. Here γ_e is the electronic and γ_N is the nuclear gyromagnetic ratio, and $\hbar$ is the reduced Planck constant. λ is the spin spin coupling constant. The expectation value

of the operator product can be worked out as usual in coupled or decoupled form [36, 37]. In the former,

$$\lambda\langle SIFM_F|\hat{S}\cdot\hat{I}|SIFM_F\rangle = \lambda\frac{\hbar^2}{2}(F(F+1) - S(S+1) - I(I+1)), \tag{16}$$

and F takes the values in Eq. (14). In the decoupled form it is well known that λ can be worked out through the Fermi contact and dipole dipole approximations, and we shall return to this calculation later.

In order to work out one of the effects of applied electromagnetic radiation on the expectation value of the hyperfine interaction Hamiltonian operator (12) it is convenient to adapt the results of Manakov et al. [38], who considered atomic systems with net electronic angular momentum. They showed that to order two in the electric field strength $\boldsymbol{E}$ of the electromagnetic field,

$$\Delta H_{SM_S} = -\frac{1}{4}A_{SM_S}|\boldsymbol{E}|^2 + \ldots, \tag{17}$$

where A_{SM_S} is made up of irreducible spherical representations of the electronic polarizability, with scalar, vector and rank two tensor components. The scalar and tensor components describe the optical Stark and optical quadratic Stark effects respectively, and the component in vectorial, or antisymmetric, polarizability describes the optical Zeeman effect in atoms. The existence of an optical Zeeman effect was independently proposed by Evans [39–42] in molecules with net electronic angular momentum. The optical Zeeman effect vanishes in linear polarization, and is maximized when the light beam is fully circularly polarized. The two optical Stark effects are finite in contrast in linear polarization.

It is convenient to develop the effect on the hyperfine Hamiltonian operator (12) of the component of A_{SM_S} responsible for the optical Zeeman effect, a component which can be written as

$$A_{SM_s} = \frac{\alpha_1}{\sqrt{2}}\frac{M_S}{\sqrt{S(S+1)}}, \tag{18}$$

where α_1 is the reduced matrix element

$$\alpha_1 = \langle S\|\hat{\alpha}_0^1\|S\rangle, \tag{19}$$

of the vectorial polarizability operator $\hat{\alpha}_0^1$ [38]. Since $\hat{\alpha}_0^1$ is a $\hat{T}$ negative, $\hat{P}$ positive, rank one irreducible tensor in spherical representation, it is directly proportional to the angular momentum operator through the gyroptic ratio [40] γ_{Π}

$$\hat{\alpha}_0^1 = \gamma_{\Pi}\hat{S}_0^1 , \tag{20}$$

$$\gamma_{\Pi} = \gamma_e\left(\frac{\langle\hat{\alpha}_0^1\rangle}{\langle\hat{m}_0^1\rangle}\right) \sim 10^{-6}\ c^3m^2J^{-2}kgm^{-1}T^{-1}. \tag{21}$$

The effect of ΔH_{SM_v} on the interaction Hamiltonian operator (12) can therefore be written as

$$\Delta\hat{H}_1 = -\gamma_e(1+x)B_Z\hat{S}_0^1 - \gamma_N\hat{I}_0^1B_Z + \lambda'\hat{S}_0^1\cdot\hat{I}_0^1 , \tag{22}$$

where the factor x is

$$x = \frac{\gamma_{\Pi}I_0}{4\sqrt{2}\epsilon_0 c\gamma_e B_Z} \sim 10^{-14}\frac{I_0}{B_Z} . \tag{23}$$

Equation (23) shows that the effect of the circularly polarized light beam is to increase the electronic gyromagnetic ratio from γ_e to $\gamma_e(1+x)$. For an intensity of $I_0 = 10^6$ watts m^{-2} the gyromagnetic ratio is changed by one part in 10^8 for a magnetic field B_Z of 1.0 T.

The expectation value of the Hamiltonian operator $\Delta\hat{H}$ is therefore changed by the circularly polarized light beam in such a way that γ_e in Eq. (12) is replaced by $\gamma_e(1+x)$, and the spin spin coupling constant is changed from λ to a value λ' to be determined as follows.

1. In the well known Fermi contact mechanism [36], the expression for spin spin coupling becomes

$$\lambda'\hat{S}\cdot\hat{I} = -\frac{2}{3}\mu_0\gamma_e(1+x)\gamma_N\delta(\boldsymbol{r_N})\hat{S}\cdot\hat{I}, \tag{24}$$

where μ_0 is the magnetic permeability of free space.

2. In the uncoupled approximation the dipole dipole mechanism for spin spin coupling gives the expression,

$$\lambda'\hat{S}\cdot\hat{I} = \frac{\mu_0}{4\pi r^3}\gamma_e(1+x)\gamma_N(1-3\cos^2\theta)S_ZI_Z, \tag{25}$$

in which $\boldsymbol{r}$ is the distance between dipoles, and θ is an orientational factor [36].

In both expressions the effect of the laser can be expressed in the mechanism considered here by changing the electronic gyromagnetic ratio from γ_e to $\gamma_e(1+x)$, thus giving a convenient estimate of the expected shifts in the spin spin splitting spectrum through either mechanism. Note that x depends on the antisymmetric polarizability $\hat{\alpha}_0^1$ which changes according to the state of the atom, as shown by Manakov et alia [38]. Near optical resonances of the atom, $\hat{\alpha}_0^1$ increases by orders of magnitude, and therefore so does x. The laser affects the original Hamiltonian (12) in several different ways therefore in this atomic model. Resonance occurs in ONMR when

$$\hbar\omega_R = \Delta H(M_F-1) - \Delta H(M_F). \tag{26}$$

where ω_R is the radio frequency from a rotating field generated by a coil as usual. Note that the resonance condition is described in terms of the F quantum number, which is made up of combinations of the S and I quantum numbers. Therefore there are several resonance frequencies in general, all of which are affected by the factor x.

As an example of a perturbation Hamiltonian in chiral molecules we consider the type $\hat{G}_{ij}E_iB_j^*$, where E_i is the oscillating electric field strength of the electromagnetic field applied to the sample tube of the NMR instrument, and B_j^* is the oscillating magnetic flux density vector. The star denotes complex conjugate. Here $\hat{G}_{ij}$ is the well known Rosenfeld tensor of semi-classical perturbation theory [36, 37], which in general has real and imaginary components. The perturbed Hamiltonian is therefore

$$\Delta\hat{H}_2 = -\hat{m}_N\cdot B_Z + \hat{G}_{ij}E_iB_j^* + \ldots + complex\ conjugate + \ldots, \tag{27}$$

whose expectation value must be also be evaluated between eigenstates of the complete electronic/nuclear wavefunction, involving [11–15] electronic and nuclear angular momentum quantum numbers simultaneously.

We develop the Hamiltonian operator component $\hat{G}_{ij}E_iB_j^*$ for a completely circularly polarized electromagnetic plane wave propagating in axis Z of the laboratory frame,

$$\Delta \hat{H}_Z = \hat{G}_{XX}E_XB_X^* + \hat{G}_{XY}E_XB_Y^* + \hat{G}_{YX}E_YB_X^* + \hat{G}_{YY}E_YB_Y^* + \textit{complex conjugate} + \ldots \tag{28}$$

In order to proceed it is necessary to convert the Cartesian components of the tensor operator $\hat{G}_{ij}$ into spherical tensorial representation in the Condon Shortley phase convention [39],

$$\hat{G}_{XX} = \frac{1}{2}\left(-\sqrt{\frac{2}{3}}(\hat{G}_0^2 + \sqrt{2}\hat{G}_0^0) + (\hat{G}_2^2 + \hat{G}_{-2}^2)\right), \quad \hat{G}_{YY} = -\frac{1}{2}\left(\sqrt{\frac{2}{3}}(\hat{G}_0^2 + \sqrt{2}\hat{G}_0^0) + \hat{G}_2^2 + \hat{G}_{-2}^2\right),$$
$$\hat{G}_{XY} = -\frac{i}{2}(\sqrt{2}\hat{G}_0^1 + \hat{G}_2^2 - \hat{G}_{-2}^2), \quad \hat{G}_{YX} = \frac{i}{2}(\sqrt{2}\hat{G}_0^2 - (\hat{G}_2^2 - \hat{G}_{-2}^2)), \tag{29}$$

In the Rosenfeld tensor there is no subscript symmetry [36, 37] in general, because it is a magnetic dipole-electric dipole property tensor. Both real and imaginary parts of $\hat{G}_{ij}$ transform in the same way from Cartesian to spherical representation.

The next step in the development is to convert the calculation into the chiral molecular point group being considered, because $\hat{G}_{ij}$ necessarily occurs only in chiral molecules. This step shows clearly that the contribution of $\hat{G}_{ij}$ to optical NMR must be site specific, because its irreducible representations in the chiral point group obviously depend on the molecular electronic structure, built around various resonating nuclei.

Because in ONMR there is always an applied magnetic field, the relevant Gruffydd form of the Wigner Eckart theorem is,

$$\langle a\alpha|\hat{G}_\beta^b|a'\alpha'\rangle = [-1]^{a+\alpha}\langle a\|\hat{G}^b\|a'\rangle V\begin{pmatrix} a & a' & b \\ -\alpha & \alpha' & \beta \end{pmatrix}, \tag{30}$$

in which the original matrix element of the operator $\hat{G}_\beta^b$ is converted into a reduced matrix element multiplied by a V coefficient. Here, the eigenstate $|a\alpha\rangle$ transforms according to the α component of the irreducible representation of a in the molecular point group. Gruffydd [34] defines $[-1]^{a+\alpha}$ and tabulates V in achiral point groups of high symmetry, using the Fano Racah phase convention, which is generated from the Condon Shortley convention by multiplication with i^l, where l is the tensor rank in spherical representation. In chiral molecular point groups, Gruffydd coefficients appear to be unavailable in general, and so the theory must remain formal, and further calculation must be numerical. This illustrates that the effect of laser light on NMR spectra in chiral molecules is richly subtle and full of potential information, as the available results have started to show [16].

Because the Hamiltonian (27) is a combination of terms in two separate operator spaces, that of $\hat{m}_N$ and $\hat{G}_{ij}$, the evaluation of matrix elements of $\hat{G}_{ij}E_iB_j^*$

in the presence of $\hat{m}_N \cdot B_Z$ and vice versa must take place in general through expressions involving the Gruffydd X coefficients [34], the equivalents in molecular point groups of the 9-j symbols in atoms. The general form of the matrix element is

$$\langle abc\alpha|\hat{Y}_\phi^f(1,2)|a'b'c'\alpha'\rangle = [-1]^{c+\alpha}V\begin{pmatrix} a & a' & f \\ -\alpha & \alpha' & \phi \end{pmatrix}\lambda(c)^{\frac{1}{2}}\lambda(c')^{\frac{1}{2}}X\begin{bmatrix} a & b & c \\ a' & b' & c' \\ d & e & f \end{bmatrix}$$
$$\times \langle a\|\hat{D}^d\|a'\rangle\langle b\|\hat{E}^e\|b'\rangle, \tag{31}$$

where the irreducible spherical tensor $\hat{Y}_\phi^f(1,2)$ is worked out in terms of its components $\hat{D}^d$ *and* $\hat{E}^e$, defined by:

$$\hat{Y}_\phi^f = [\hat{D}^d \otimes \hat{E}^e]_\phi^f = \sum_{\delta\epsilon}\langle d\delta e\epsilon|def\phi\rangle\hat{D}_\delta^d\hat{E}_\epsilon^e = \lambda(f)^{\frac{1}{2}}\sum_{\delta\epsilon}V\begin{pmatrix} d & e & f \\ \delta & \epsilon & \phi \end{pmatrix}\hat{D}_\delta^d\hat{E}_\epsilon^e. \tag{32}$$

This type of algebra must be applied to find the matrix elements both of $\hat{m}_N \cdot B_Z$ and of $\hat{G}_{ij}E_iB_j^*$, because the spaces 1 and 2 are interdependent, i.e. the wave function has both nuclear and electronic character in general.

Progress can be made against contemporary lack of data on Gruffydd V, W, and X coefficients in chiral molecular point groups by using considerations of symmetry as in the preceding section. In this way it is deduced that the $\hat{G}_{ij}$ tensor mechanism considered here depends on the polarization of the laser light. This can be illustrated straightforwardly by reducing the relevant part of the Hamiltonian (27) to the Cartesian form

$$\Delta \hat{H}_Z = \hat{G}_{XX}E_XB_X^* + \hat{G}_{YY}E_YB_Y^* + \frac{1}{2}(\hat{G}_{XY} - \hat{G}_{YX})(E_XB_Y^* - E_YB_X^*)$$
$$+ \frac{1}{2}(\hat{G}_{XY} + \hat{G}_{YX})(E_XB_Y^* + E_YB_X^*) + \ldots, \tag{33}$$

which contains the trace, antisymmetric, and symmetric, parts of the relevant field tensor $E_iB_j^*$. This can be further reduced for right and left circular polarizations as follows,

$$E_XB_X^*(\textit{right}) = -E_XB_X^*(\textit{left}) = E_0B_0i, \quad E_YB_Y^* = E_XB_X^*,$$
$$E_XB_Y^* = -E_YB_X^* = E_0B_0 \ (\textit{right and left}), \tag{34}$$

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Chapter 14

MANIFESTLY COVARIANT THEORY OF THE ELECTROMAGNETIC FIELD: LONGITUDINAL MAGNETIC FIELDS IN NON-CONDUCTING AND CONDUCTING MEDIA, REFLECTION AND REFRACTION

M. W. Evans

Abstract

It is shown that the conjugate product of transverse electric field solutions of Maxwell's equations produces a magnetic field, which is longitudinally polarized in non-conducting and conducting media. The electromagnetic field in these media is therefore manifestly covariant, in that there are three physically meaningful space-like polarizations and one time-like polarization. In non-conducting media, the longitudinal magnetic field is uniform, independent of time, and solenoidal. In conducting media it is independent of time, but is damped exponentially. The phenomena of optical reflection and refraction are considered at an interface between two non-conducting media, and Snell's law is expressed in terms of these novel longitudinal magnetic fields.

1. Introduction

It has recently been shown [1–8] that the well known conjugate product of the transverse oscillating solution, $\boldsymbol{E}^{(1)}$, of Maxwell's equations with its complex conjugate $\boldsymbol{E}^{(2)}$ produces a longitudinal magnetic flux density, $\boldsymbol{B}^{(3)}$, in vacuo. This observation led to the deduction [4] that if $\boldsymbol{B}^{(3)}$ be non-zero in vacuo, then there is also present a non-zero but pure imaginary longitudinal electric field strength $i\boldsymbol{E}^{(3)}$. The longitudinal and transverse fields in free space have recently [6–8] been incorporated into a novel, manifestly covariant theory of electrodynamics based on the four-vectors E_μ and B_μ in space-time. Both E_μ and B_μ signal the presence of four physically meaningful photon polarizations in the quantum field, three space-like and one time-like. In conventional electrodynamics [9–17] the two transverse space-like components are accepted as physically meaningful, and the other two polarizations discarded as being "physically meaningless", or "unphysical". The conventional theory is therefore not manifestly covariant, because although Maxwell's equations are Lorentz covariant, only two out of the four photon polarizations are accepted as physical. In the manifestly covariant theory [1–8] on the other hand, all four photon polarizations are used as physically meaningful.

A radical departure such as this from conventional theory requires careful justification, and in Refs. [1–8], copious detail is presented of the reasoning for and self consistency of the new manifestly covariant electrodynamics (MCE). Before proceeding to the subject of this paper, which is the application of MCE to simple optical phenomena in condensed media, a brief summary of the basis for MCE is given as follows.

Conventional electrodynamics becomes untenable in view of the equation,

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}}{E^{(0)} c i} = B^{(0)} \boldsymbol{k}, \tag{1}$$

in vacuo [1–8]. Here $E^{(0)}$ is the scalar amplitude of the electric field strength of the electromagnetic wave in vacuo, and is associated with the time-like polarization (0) in MCE. In Eq. (1) c is the speed of light in vacuo. The oscillating, complex, electric field $\boldsymbol{E}^{(1)}$ is associated with transverse space-like polarization (1), and its complex conjugate $\boldsymbol{E}^{(2)}$ with transverse space-like polarization (2). The magnetic field $\boldsymbol{B}^{(3)}$ has longitudinal space-like polarization (3). As usual, for circular polarization,

$$\boldsymbol{E}^{(1)} = E^{(0)} \hat{\boldsymbol{e}}^{(1)} e^{i\phi}, \tag{2}$$

$$\boldsymbol{E}^{(2)} = E^{(0)} \hat{\boldsymbol{e}}^{(2)} e^{-i\phi}, \tag{3}$$

where $\hat{\boldsymbol{e}}^{(1)}$ and $\hat{\boldsymbol{e}}^{(2)}$ are unit polarization vectors in the circular basis,

$$\hat{\boldsymbol{e}}^{(1)} = \frac{\boldsymbol{i} - i\boldsymbol{j}}{\sqrt{2}}, \quad \hat{\boldsymbol{e}}^{(2)} = \frac{\boldsymbol{i} + i\boldsymbol{j}}{\sqrt{2}}, \tag{4}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are Cartesian unit vectors in X and Y for propagation direction Z of the electromagnetic plane wave in vacuo. The phase, as usual, is

$$\phi = \omega t - \boldsymbol{\kappa} \cdot \boldsymbol{r}, \tag{5}$$

where ω is the angular frequency at instant t and $\boldsymbol{\kappa}$ the wave vector at position $\boldsymbol{r}$. Using the well known vacuum relation,

$$E^{(0)} = cB^{(0)}, \tag{6}$$

it is easily shown [1–8] that the well known conjugate product,

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = E^{(0)2} i\boldsymbol{k} = E^{(0)} ci\boldsymbol{B}^{(3)}, \tag{7}$$

can be expressed as a longitudinal, phase independent, magnetic flux density $\boldsymbol{B}^{(3)}$ which is physically meaningful, since $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ are physically meaningful and $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$ is directly proportional to the antisymmetric part of the light intensity tensor in vacuo [9–14]. In conventional electrodynamics (both classical and quantum) $\boldsymbol{B}^{(3)}$ must be discarded as being "unphysical", in direct conflict with Eq. (1). Conventional electrodynamics is also internally inconsistent, because quantum field theory [5–8] leads through the well known Gupta Bleuler condition back to the classical results,

$$|\boldsymbol{B}^{(3)}| - B^{(0)} = 0, \tag{8a}$$

$$\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} = B^{(0)2}, \tag{8b}$$

in our notation. In conventional classical field theory [15] these solutions of Maxwell's equations in vacuo are discarded as "unphysical", despite the fact that admixtures of photon states (0) and (3), described by creation and annihilation operators [16], are deemed to be accepted [17] as physical states in the quantum theory. In MCE on the other hand, Eqs. (8a) and (8b) are consistent with Eq. (1). Both $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ change sign with the sense of circular polarization, and both vanish therefore in linear or incoherent polarization.

A few of the numerous theoretical consequences of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ in free space have been considered in detail in Refs. [6] to [8]. In Ref. [6], it was shown that $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ do not contribute to electromagnetic energy density; and are consistent therefore with the conventional Poynting theorem on a time averaged basis. The use of a four-vector description, E_μ and B_μ, in Ref. [6] is necessitated by four physically meaningful polarizations, and this leads to fundamental changes in MCE as compared with conventional electrodynamics. For example, the Stokes parameters were expressed in Ref. [6] in terms of Dirac rather than the usual Pauli matrices, and two of the Maxwell equations were necessarily expressed in manifestly covariant form, viz.,

$$\nabla \cdot \boldsymbol{E} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t} = 0, \tag{9}$$

and

$$\nabla \cdot \boldsymbol{B} + \frac{1}{c}\frac{\partial B^{(0)}}{\partial t} = 0, \tag{10}$$

rather than the usual

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{B} = 0. \tag{11}$$

In Refs. [4] and [6] several experimental consequences were discussed of the real and physical $\boldsymbol{B}^{(3)}$ which might, for example, produce light induced magnetization and polarization proportional to the square root of light intensity. Experimental evidence was presented in Ref. [4] for light induced Faraday rotation [18] which was found [4] to be proportional to the square root of light intensity, i.e to be directly proportional to $\boldsymbol{B}^{(3)}$, as expected. In Ref. [6], the well known phenomena of circular dichroism, ellipticity, optical absorption, and the Kerr effect were shown to be due to $\boldsymbol{B}^{(3)}$, and all four Stokes parameters expressed in terms of this longitudinal field. It was shown [4] that four photon polarizations are consistent with only two photon helicities (+1 and -1), because the two helicities can be expressed either in terms of (0) and (3) polarizations or (1) and (2) polarizations. In terms of (0) and (3) polarizations the link with photon helicity is given, for example, by [1, 2],

$$\hat{B}^{(3)} = B^{(0)} \frac{\hat{J}}{\hbar}, \tag{12}$$

where $\hat{J}$ is the photon angular momentum, whose eigenvalues are $M_J\hbar$, where $\hbar$ is the reduced Planck constant and $M_J = +1$ or -1. For a single photon,

$$\boldsymbol{B}^{(3)} = \langle \psi | \hat{B}^{(3)} | \psi \rangle, \tag{13}$$

where $|\psi\rangle$ is an eigenstate of the quantized field. The eigenvalues +1 and -1 take the same values as the photon helicities +1 and -1, and $\boldsymbol{B}^{(3)}$ is linked to polarizations (1) and (2) through Eq. (1) either in the classical or quantum field theory [4]. (In (1) and (2) the helicities are described in terms of the usual right and left circular polarizations.)

In Ref. [7], it was shown that the Lorentz force equation contains extra terms due to $\boldsymbol{B}^{(3)}$ in MCE compared with conventional theory, and in Ref. [8], it was shown that Eq. (1) is invariant under the basic symmetries of physics, $\hat{C}$, $\hat{P}$, and $\hat{T}$ charge conjugation, parity inversion and motion reversal respectively. Therefore $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ do not violate the fundamental symmetry principles of physics applied in vacuo or in matter.

In this paper we initiate the application of classical MCE to matter, which is considered as continuous in nature, so that molecular and atomic structural considerations are not considered explicitly, as is the custom in the simple, classical, electrodynamical theory of matter. In Sec. 2, MCE is applied in non-conducting media to the optical phenomena of reflection and refraction, described by Snell's law [19]. It is shown that longitudinal $\boldsymbol{B}^{(3)}$ can be used to describe both phenomena in a manner which is consistent with experimental data and the conventional theory, but which gives extra information in the form of an equation involving changes of $\boldsymbol{B}^{(3)}$ at the interface. This equation does not appear in the conventional theory, but is shown to be consistent with the experimentally observed phenomena of polarization by reflection at Brewster's angle [19], and total internal reflection when the angle of refraction is 90° [19]. These well known experimental phenomena of physical optics are therefore accommodated consistently within MCE.

In Sec. 3, MCE is applied to a simple theory of electromagnetic propagation in a conducting medium, and it is shown that the conjugate product of transverse electric fields produces an exponentially damped longitudinal magnetic field which is no longer spatially uniform, because the transverse electric fields which define it are no longer uniform [19]. For a medium of high conductivity therefore, the novel longitudinal magnetic field is damped out very quickly to effectively zero just below the surface of the conductor.

2. MCE in Non-Conducting Matter, Reflection and Refraction

In applying MCE to non-conducting matter we follow the methodology of a classic paper by Kielich [20] on non-linear multipole magnetization in non-conducting matter, but restrict our considerations to reflection and refraction in continua, in order to show simply how MCD differs from the conventional theory of electrodynamics and physical optics.

Kielich [20] incisively used the microscopic equations of electrodynamics, introduced by Lorentz [21], and developed by van Vleck [22] and Rosenfeld [23], who introduced quadrupolar polarization from Maxwell's macroscopic equations. Lorentz originally made the transition from his microscopic equations to the phenomenological macroscopic equations of Maxwell through the time and space averaging of microscopic field quantities over physically infinitesimal regions of matter. Mazur and Nijboer [24] showed that this procedure is equivalent to statistical ensemble averaging and developed a self consistent method of linking the microscopic Lorentz equations to the macroscopic Maxwell equations. De Groot and Vlieger [25] have developed a generally applicable and covariant method for deriving the macroscopic Maxwell equations in matter from atomic field equations. Kielich [20] adapted the methods of Mazur and Nijboer [24] and provided a multipole expansion of charge and current density to obtain generally applicable equations for the permittivity and permeability tensors of matter on the basis of the Maxwell Lorentz equations and generalized the results of Langevin [26] and van Vleck [22] for multipole magnetic polarization.

In this paper we follow Kielich's method of transition from the Lorentz theory of the electron to the macroscopic Maxwell equations in continuous matter. We do not attempt to treat matter in terms of its constituent atoms and molecules, but this can be done systematically [20] at a later theoretical stage.

In S.I. units, Lorentz's microscopic electromagnetic field equations in matter are,

$$\nabla\times\boldsymbol{e} = -\frac{\partial\boldsymbol{h}}{\partial t},\quad \nabla\cdot\boldsymbol{h} = 0,\quad \nabla\times\boldsymbol{h} = \frac{\partial\boldsymbol{e}}{\partial t} + \boldsymbol{j},\quad \nabla\cdot\boldsymbol{e} = \rho, \tag{14}$$

where $\boldsymbol{e}$ and $\boldsymbol{h}$ are the microscopic electric and magnetic field strengths, ρ is the electric charge density and $\boldsymbol{j}$ the electric current density. In general, the charge and current densities appearing in these equations can be expressed in terms of N identical microsystems, representing atoms, molecules, ions, etc. However, for our present purposes, we regard matter as continuous, and directly average [20] the microscopic field equations to give the macroscopic field equations in continuous matter,

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t},\quad \nabla\cdot\boldsymbol{B} = 0, \tag{15a}$$

$$\nabla\times\boldsymbol{B} = -\frac{\partial\boldsymbol{E}}{\partial t} + \langle\boldsymbol{j}\rangle,\quad \nabla\cdot\boldsymbol{E} = \langle\rho\rangle. \tag{15b}$$

Here, $\boldsymbol{E} = \langle\boldsymbol{e}\rangle$, where $\langle\ \rangle$ denotes an ensemble average [20], is the macroscopic electric field strength; and $\boldsymbol{B} = \langle\boldsymbol{h}\rangle$ is the macroscopic magnetic flux density. Equations (15b) can be rewritten as

$$\nabla\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t} + \langle\boldsymbol{j}\rangle,\quad \nabla\cdot\boldsymbol{D} = \langle\rho\rangle, \tag{16}$$

where $\boldsymbol{D}$ is the macroscopic electric displacement and $\boldsymbol{H}$ the macroscopic magnetic field strength, quantities defined in S.I. by

$$\boldsymbol{D} = \epsilon\boldsymbol{E},\quad \boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}. \tag{17}$$

Here we have assumed that the continuous matter is isotropic and non-conducting, so that its permittivity ϵ and permeability μ are scalars.

We are now in a position to apply the methods of MCE [6-8] to the macroscopic equations (16) and (17). In covariant form, Eqs. (16) and (17) become,

$$\frac{\partial f_{\lambda\mu}}{\partial x_\mu} = i_\lambda, \tag{18}$$

in Minkowski space-time, with the four quantities, $i_\lambda \equiv \left(\frac{1}{c}\langle\rho v_X\rangle, \frac{1}{c}\langle\rho v_Y\rangle, \frac{1}{c}\langle\rho v_Z\rangle, i\langle\rho\rangle\right)$, $x_\lambda \equiv (X, Y, Z, ict)$, and

$$f_{\lambda\mu} \equiv \begin{bmatrix} 0 & \frac{H_Z}{c} & -\frac{H_Y}{c} & -iD_X \\ -\frac{H_Z}{c} & 0 & \frac{H_X}{c} & -iD_Y \\ \frac{H_Y}{c} & -\frac{H_X}{c} & 0 & -iD_Z \\ iD_X & iD_Y & iD_Z & 0 \end{bmatrix} \tag{19}$$

In MCE, all four possible polarizations of the electric displacement $\boldsymbol{D}$ and the magnetic field strength $\boldsymbol{H}$ are considered to be physically meaningful, and this is expressed [6] explicitly by defining electric and magnetic fields as four-vectors. Thus,

$$D_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{f}_{\nu\rho}\delta_\sigma, \tag{20a}$$

$$H_\mu \equiv -\frac{c}{2}\epsilon_{\mu\nu\rho\sigma}f_{\nu\rho}\delta_\sigma, \tag{20b}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric unit tensor in four dimensions, and δ_σ is the unit vector,

$$\delta_\sigma \equiv (0, 0, 1, -i). \tag{21}$$

We define [6] the four-vectors D_μ and H_μ as,

$$D_\mu \equiv (D^{(1)}, D^{(2)}, D^{(3)}, -iD^{(0)}), \qquad H_\mu \equiv (H^{(1)}, H^{(2)}, H^{(3)}, -iH^{(0)}). \tag{22}$$

The definitions (20a) and (20b) provide a manifestly covariant description of the macroscopic electric displacement and magnetic field strength in continuous matter, a description in which there are four physically meaningful field polarizations: two transverse space-like; one longitudinal space-like, and one time-like. In analogy with Eq. (1) in vacuo the two transverse polarizations (1) and (2) are linked to the longitudinal space-like polarization (3) through the equation,

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}}{iE^{(0)}v}, \tag{23}$$

where v is the velocity of light in the continuous matter. In Eq. (23) the electric field strengths are related to the electric displacements $\boldsymbol{D}^{(1)}$ and $\boldsymbol{D}^{(2)}$ by,

$$\boldsymbol{E}^{(1)} = \frac{1}{\epsilon_0}(\boldsymbol{D}^{(1)} - \boldsymbol{P}^{(1)}), \qquad \boldsymbol{E}^{(2)} = \frac{1}{\epsilon_0}(\boldsymbol{D}^{(2)} - \boldsymbol{P}^{(2)}), \tag{24}$$

where $\boldsymbol{P}^{(1)}$ and $\boldsymbol{P}^{(2)}$ are polarizations. In analogy with Eq. (1), $\boldsymbol{D}^{(1)}$ is the complex conjugate of $\boldsymbol{D}^{(2)}$ and $\boldsymbol{P}^{(1)}$ the complex conjugate of $\boldsymbol{P}^{(2)}$. There are also analogous relations of the type,

$$\boldsymbol{B}^{(3)} = \mu_0\boldsymbol{H}^{(3)} + \boldsymbol{M}^{(3)}, \tag{25}$$

between the magnetic field flux density, where $\boldsymbol{M}^{(3)}$ is the macroscopic magnetization.

In MCE therefore continuous matter sustains a longitudinal magnetic flux density defined by equations (23) and (24). This is precisely analogous with the fact that in MCD in vacuo, there is a longitudinal $\boldsymbol{B}^{(3)}$ defined by Eq. (1). In the conventional approach [19], the fields $\boldsymbol{B}^{(3)}$ and $\boldsymbol{H}^{(3)}$ are discarded, despite the fact that the transverse fields from which they are defined are used routinely. For example, the well known inverse Faraday effect (21) is usually expressed in terms of $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$ in matter, using a molecular theory and molecular property tensors [28]. Equation (23) shows that $\boldsymbol{B}^{(3)}$ has the $\hat{C}$, $\hat{P}$, $\hat{T}$ symmetries and units of magnetic flux density in continuous matter, and since $\boldsymbol{B}^{(3)}$ is non-zero, it is a physically meaningful magnetic flux density in tesla.

If not, then electrodynamics becomes arbitrary, because a real quantity with all the known attributes of magnetic flux density would have to be regarded as having no meaning in physics (i.e., as "unphysical"). Equation (23) therefore makes conventional electrodynamics in continuous matter untenable, and by direct extrapolation, it also makes untenable the conventional electrodynamics of particulate, for example molecular, matter. A consistent electrodynamics, be this classical or quantized, must be manifestly covariant by Eq. (23).

Equations (20), as shown in detail in Ref. [6], consistently define the four-vectors H_μ and D_μ in terms of the four-tensor $f_{\lambda\mu}$, which is the four-curl of the relevant vector potential, another four-vector [10] of Minkowski space-time. Hereby, H_μ and D_μ become Pauli Lubansky types [6–8], and the manifestly covariant description (20) also satisfies [6–8] the geometrical requirements of the inhomogeneous Lorentz group in contemporary relativistic field theory. By defining H_μ and D_μ through $f_{\lambda\mu}$, they are automatically gauge invariant as required by the fundamentals of contemporary gauge theory. We emphasize that definitions such as (20) are immediately necessitated by Eq. (23), which shows that there are four physically meaningful field polarizations in continuous matter.

2.1. Reflection and Refraction

In order to illustrate a use of MCE in non-conducting matter we provide an example using the everyday phenomena of reflection and refraction in physical optics by considering an interface between two non-conducting continua. This is well described for example in the classic text by Jackson [19] on conventional, classical electrodynamics, pp. 219 ff. The kinematics of reflection and refraction are described in the physical optics of continuous matter by Snell's law [19],

$$\frac{\sin i}{\sin r} = \frac{n'}{n} = \left(\frac{\mu'\epsilon'}{\mu\epsilon}\right)^{\frac{1}{2}}, \tag{26}$$

where i is the incident angle and r the angle of refraction, n' is the refractive index of the layer (b) through which the light is refracted, n that of the layer (a) through which the light arrives at the boundary. Here μ' and ϵ' are the permeability and permittivity, respectively of layer (b), and μ and ϵ the equivalents in layer (a).

In the MCE of reflection and refraction of this type, it is necessary to take into account the existence of Eq. (23) in both layers for the incident, reflected, and refracted light. For the incident light beam,

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}}{E^{(0)} v i}, \tag{27}$$

where $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ are complex conjugates, and v is the velocity of light in layer (a). Here $E^{(0)}$ is defined as the electric field strength amplitude in vacuo. Similarly, for the refracted light beam,

$$\boldsymbol{B}^{(3)\prime} = \frac{\boldsymbol{E}^{(1)\prime} \times \boldsymbol{E}^{(2)\prime}}{E^{(0)} v' i}, \tag{28}$$

where the primed symbols relate to layer (b); and for the reflected beam in layer (a),

$$\boldsymbol{B}^{(3)\prime\prime} = \frac{\boldsymbol{E}^{(1)\prime\prime} \times \boldsymbol{E}^{(2)\prime\prime}}{E^{(0)} v'' i}, \qquad v'' = v. \tag{29}$$

Equations (27) to (29) consistently define the longitudinal magnetic flux densities of the incident, refracted, and reflected beams.

The electric field strengths appearing in the numerators (conjugate products) on the right hand sides of equations (27) to (29) are inter-related by the boundary conditions [19] on transverse components of the conventional electromagnetic field. Thus, for the incident beam,

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = E^{(0)2}\, i\boldsymbol{k}, \tag{30}$$

for the refracted beam

$$\boldsymbol{E}^{(1)\prime} \times \boldsymbol{E}^{(2)\prime} = \alpha\beta E^{(0)2}\, i\boldsymbol{k}, \tag{31}$$

and for the reflected beam

$$\boldsymbol{E}^{(1)\prime\prime} \times \boldsymbol{E}^{(2)\prime\prime} = \gamma\delta E^{(0)2}\, i\boldsymbol{k}, \tag{32}$$

where

$$\alpha = \frac{2}{1 + \frac{\mu \tan i}{\mu' \tan r}}, \qquad \beta = 2\left(\frac{\mu\epsilon}{\mu'\epsilon'}\right)^{\frac{1}{2}} \frac{\sin 2i}{\sin 2r + \frac{\mu}{\mu'} \sin 2i},$$

$$\gamma = \frac{1 - \frac{\mu \tan i}{\mu' \tan r}}{1 + \frac{\mu \tan i}{\mu' \tan r}}, \qquad \gamma = \frac{\frac{\mu}{\mu'} \sin 2i - \sin 2r}{\sin 2r + \frac{\mu}{\mu'} \sin 2i},$$

are well known [19] geometrical factors. In each layer, we have the relation,

$$B^{(0)} = \frac{1}{c}\left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)^{\frac{1}{2}} E^{(0)} \equiv \frac{E^{(0)}}{v}, \tag{33}$$

between the time-like components $B^{(0)}$ and $E^{(0)}$.

Equations (27) to (33) show that MCE is consistent with the experimental phenomena of polarization by reflection and total internal reflection. In the former [19], $\delta = 0$ at Brewster's angle, so that $\boldsymbol{B}^{(3)\prime}$ vanishes. This is consistent with the fact that the reflected wave for $\delta = 0$ is linearly polarized perpendicular to the plane of refraction defined by the angles i and r. In linear polarization, the conjugate product $\boldsymbol{E}^{(1)\prime} \times \boldsymbol{E}^{(2)\prime}$ defining $\boldsymbol{B}^{(3)\prime}$ vanishes [1–8]. In total internal reflection, the angle of refraction is 90° and there is no refracted light beam. Total internal reflection can be described by MCE through the boundary condition,

$$(B^{(0)} - B^{(0)\prime\prime}) \cos i = B^{(0)\prime} \cos r, \tag{34}$$

which provides a supplementary to Snell's law,

$$\frac{\cos i}{\cos r} = \frac{B^{(0)\prime}}{B^{(0)} - B^{(0)\prime\prime}}. \tag{35}$$

This boundary condition arises because $\boldsymbol{B}^{(3)}$, $\boldsymbol{B}^{(3)\prime}$ and $\boldsymbol{B}^{(3)\prime\prime}$ are longitudinal, and in the plane defined by i and r, so that Eq. (34) simply expresses the fact that the longitudinal magnetic flux density in layer (b) is equal to the vector sum in the plane of r and i of the magnetic flux densities in layer (a). There are

no perpendicular components of these longitudinal magnetic fields to consider.

When $r = 90°$ in Eq. (35), the left hand side is infinite for a given i, the latter being defined by Snell's law, Eq. (26), as usual [19]. This means that $B^{(0)} = B^{(0)\prime\prime}$ for total internal reflection, and that $B^{(0)\prime} = 0$. The description of this phenomenon in MCE is therefore consistent with experimental observation. Equation (35) is a new law of reflection and refraction in physical optics which is the result of Eqs. (27) to (29), i.e. is actually the result of conventional electrodynamics, converted by these equations into manifestly covariant electrodynamics. The supplementary to Snell's law, Eq. (35), is not present in conventional electrodynamics, despite the fact that it was derived from conjugate products (30) to (32), and can obviously be rewritten in terms of these directly. This is one of the clearest indications of the need for manifestly covariant electrodynamics, because if $B^{(0)}$, $B^{(0)\prime}$, and $B^{(0)\prime\prime}$ be unphysical, as (incorrectly) asserted then the ratio on the left hand side of the new equation (35) is also unphysical, a result which is unsustainable because it conflicts directly with Snell's law, which is well known to define r and i as physically meaningful angles, the angle of incidence, and the angle of refraction.

We conclude that MCE is consistent with the experimental phenomena of reflection and refraction and provides a new law, Eq. (35), for these phenomena.

3. MCE in Conducting Continuous Matter

In continuous matter which is conducting, Maxwell's macroscopic equations are supplemented by Ohm's law [19],

$$\boldsymbol{J} = \sigma \boldsymbol{E}, \tag{36}$$

where σ is the conductivity. The conventional electrodynamical theory in this context is described by the Maxwell curl equations (in S.I. units),

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t}, \tag{37}$$

and the conventional divergence equations,

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}, \tag{38}$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{D} = 0. \tag{39}$$

Jackson [19] gives the general solutions of this set of equations as a sum of longitudinal and transverse components,

$$\boldsymbol{E} = \boldsymbol{E}_{long} + \boldsymbol{E}_{trans}, \quad \boldsymbol{B} = \boldsymbol{B}_{long} + \boldsymbol{B}_{trans}. \tag{40}$$

Thus, it is well accepted that longitudinal components of Maxwell's equations exist in conducting continua, even within the context of conventional electrodynamics. The longitudinal magnetic field in the conducting continuum is conventionally [19] uniform and solenoidal, so that equation (39) is obeyed. The longitudinal electric field is a function of time and is almost instantaneously damped exponentially to zero.

It is important to note that the longitudinal component of the magnetic field defined in Jackson's equations (7.71) [19] is independent of the conductivity, time, and spatial coordinates, and that Jackson has implicitly accepted that Maxwell's equations give rise to physically meaningful longitudinal components which are phase independent in the conducting continuum. This is another sign that the conventional approach to electrodynamics is internally inconsistent, because longitudinal solutions of the type just described are independent of conductivity, and *therefore can also exist in a continuum of zero conductivity, such as a vacuum, or non conductor.* Despite this, it is conventionally assumed that this longitudinal field is identically zero, while there is nothing in Maxwell's equations to support this assumption. In general these longitudinal solutions are non-zero [1–8].

Equation (1) is the keystone of MCE, and a consistent development if MCE in conducting continua must take account of the fact that the conjugate product of the transverse electric fields generates a longitudinal magnetic flux density in the conducting continuum. Assuming that the transverse electric fields take the form [19],

$$\boldsymbol{E}^{(1)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{-\beta Z}e^{i(\kappa Z - \omega t)}, \quad \boldsymbol{E}^{(2)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{-\beta Z}e^{-i(\kappa Z - \omega t)}, \tag{41}$$

their conjugate product produces a magnetic field according to the equation,

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}}{E^{(0)} v_c i}, \tag{42}$$

where v_c is the speed of propagation the electromagnetic plane wave in the conducting continuum. We have,

$$v_c = c\left(\frac{\mu_0 \epsilon_0}{\mu \epsilon}\right)^{\frac{1}{2}}\left[1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2\right]^{\frac{1}{4}}, \tag{43}$$

The magnetic field from Eq. (42) is damped exponentially,

$$\boldsymbol{B}^{(3)} = B^{(0)} \exp\left(-\frac{Z}{Z_0}\right)\boldsymbol{k}, \quad Z_0 = \frac{1}{2}\left(\frac{2}{\omega \mu \sigma}\right)^{\frac{1}{2}}, \tag{44}$$

and satisfies the Maxwell curl equation,

$$\nabla \times \boldsymbol{E}^{(3)} = \frac{\partial \boldsymbol{B}^{(3)}}{\partial t}, \tag{45}$$

where

$$\boldsymbol{E}^{(3)} = E^{(0)} \exp\left(-\frac{\sigma t}{\epsilon}\right)\boldsymbol{k}, \tag{46}$$

is the longitudinal electric field defined by Jackson [19]. The damped longitudinal magnetic field can be written as

$$\boldsymbol{B}^{(3)} = B^{(0)} \exp\left(-2\left(\frac{\omega \mu \sigma}{2}\right)^{\frac{1}{2}} Z\right)\boldsymbol{k}, \tag{47}$$

where

$$Z_0 = \left(\frac{2}{\omega \mu \sigma}\right)^{\frac{1}{2}}, \tag{48}$$

is the skin depth [19] in S.I. units. Its divergence is non-zero,

$$\nabla \cdot \boldsymbol{B}^{(3)} = -2\left(\frac{\omega\mu\sigma}{2}\right)^{\frac{1}{2}} B^{(0)} \exp\left(-2\left(\frac{\omega\mu\sigma}{2}\right)^{\frac{1}{2}} Z\right), \tag{49}$$

and at first sight this appears to conflict with the divergence equation of magnetostatics,

$$\nabla \cdot \boldsymbol{B} = 0. \tag{50}$$

However, this equation, the first equation of magnetostatics [19] applies only when the frequency is zero, by definition. It is immediately seen from Eq. (49) that if we set the angular frequency to zero, the divergence of $\boldsymbol{B}^{(3)}$ vanishes.

MCE applied to conducting continua is therefore consistent with what is known from the conventional approach [19], but gives the additional equation (44) describing a rapidly damped magnetic field, which is effectively zero just below the surface of the conductor. This is consistent with the behavior of waves in conducting media, and with the fact that the longitudinal and non-uniform field $\boldsymbol{B}^{(3)}$ is derived from the conjugate product of transverse electric components of these waves, and is therefore damped in the same way at finite ω. For zero ω the magnetic field is solenoidal and time independent, but is then identically zero because there are no transverse components at zero frequency. As discussed by Jackson [19], in a good conductor, current flows at high frequency only on the surface of a conductor, and in its interior there are no magnetic or electric fields, either transverse or longitudinal. Near the surface, however, there exists the novel magnetic field $\boldsymbol{B}^{(3)}$ in MCE which is not predicted in conventional electrodynamics. This might give rise to interesting surface effects which could be detected experimentally with an instrument such as a SQUID detector.

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Chapter 15

CRITICISMS OF THE DIAGRAMMATIC APPROACH TO COMPLETE EXPERIMENT SYMMETRY[1]

M. W. Evans

Abstract

The conservation or violation of natural discrete symmetries such as $\hat{P}$, $\hat{T}$ and $\hat{C}$ must be deduced from the mathematical equations describing physical laws. It is shown that recent attempts to by-pass these equations with simple diagrams lead to spurious and as often as not, erroneous, results.

1. Introduction

The use of simple diagrams to explain complete experiment symmetry has been proposed [1, 2] in an attempt to describe without mathematical calculation magneto and electro optic phenomena such as the Faraday effect. It is shown that these methods are based on a subjective choice of parameters, and produce fortuitous or erroneous results. This problem is illustrated with reference to beta decay, natural optical activity, the Faraday effect, magnetochiral birefringence, and longitudinal solutions in vacuo of the Proca and Maxwell equations.

2. Criticism of the Diagrammatic Method

2.1. Parity Violation in Beta Decay

Simple diagrams based on the complete experiment argument have been used [3] to illustrate parity violation in beta decay, an experimentally verified phenomenon. The diagrammatic method is based on the idea that the discrete symmetries of nature are conserved in a complete experiment [1–3]. If this is observed not to be the case, then there is symmetry violation. The three discrete symmetries under discussion are parity inversion ($\hat{P}$); motion reversal ($\hat{T}$) and charge conjugation ($\hat{C}$), well known and discussed in many textbooks, for example, Ref. [3]. The diagrammatic argument [3] for parity violation in beta decay uses three variables: 1) a polarizing magnetic flux density ($\boldsymbol{B}$); 2) a nuclear spin; 3) a direction vector for electron emission from polarized cobalt 60 nuclei. A simple diagram is used [3] to show that $\hat{P}$ reverses (3) but not (1) and (2). On this basis it is argued that the complete experiment and its "mirror

[1] "There is no certainty in sciences when mathematics cannot be applied." Leonardo da Vinci, (Windsor, Royal Library, 19,084).

image" are not superposable and that the observed electron emission violates parity.

While this sketch leads fortuitously to the right conclusion, it cannot be extended to $\hat{T}$ and is therefore unreliable. The application of $\hat{T}$ has the effect of reversing (1) and (2), leaving (3) unchanged. The application of $\hat{C}$ reverses (1) but leaves (2) and (3) unchanged. Experimentally, there is no $\hat{T}$ violation but there *is* $\hat{C}$ violation, and the diagrammatic method fails. It is not meant by the author [3] to be anything other than a schematic analogy to aid understanding. The subjective nature of this diagram is revealed if we replace the direction vector used to represent the electron emission (element (3) of the diagram) with a propagation vector, representing the electronic linear momentum. This, albeit arbitrary, combination of elements gives the right result, because $\hat{T}$ now reverses all three diagrammatic elements, so that on this (spurious) basis, there is no $\hat{T}$ violation, and the diagram fortuitously agrees with the experimental data. Since helicity (vide infra) is defined as the projection of electron spin on electron linear momentum, it seems more appropriate to use electron linear momentum in the sketch than to use electron direction, but even so, the diagrammatic argument remains subjective, because it is not known, without mathematical analysis, how the chosen diagrammatic elements contribute (if at all) to the description of beta decay and parity violation therein.

The analytical description of $\hat{P}$ violation in beta decay uses a physical law. It is described by Ryder [3] through.

$$\langle\boldsymbol{\lambda}\rangle \xrightarrow{\hat{P}} -\langle\boldsymbol{\lambda}\rangle, \tag{1}$$

where $\langle\boldsymbol{\lambda}\rangle$ is the mean helicity of the emitted beta particles. To test for complete experiment symmetry, the operators $\hat{P}$, $\hat{T}$, and $\hat{C}$ must be applied to the mathematical description of the relevant physical law. (For example $\hat{P}$, $\hat{T}$, and $\hat{C}$ applied to the Maxwell equations leaves them unchanged, and the same physical laws are regained. Maxwell's equations therefore conserve $\hat{P}$, $\hat{T}$ and $\hat{C}$.) Equation (1) indicates that in order for beta decay to conserve parity in the complete experiment, the mean value of the helicity must vanish. The reason is that in the $\hat{P}$ inverted experiment the mean helicity would have to be equal in magnitude but opposite in sign to the original. If parity is conserved, this is possible if and only if the mean helicity is zero. (In analogy, the mean of a polar vector, (one that is negative to $\hat{P}$) in an isotropic sample is zero.) Equation (1) is true if parity is conserved, and not true otherwise. Since Eq. (1) is not observed experimentally, $\hat{P}$ is violated. What is actually observed is a preferential direction of electron emission.

The mathematical description (1) also explains why there is no $\hat{T}$ violation in beta decay. Helicity is $\hat{T}$ positive, and

$$\langle \lambda \rangle \overset{\hat{T}}{\rightarrow} \langle \lambda \rangle, \tag{2}$$

and the sign of the mean helicity is not reversed. This means that $\langle\lambda\rangle$ can be non zero, as is observed experimentally, without $\hat{T}$ violation. (In analogy, the mean of a $\hat{T}$ positive quantity can be non-zero in an isotropic molecular liquid, for example the mass of an isotropic liquid is non-zero because it is calculated from the mean of individually non-zero molecular masses, which are $\hat{T}$ positive scalars.)

The mathematical description is simpler, and gives the correct result, that the observation of a preferential direction for $\langle\lambda\rangle$ in beta decay violates $\hat{P}$ but conserves $\hat{T}$. The diagrammatic argument, if used without care or without mathematical development, would lead to the incorrect conclusion that $\hat{T}$ is violated, as discussed.

The $\hat{C}\hat{P}\hat{T}$ theorem [3] states that $\hat{C}\hat{P}\hat{T}$ is always conserved in field theory with local interactions, so that the discrete symmetry operator $\hat{C}\hat{T}$ must be violated if $\hat{P}$ is violated. The diagrammatic method, however, goes wrong in that it implies that $\hat{C}$ and $\hat{T}$ are violated individually so that $\hat{C}\hat{T}$, on this basis, is conserved, an incorrect and inconsistent conclusion, because the same diagram implies that $\hat{P}$ is violated. If the $\hat{C}\hat{P}\hat{T}$ theorem is accepted, then the operator $\hat{C}\hat{T}$ must have the same effect as $\hat{P}$ on electron helicity,

$$\langle \lambda \rangle \overset{\hat{C}\hat{T}}{\rightarrow} \langle -\lambda \rangle, \tag{3}$$

Therefore $\hat{P}$ violation in beta decay is accompanied by $\hat{T}$ conservation and $\hat{C}$ violation. The product $\hat{C}\hat{P}$ must be conserved however, because $\hat{T}$ is conserved,

$$\langle \lambda \rangle \overset{\hat{C}\hat{P}}{\rightarrow} \langle \lambda \rangle, \tag{4}$$

This means that for $\hat{C}\hat{P}$ conservation in beta decay a right handed electron becomes a left handed positron; alternatively, a left handed electron becomes a right handed positron.

The diagrammatic method gives no indication of this, which is a consequence of the analytically derived $\hat{C}\hat{P}\hat{T}$ theorem.

2.2. Natural Optical Activity

Barron has attempted to extend [1, 2] the use of simple diagrams to molecular-optical phenomena such as natural optical activity and the Faraday effect. For natural optical activity:

$$\begin{bmatrix} \lambda \\ \Delta\theta \\ \boldsymbol{\kappa} \\ R \end{bmatrix} \overset{\hat{P}}{\rightarrow} \begin{bmatrix} -\lambda \\ -\Delta\theta \\ -\boldsymbol{\kappa} \\ S \end{bmatrix}, \quad \begin{bmatrix} \lambda \\ \Delta\theta \\ \boldsymbol{\kappa} \\ R \end{bmatrix} \overset{\hat{T}}{\rightarrow} \begin{bmatrix} \lambda \\ \Delta\theta \\ -\boldsymbol{\kappa} \\ R \end{bmatrix}. \tag{5}$$

Here λ, the helicity of the light probe, is described [1, 2] as the "screw sense". Rotation of the plane of the probe occurs through an angle $\Delta\theta$, which is assumed to be a pseudo-scalar. The propagation direction of the light beam is represented by a $\hat{P}$, and $\hat{T}$ negative propagation vector $\boldsymbol{\kappa}$. The symbol $R \overset{\hat{P}}{\rightarrow} S$ represents the two structural enantiomers of the sample, interrelated by $\hat{P}$. It is claimed [2] on the basis of construction (5) that natural optical activity conserves $\hat{P}$, and $\hat{T}$. The operator $\hat{C}$ is not considered [1, 2]. There are several spurious sides to this argument.

1. The helicity λ is defined as [3],

$$\lambda \equiv \boldsymbol{\sigma} \cdot \frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|}, \tag{6}$$

i.e is the projection of spin, $\boldsymbol{\sigma}$, ($\hat{P}$ positive, $\hat{T}$ negative) on to the propagation vector, $\boldsymbol{\kappa}$ ($\hat{P}$ and $\hat{T}$ negative). The spin in this case is the sense of rotation of the light beam (clockwise or anticlockwise), and is represented by an axial vector. Therefore $\boldsymbol{\kappa}$, a polar vector, is included twice in the construction. This is subjective, there is no way of knowing how many times $\boldsymbol{\kappa}$ should appear. Presumably, since the other variables appear once, $\boldsymbol{\kappa}$ should appear once, but there is no certainty that this would lead to anything meaningful.

2. The well known potential four-vector A_μ is arbitrarily excluded. In field theory electromagnetism is introduced through A_μ as a consequence of gauge transformation of the second kind [4]. If A_μ is absent, there are no fields present. The quantities λ and $\boldsymbol{\kappa}$ represent only the spatio-temporal aspects of electromagnetism. The exclusion of A_μ is subjective.

3. From construction (5), $\hat{T}$ reverses $\boldsymbol{\kappa}$ but leaves all other quantities unchanged. Without further analysis, it appears that $\hat{T}$ produces a result which is distinguishable from the original experiment, so that there is $\hat{T}$ violation, an obviously incorrect conclusion. This is rationalized however [1, 2] on the grounds that "it corresponds simply to reversing the direction of propagation of the light beam". However, if this is done in a genuine investigation of natural optical activity, the observer would see

a reversal in $\Delta\theta$, in contradiction to construction (5), in which $\hat{T}$ does not affect $\Delta\theta$ because the latter is taken to be a pseudo-scalar. The reason for the reversal of $\Delta\theta$ is that with the beam travelling away from the static (i.e. $\hat{T}$ invariant) observer, a clockwise $\Delta\theta$, for example, is observed. Simple reversal of the beam direction means that the static observer will see an anticlockwise rotation, i.e. the sign of $\Delta\theta$ is *changed* by simple reversal of beam direction. Therefore the motion reversed complete experiment does not correspond to simply reversing the beam direction. In the former $\Delta\theta$ is unchanged, in the latter $\Delta\theta$ is reversed.

4. The most striking anomaly is that in deriving construction (5), knowledge of the pseudo-scalar nature of $\Delta\theta$ has been assumed. (On page 31 of Ref. [2] for example, it is stated that "Thus under $\hat{P}$, the screw sense of the helical pattern of electric field vectors in the medium is inverted (*the optical rotation angle being a pseudo-scalar*)..."). The argument is therefore circular: in order for $\Delta\theta$ to be classified as a pseudo-scalar, an analytical (mathematical) knowledge of natural optical activity is required, whereas construction (5) is intended to supersede a mathematical analysis of the same phenomenon. This renders the initial construction meaningless.

5. If we attempt to describe the "complete experiment" with $A_\mu \equiv (\mathbf{A}, \phi)$ and λ we obtain,

$$\begin{bmatrix} (\mathbf{A}, \phi) \\ \lambda \\ \Delta\theta \\ R \end{bmatrix} \xrightarrow{\hat{P}} \begin{bmatrix} (-\mathbf{A}, \phi) \\ -\lambda \\ -\Delta\theta \\ S \end{bmatrix}, \quad \begin{bmatrix} (\mathbf{A}, \phi) \\ \lambda \\ \Delta\theta \\ R \end{bmatrix} \xrightarrow{\hat{T}} \begin{bmatrix} (\mathbf{A}, \phi) \\ \lambda \\ \Delta\theta \\ R \end{bmatrix}. \tag{7}$$

We have again used knowledge of the pseudo-scalar nature of $\Delta\theta$, but for the sake of illustration, we see that this time the $\hat{T}$ reversed experiment appears *indistinguishable* from the original, appearing this time to imply $\hat{T}$ conservation. This is a fortuitously correct result, but one which is obtained in a spurious way. In other words, two different scientists, one who chose to use construction (5), the other construction (7), would arrive at opposite conclusions, showing that the procedure is subjective. It is subjective because it is based on an arbitrary choice of parameters. Scientist one has chosen to use λ and $\boldsymbol{\kappa}$; scientist two has chosen to use A_μ and λ. In addition, the original purpose of the approach [1, 2] is undermined completely by the necessity to use knowledge of the pseudo-scalar nature of $\Delta\theta$. As we shall see from the following analysis of the Faraday effect, it is by no means obvious why $\Delta\theta$ should take this symmetry ($\hat{P}$ negative, $\hat{T}$ positive). In the Faraday effect $\Delta\theta$ has the opposite

symmetry ($\hat{P}$ positive, $\hat{T}$ negative).

From the standard mathematical description of natural optical activity, given by Rosenfeld [5], $\Delta\theta$ is a pseudo-scalar because it is proportional to the pseudo-scalar trace of the ensemble averaged electric dipole-magnetic dipole molecular property tensor of a structurally chiral molecule. This is made clear by Barron himself [2]. The correct (and simple) way to describe the complete experiment symmetry of natural optical activity is to apply $\hat{P}$, $\hat{T}$, and $\hat{C}$ in turn to each symbol appearing in Rosenfeld's equation [5]. It is found that the same equation is regained in each case, i.e. application of $\hat{P}$, $\hat{T}$, and $\hat{C}$ leads in each case to the same law of physics. Thus, $\hat{P}$, $\hat{T}$, and $\hat{C}$ are conserved and the $\hat{C}\hat{P}\hat{T}$ theorem satisfied.

2.3. The Faraday Effect

The diagrammatic construct used by Barron [1, 2] for the Faraday effect is,

$$\begin{bmatrix} \mathbf{B} \\ \boldsymbol{\kappa} \\ \lambda \\ \Delta\theta \\ R \end{bmatrix} \xrightarrow{\hat{P}} \begin{bmatrix} \mathbf{B} \\ -\boldsymbol{\kappa} \\ -\lambda \\ \Delta\theta \\ S \end{bmatrix}, \quad \begin{bmatrix} \mathbf{B} \\ \boldsymbol{\kappa} \\ \lambda \\ \Delta\theta \\ R \end{bmatrix} \xrightarrow{\hat{T}} \begin{bmatrix} -\mathbf{B} \\ -\boldsymbol{\kappa} \\ \lambda \\ \Delta\theta \\ R \end{bmatrix}, \tag{8}$$

a complicated combination of five variables: one external field variable ($\mathbf{B}$); three light beam variables ($\boldsymbol{\kappa}$, $\Delta\theta$, and λ); and one sample variable (R), where we have allowed for the fact that the Faraday effect can occur in chiral as well as in achiral, samples. The experimentally based mathematical description [6] of the Faraday effect is much simpler,

$$\Delta\theta = \bar{V}_B B_Z , \tag{9}$$

where $\bar{V}_B$ represents an ensemble averaged scalar, the Verdet constant. This is a $\hat{P}$, and $\hat{T}$ positive quantity. The following points emerge.

1. Only by making use of Eq. (9) do we realize that $\Delta\theta$ takes the $\hat{P}$ positive, $\hat{T}$ negative symmetries of B_Z, and is not a pseudo-scalar as in natural optical activity. This knowledge is again used [1, 2] in the original construction (8), a construction which is intended to provide this knowledge. The argument is again circular and therefore meaningless.

2. The variable $\boldsymbol{\kappa}$ is again used twice and A_μ is neglected.
3. The discussion of construction (8) on page 33 of Ref. [2] is therefore spurious.

It appears from the construction (8) that both $\hat{P}$ and $\hat{T}$ lead to distinguishable situations, giving the incorrect impression of both $\hat{P}$ and $\hat{T}$ violation. In this case, the use of A_μ does not improve matters,

$$\begin{bmatrix} \boldsymbol{B} \\ (\boldsymbol{A}, \phi) \\ \lambda \\ \Delta\theta \\ R \end{bmatrix} \xrightarrow{\hat{P}} \begin{bmatrix} \boldsymbol{B} \\ (-\boldsymbol{A}, \phi) \\ -\lambda \\ \Delta\theta \\ S \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{B} \\ (\boldsymbol{A}, \phi) \\ \lambda \\ \Delta\theta \\ R \end{bmatrix} \xrightarrow{\hat{T}} \begin{bmatrix} -\boldsymbol{B} \\ (\boldsymbol{A}, \phi) \\ \lambda \\ -\Delta\theta \\ R \end{bmatrix}, \tag{10}$$

and we have again used knowledge of the symmetry of $\Delta\theta$. An acceptable diagrammatic representation of the Faraday effect is, for example,

$$\begin{bmatrix} \boldsymbol{B} \\ \Delta\theta \\ \overline{V}_B \end{bmatrix} \xrightarrow{\hat{P}} \begin{bmatrix} \boldsymbol{B} \\ \Delta\theta \\ \overline{V}_B \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{B} \\ \Delta\theta \\ \overline{V}_B \end{bmatrix} \xrightarrow{\hat{T}} \begin{bmatrix} -\boldsymbol{B} \\ -\Delta\theta \\ \overline{V}_B \end{bmatrix}, \tag{11}$$

which simply summarizes Eq. (9). Only through this equation do we realize that the Faraday effect can be described through one field variable ($\boldsymbol{B}$); one beam variable ($\Delta\theta$) and one sample variable ($\overline{V}_B$).

By applying $\hat{P}$, $\hat{T}$, and $\hat{C}$ to Eq. (7) we obtain,

$$\hat{T}: \quad -\Delta\theta = \overline{V}_B\,(-B_Z), \tag{12}$$

$$\hat{P}: \quad \Delta\theta = \overline{V}_B\, B_Z, \tag{13}$$

$$\hat{C}: \quad \Delta\theta = (-\overline{V}_B)\,(-B_Z), \tag{14}$$

and in each case the same law of physics is regained. The Faraday effect conserves $\hat{P}$, $\hat{T}$ and $\hat{C}$. In so doing we realize that $\Delta\theta$ has the same $\hat{P}$ and $\hat{T}$ symmetries as B_Z, but the opposite $\hat{C}$ symmetry. This is consistent with definition of $\hat{C}$ [3], which reverses the signs of charges in classical physics,

but leaves all spatio-temporal variables unchanged. The Verdet constant is $\hat{C}$ negative because it is an ensemble average over $\hat{C}$ negative molecular property tensors. There is no way of arriving at this conclusion from a construction such as (8).

If we attempt to assert that there is an electrically induced Faraday effect,

$$\Delta\theta =?\ \overline{V}_E\, E_Z, \tag{15}$$

then $\overline{V}_E$, from perturbation theory [7], must be an ensemble average over $\hat{P}$, and $\hat{T}$ *negative* molecular property tensors or products thereof. This ensemble average is inevitably zero for all E_Z, and there is no observed effect. Barron [1, 2] attempts to rationalize this result by noting that $\boldsymbol{\kappa}$ and $\boldsymbol{E}$ have opposite $\hat{T}$ symmetries, whereas $\boldsymbol{B}$ and $\boldsymbol{\kappa}$ have the same $\hat{T}$ symmetries. (He uses this spurious argument again in the description of magnetochiral birefringence, *vide infra*.) We could just as well assert that since $\boldsymbol{\lambda}$ and $\boldsymbol{E}$ have the same $\hat{T}$ symmetries, there is an electric Faraday effect, an erroneous conclusion. On these, albeit spurious, grounds, there would be no magnetic Faraday effect because $\boldsymbol{\lambda}$ and $\boldsymbol{B}$ have different $\hat{T}$ symmetries, again an erroneous result. There is no a priori method of knowing which variable to consider, $\boldsymbol{\lambda}$ or $\boldsymbol{\kappa}$. On the other hand the mathematical approach gives an objective result. It is noteworthy that in the mathematical description (9), neither $\boldsymbol{\lambda}$ nor $\boldsymbol{\kappa}$ appears explicitly, and neither does A_μ. These field variables have been incorporated in the derivation of Eq. (9), and the latter is therefore a sufficient as well as necessary description of the Faraday effect, a result which has been known for over 150 years. Note that Eq. (9) is therefore already a description of the "complete experiment", and is a law of physics which is invariant to natural discrete symmetry operators.

Barron's consideration of the Faraday effect is based on the work of Rinard and Calvert [8]. The main conclusion of this paper is that for $\hat{P}$ conservation the sense of rotation of the polarization vector of the electromagnetic plane wave must be independent of the direction of propagation of the light beam. This statement is equivalent to,

$$\boldsymbol{\lambda} \xrightarrow{\hat{P}} \boldsymbol{\lambda}, \tag{16}$$

where $\boldsymbol{\lambda}$ is the beam helicity. In the Faraday effect therefore, $\boldsymbol{\lambda} \xrightarrow{\hat{P}} -\boldsymbol{\lambda}$ and $\boldsymbol{B} \xrightarrow{\hat{P}} \boldsymbol{B}$. This is essentially all we are told by Rinard and Calvert, and it is not possible to deduce anything further about the Faraday effect. The criticisms which apply to Barron's method also apply to the work of these authors. Rinard and Calvert [8], however, also consider $\hat{C}$, whereas Barron [1, 2] does not. The former, however, come to no clear conclusion: "The effect of $\hat{C}$ inversion cannot be uniquely anticipated since it cannot be proven that the antimatter medium would not cause the left hand rotation." This obscure diagrammatic finding is contrasted with the simple, but fundamental, Eq. (14) of this note, which shows that the Faraday effect conserves $\hat{C}$. That it must do so follows from the $\hat{C}\hat{P}\hat{T}$ theorem, because the Faraday effect conserves $\hat{T}$, and $\hat{P}$, and thus $\hat{P}\hat{T}$. The diagrams used by Rinard and Calvert [8] fail to produce this result, showing that the diagrams are limited in utility for $\hat{C}$, even when they happen to produce the correct result, fortuitously, for $\hat{P}$ and $\hat{T}$.

It is intuitively expected that there is no electric Faraday effect, because the electric field, being a polar vector, has no rotational nature, and cannot induce a rotation, $\Delta\theta$. Although intuition, because it is subjective, is often confounded in physics, a cursory examination of the hypothetical Eq. (15) shows that the right hand side must have the following properties.

$$\overline{V}_E E_Z \stackrel{\hat{P}}{\rightarrow} (-\overline{V}_E)(-E_Z), \qquad \overline{V}_E E_Z \stackrel{\hat{T}}{\rightarrow} (-\overline{V}_E)(E_Z), \qquad \overline{V}_E E_Z \stackrel{\hat{C}}{\rightarrow} (-\overline{V}_E)(-E_Z). \tag{17}$$

From this, it follows that any $\Delta\theta$ induced by an electric field in the Faraday effect would have to have the same symmetries as $\Delta\theta$ in the observed magnetic Faraday effect, i.e. $\hat{P}$ positive, $\hat{T}$ negative, $\hat{C}$ positive. Otherwise there would be violation of some kind of discrete natural symmetry. The electric field $\boldsymbol{E}$ is, in contrast, $\hat{P}$ negative, $\hat{T}$ positive. Thus $\Delta\theta$ cannot take the spatio-temporal symmetries (i.e. P and T) of the electric field without violation. This is a second reason why the electric Faraday effect is not observed. The other reason, given already, is that $\overline{V}_E$ is $\hat{P}$ and $\hat{T}$ negative and ensemble averages to zero.

These arguments have nothing to do with Barron's claim [1, 2] that the electric Faraday effect vanishes because $\boldsymbol{\kappa}$ and $\boldsymbol{E}$ have opposite $\hat{T}$ symmetries. This claim is subjective, and therefore spurious and correct only fortuitously.

2.4. Magnetochiral Birefringence

Similar use of the diagrammatic approach has been advocated [9, 10] for the magnetochiral effect, first proposed in crystals by Portigal and Burstein [11], and in liquids by Woźniak and Zawodny [12] and Wagnière and Meier [13]. The argument again relies on the subjective comparison of $\boldsymbol{\kappa}$ and $\boldsymbol{B}$, and is therefore spurious. The objective description of the phenomenon is based on perturbation theory applied to molecular property tensors. The correct explanation of the symmetry of the effect has been given in simple terms by Wagnière [14], using mathematical methods.

2.5. Longitudinal Solutions of the Proca and Maxwell Equations in Vacuo

It has been shown recently [15–17] that solutions of Maxwell's equations in free space form a cyclic Lie algebra typified by,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} - iB^{(0)}\boldsymbol{B}^{(3)}, \tag{18}$$

and cyclic permutations, where the complex conjugates $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are the usual transverse, oscillating, magnetic fields in vacuo, and where $\boldsymbol{B}^{(3)}$ is longitudinal and real,

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}, \tag{19}$$

Here $\boldsymbol{k}$ is an axial unit vector and $B^{(0)}$ the scalar amplitude of the magnetic component of the circularly polarized light beam. The field $\boldsymbol{B}^{(3)}$ has been observed experimentally [18] in the inverse Faraday effect (bulk magnetization by circularly polarized light).

Maxwell's field equations in vacuo represent a law of physics which conserves $\hat{T}$, $\hat{P}$, and $\hat{C}$. After applying these operators the same equations and law of physics are regained unchanged. The equations are unchanged, and it follows that all solutions of the equations are unchanged by $\hat{P}$, $\hat{T}$, and $\hat{C}$. Therefore $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ all conserve $\hat{P}$, $\hat{T}$, and $\hat{C}$. The solutions are linked together by Eqs. (18), which also conserve $\hat{P}$, $\hat{T}$, and $\hat{C}$. This can be checked by operating on each symbol by $\hat{P}$, $\hat{T}$, and $\hat{C}$ in turn, noting that $B^{(0)}$ is $\hat{C}$ negative [17]. In contemporary language, electromagnetism, having U(1) gauge symmetry, conserves the discrete symmetries of nature and obeys the $\hat{C}\hat{P}\hat{T}$ theorem [3, 4, 19]. The same conclusion holds for the Proca equation [17] of electromagnetism, in which the photon has a tiny but finite mass. Longitudinal solutions of the Proca and Maxwell equations are identical for all practical purposes in the zero frequency limit [17].

In this context, Barron's diagrammatic methods go seriously astray [20, 21], because they lead to the result that $\boldsymbol{B}^{(3)}$, a solution of Maxwell's equations, violates $\hat{C}$ and $\hat{C}\hat{P}\hat{T}$. The inverse Faraday effect shows that $\boldsymbol{B}^{(3)}$ is a physical magnetic field, and therefore a solution of Maxwell's equations. This much is clear from the relation between $\boldsymbol{B}^{(3)}$ and the well known conjugate product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$, which is usually used to describe [22] the inverse Faraday effect,

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = icE^{(0)}\boldsymbol{B}^{(3)} - ic^2B^{(0)}\boldsymbol{B}^{(3)}, \tag{20}$$

Experimental evidence for $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is evidence for $\boldsymbol{B}^{(3)}$. Therefore the diagrammatic arguments conflict both with experiment and basic field theory. The source of this erroneous result is Barron's spurious comparison [20, 21] of $\boldsymbol{B}^{(3)}$ with $\boldsymbol{\kappa}$. He asserts that since $\boldsymbol{B}^{(3)}$ and $\boldsymbol{\kappa}$ have opposite $\hat{C}$ symmetry, $\boldsymbol{B}^{(3)}$ must vanish. The correct analytical description of $\boldsymbol{B}^{(3)}$ is given by Eqs. (18), (19) and (20), all of which conserve $\hat{P}$, $\hat{T}$, and $\hat{C}$.

The diagrammatic approach to complete experiment symmetry is subjective, and significantly, is confined almost entirely to the work of Rinard and Calvert [8] and Barron [2, 10]. The overwhelming majority prefers to use the standard mathematical description of physical laws, because the latter is objective and reliable. A simple diagram is useful as an aid to understanding, but is no more than this. The discovery in the late fifties of parity violation can be described analytically, as in a textbook such as that of Ryder [3]. The sketch used by this author to illustrate the phenomenon is clearly meant to be schematic, because it refers to the parity operation (and illustrates it) as a "mirror image" or "reflection". The same author makes it clear however, that parity inversion is more accurately defined as the inversion of coordinates, i.e. (X, Y, Z) to (-X, -Y, -Z). This is not a reflection through a plane.

Conclusion

A valid mathematical description of a natural phenomenon is objective, such an equation represents a law of physics. This law is a necessary and sufficient description of the complete experiment. If an equation is unchanged by application of a discrete symmetry operator, e.g. $\hat{P}$, then the law of physics is unchanged under parity inversion of the complete experiment and parity is conserved in the natural phenomenon described by the law. If not, parity is violated. The law of physics describing a complete experiment is embodied in an equation. The latter therefore defines the complete experiment, and the converse approach, in which an attempt is made to use simple diagrams to evaluate natural phenomena without an objective analysis (i.e. without an equation, and without a law of physics) is inevitably subjective. The diagram may produce the same result as the law, but as often may not, and is therefore unreliable. The root cause of this is that without knowledge of the appropriate law of physics, it is not possible in general to know how to define the complete experiment, i.e. it is not possible to know which variables are relevant to the law of physics and which are not. The variables appearing in the diagrammatic approach to complete experiment symmetry have to be chosen subjectively, or arbitrarily. A different choice of variables may or may not lead to different conclusions.

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Chapter 16

THE PHOTON'S MAGNETOSTATIC FLUX QUANTUM: ITS ROLE IN CIRCULAR DICHROISM AND THE ELECTRICAL KERR EFFECT

M. W. Evans

Abstract

It has recently been deduced theoretically that the photon generates a magnetostatic flux quantum ($\hat{B}^{(3)}$), a concept which appears to be fundamental to physical optics. Considering the classical $\boldsymbol{B}^{(3)}$ it is shown that it is directly responsible for several well known phenomena, including circular dichroism and the development of beam ellipticity in the electrical Kerr effect. The reason is that $\boldsymbol{B}^{(3)}$ is directly proportional to the well known Stokes parameter S_3, a parameter whose origin resides therefore in the photon's elementary magnetic field.

1. Introduction

In free space relativistic quantum electrodynamics it is well known [1,2] that the photon in the manifestly covariant Lorentz gauge has four polarizations, one time-like ((0)), two transverse space-like ((1) and (2)), and one longitudinal space-like ((3)). These conclusions emerge directly from integration of the d'Alembert equation

$$\Box A_\mu = 0, \tag{1}$$

which is quantized in the Lorentz gauge as the well known [1, 2] Gupta Bleuler condition

$$\partial_\mu \hat{A}^{(+)\mu}|\chi\rangle = 0. \tag{2}$$

Here A_μ is the potential four-vector, and $\hat{A}^{(+)\mu}$ its equivalent operator. $|\chi\rangle$ is an eigenstate of the photon field. *Equation (2) leads directly to the result that admixtures of time-like and longitudinal space-like photon polarizations are physically meaningful in free space.* This is stated [2] using photon annihilation and creation operators through the equations [2]

$$(\hat{a}^{(0)} - \hat{a}^{(3)})|\chi\rangle = 0, \tag{3a}$$

$$\langle\chi|\hat{a}^{(0)\dagger}\hat{a}^{(0)}|\chi\rangle = \langle\chi|\hat{a}^{(3)\dagger}\hat{a}^{(3)}|\chi\rangle. \tag{3b}$$

Furthermore, the Hamiltonian operator in the quantized field is proportional to an integral [2] over the sum

$$\sum_{\lambda=1}^{3} (\hat{a}^{(\lambda)\dagger}\hat{a}^{(\lambda)} - \hat{a}^{(0)\dagger}\hat{a}^{(0)}), \tag{4}$$

so that the contributions from the (3) and (0) photon states cancel, leaving only those from the transverse (1) and (2) states. *In other words, the (0) and (3) states do not contribute to the beam energy.*

Recently, it has been shown [3–6] that the classical interpretation of Eqs. (3a), (3b) and (4) is that there exist *longitudinal* solutions of Maxwell's equations in free space, given by

$$i(E^{(0)} - |\boldsymbol{E}^{(3)}|) = 0, \text{ or } i\boldsymbol{E}^{(3)} = iE^{(0)}\hat{\boldsymbol{e}}^{(3)}, \quad B^{(0)} - |\boldsymbol{B}^{(3)}| = 0, \text{ or } \boldsymbol{B}^{(3)} = B^{(0)}\hat{\boldsymbol{e}}^{(3)}. \tag{5}$$

Here $E^{(0)}$ and $B^{(0)}$ are the scalar amplitudes of the electric and magnetic components of the plane wave in free space, and $\hat{e}^{(3)}$ is a unit vector in the circular basis in the propagation axis of the plane wave. Equations (5) are the classical counterparts of Eq. (3a). The classical counterparts of Eq. (3b) are,

$$-E^{(0)2} = -\boldsymbol{E}^{(3)}\cdot\boldsymbol{E}^{(3)}, \quad B^{(0)2} = \boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)}, \tag{6}$$

and it is clear from the structure of Eq. (6) and from Eq. (4) of the quantum field that the fields $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ do not contribute to the classical electromagnetic energy density. The longitudinal fields $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ are associated with the (3) space-like polarization of the photon in the quantum field theory; and $E^{(0)}$ and $B^{(0)}$ with the time-like photon polarization (0) in the quantum field [4–6].

Furthermore, it has been shown [4–6] that the two helicities of the photon [2] can be related to the (0) and (3) polarizations of the photon in free space through

$$\hat{B}^{(3)} = B^{(0)}\frac{\hat{J}}{\hbar}, \tag{7a}$$

where the classical $\boldsymbol{B}^{(3)}$ is the expectation value,

$$\boldsymbol{B}^{(3)} = \langle\chi|\hat{B}^{(3)}|\chi\rangle, \tag{7b}$$

and $\hat{J}$ is the angular momentum boson operator describing the photon field. For one photon, the eigenvalues of the boson operator J are the helicities, +1 and -1 [1, 2]. The helicities are usually interpreted through the transverse polarizations (1) and (2), as the left and right circularly polarized components of the classical plane wave in free space.)

Therefore $\boldsymbol{B}^{(3)}$ is a novel fundamental quantity in physical optics, and in quantum and classical field theory. In this Letter we continue a systematic survey of the consequences of $\hat{B}^{(3)}$, consequences which include the re-interpretation of well known phenomena in terms of $\hat{B}^{(3)}$ and the theoretical prediction of novel spectroscopies such as optical NMR [7]; optical Zeeman effects [8, 9]; an optical Faraday effect and optical MCD [10]; and other phenomena of light induced magnetism [11] such as the inverse Faraday effect. In Sec. 2 we develop the relation between the magnitude of $\boldsymbol{B}^{(3)}$ and the Stokes parameter S_3 [12–14] of circularly polarized electromagnetic radiation at all frequencies of the plane wave. This key equation allows S_3 to be replaced wherever it occurs [12, 13] in physical optics by the novel flux quantum $\hat{B}^{(3)}$ in its classical form $\boldsymbol{B}^{(3)}$. The latter is an axial vector [3–6, 8–11] in the propagation axis Z of the plane wave, a vector which is positive to parity inversion $\hat{P}$ and negative to motion reversal $\hat{T}$, and which has the units of tesla. It must not be confused with the well known ("standard IUPAC") oscillating, transverse magnetic $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ vectors of the electromagnetic plane wave. The vector $\boldsymbol{B}^{(3)}$ changes sign with circular polarization of the wave, vanishes in linear polarization, is purely real, and is independent of both the frequency (ω) and propagation vector ($\boldsymbol{\kappa}$). This is because it is proportional to the conjugate vector product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)*}$, where $\boldsymbol{E}^{(1)}$ is the oscillating electric vector of the wave, and where $\boldsymbol{E}^{(1)*}$ is the complex conjugate of $\boldsymbol{E}^{(1)}$ [3–6]. In this respect, $\boldsymbol{B}^{(3)}$ is a relative of the Poynting vector, $\boldsymbol{N}$, which is proportional to the vector product $\boldsymbol{E}^{(1)} \times \boldsymbol{B}^{(1)*}$ of the transverse components of the wave. However, it must be noted with particular care that the well known $\boldsymbol{N}$ and the novel $\boldsymbol{B}^{(3)}$ are fundamentally different in $\hat{P}$ *and* $\hat{T}$ symmetries, the former ($\boldsymbol{N}$) is a flux of *energy* density, the latter ($\boldsymbol{B}^{(3)}$) a flux of *magnetic* density, whose quantized counterpart is given by Eq. (7).

Section 3 illustrates the relation demonstrated in Sec. 2 between $|\boldsymbol{B}^{(3)}|$ and S_3 with two examples, circular dichroism in chiral material [3–6], and the development of measuring beam ellipticity in the well known electrical Kerr effect [15]. In both cases it is demonstrated in a straightforward way that the origin of both phenomena resides in $|\boldsymbol{B}^{(3)}|$, or more rigorously, in the quantized field operator $\hat{B}^{(3)}$. Thus ordinary, everyday, circular dichroism is shown for the first time to be magnetic in origin because it is described by changes in the magnetic $\boldsymbol{B}^{(3)}$ field as the latter traverses a chiral ensemble. A similar conclusion is drawn for ellipticity in the Kerr effect, and in general, whenever the well known Stokes parameter S_3 appears in physical optics, it indicates the

influence of $|\boldsymbol{B}^{(3)}|$. In quantum field theory, Tanas and Kielich have recently [16] discussed the significance of the four Stokes operators of the quantized field, and the present author [3–6, 8–11] has shown that the Stokes operator S_3 is proportional to the magnetostatic flux quantum $\hat{B}^{(3)}$ defined in Eq. (7).

A discussion ends the Letter on an inductive note, in which the origin of beam ellipticity in the classical and quantized field is traced to the novel vector $\boldsymbol{B}^{(3)}$ and quantum operator $\hat{B}^{(3)}$ respectively.

2. The Classical Relation Between $\boldsymbol{B}^{(3)}$ and S_3 in Free Space

In free space, the link between the classical $\boldsymbol{B}^{(3)}$ and the classical parameter S_3 introduced by Sir George Stokes in 1852 [14] is forged through the conjugate product

$$\boldsymbol{\Pi}^{(A)} = \boldsymbol{E} \times \boldsymbol{E}^* = \pm 2E_0^2 i\boldsymbol{k}, \tag{8}$$

which is *negative* to $\hat{T}$ and positive to $\hat{P}$ [17]. Here $\boldsymbol{k}$ must be a unit *axial* vector, and E_0 is the usual electric field strength amplitude of the plane wave [12, 13], a scalar quantity. The quantity i is the root of minus one, indicating that $\boldsymbol{E} \times \boldsymbol{E}^*$ is purely imaginary. The $\pm$ sign in Eq. (8) indicates that it changes sign with circular polarization, and vanishes in linear polarization. Using the fundamental free space electrodynamical relations,

$$E_0 = cB_0, \tag{9}$$

and

$$I_0 = \frac{1}{2}\epsilon_0 c E_0^2, \tag{10}$$

the conjugate product can be written in terms of the vector as follows:

$$\boldsymbol{B}^{(3)} = B_0\boldsymbol{k}, \tag{11}$$

$$\boldsymbol{\Pi}^{(A)} = \pm\left(\frac{8I_0 c}{\epsilon_0}\right)^{\frac{1}{2}}\boldsymbol{B}^{(3)} i. \tag{12}$$

Here I_0 is the beam intensity in watts per square meter, ϵ_0 is the S.I. permittivity in vacuo,

$$\epsilon_0 = 8.854 \times 10^{-12} J^{-1} c^2 m^{-1}, \tag{13}$$

and c the speed of light. It follows that $\boldsymbol{B}^{(3)}$ is related to I_0 by

$$\boldsymbol{B}^{(3)} = \left(\frac{2I_0}{\epsilon_0 c^3}\right)^{\frac{1}{2}} \boldsymbol{k} \sim 10^{-7} I_0^{\frac{1}{2}} \boldsymbol{k}, \tag{14}$$

i.e., is proportional to the square root of the intensity.

It is well known [12–14] that the Stokes parameter S_3 is the *real* quantity defined for different circular polarity by

$$\boldsymbol{\Pi}^{(A)} = \pm i S_3 \boldsymbol{k}, \tag{15}$$

and it follows immediately that the magnitude of the novel $\boldsymbol{B}^{(3)}$ vector is directly proportional to the *scalar* S_3, with

$$\boldsymbol{B}^{(3)} = \left(\frac{\epsilon_0}{8 I_0 c}\right)^{\frac{1}{2}} S_3 \boldsymbol{k}, \tag{16}$$

and

$$|\boldsymbol{B}^{(3)}| = \left(\frac{\epsilon_0}{8 I_0 c}\right)^{\frac{1}{2}} S_3. \tag{17}$$

The third Stokes parameter can also be defined through the antisymmetric part of the Fedorov light beam tensor [17], whose components can be expressed in terms of the Stokes parameters.

3. Applications in Material Media to Circular Dichroism and the Kerr Effect

Equation (17) of the preceding section implies that S_3 can be replaced wherever it occurs in free space physical optics by the quantity $(8I_0 c/\epsilon_0)^{\frac{1}{2}}|\boldsymbol{B}^{(3)}|$, which is a scalar. In material media, as opposed to free space, Kielich [18, 19] has shown that in the classical theory of the electromagnetic field all susceptibility tensors occurring in the theory of natural electronic optical activity, for example, are affected by frequency (ω) and spatial ($\boldsymbol{\kappa}$) dispersion, where ω is the angular frequency of the plane wave and where $\boldsymbol{\kappa}$ is its wave vector. The wave vector becomes [18, 19] a function of the light refractive index, and in an intense optical field the latter is no longer a scalar but a second rank tensor. In this situation, Kielich [18, 19] has shown that the diagonal and non-diagonal components then determine two distinct processes of self-induced optical anisotropy in the medium. Similar results apply to optically active media [19]. The theory of nonlinear refractivity and nonlinear optical activity developed by Kielich [19] relies, however, on the traditional electrodynamical viewpoint, which uses only the transverse fields (space-like polarizations (1) and (2)) of Sec. 2. Additional contributions are expected, however, from the longitudinal fields $i\boldsymbol{B}^{(3)}$ and $\boldsymbol{B}^{(3)}$. Any occurrence, for example, of S_3 in these theories can be replaced by a term proportional to the magnitude of $|\boldsymbol{B}^{(3)}|$. The Maxwell equations in the medium differ from those in free space through the occurrence of polarization and magnetization, treated through various molecular property tensors [15]. However, in the general theory of natural optical activity in chiral media, which can be derived from Rayleigh refringent scattering theory [15], it is well known that whatever the nature of the several molecular property tensors participating in the polarization and magnetization of the medium, the observable of the well known phenomenon of circular dichroism also has pseudo-scalar symmetry. In the theory of circular dichroism it is well known that [15], for different enantiomers for a given circular polarity or vice versa,

$$\frac{I_R - I_L}{I_R + I_L} = \pm \frac{S_3}{S_0}, \tag{18}$$

where S_3 is the zero'th order Stokes parameter, $S_0 = 2E_0^2$, and where I_R and I_L the intensities of right and left circularly polarized radiation transmitted by structurally chiral material, with

$$I_0 = I_R + I_L, \tag{19}$$

for the transmitted total beam intensity.

From Eqs. (17) and (18) we derive the important new result for left and right circular polarity

$$|\mathbf{B}^{(3)}| = \mp\left(\frac{2}{\epsilon_0 c^3 I_0}\right)^{\frac{1}{2}}(I_R - I_L), \tag{20}$$

which reveals the fundamental origin of the phenomenon of circular dichroism at all electromagnetic frequencies, because it shows that the well known and widely used observable $(I_R - I_L)$ is proportional to the magnitude of $\mathbf{B}^{(3)}$, the classical equivalent of the operator $\hat{B}^{(3)}$ (Note carefully that the vector $\mathbf{B}^{(3)}$ is *axial*; and that its magnitude, $|\mathbf{B}^{(3)}|$, is a *scalar*.)

The origin of circular dichroism resides, therefore [4], in the photon's longitudinal magnetostatic flux quantum, $\hat{B}^{(3)}$.

In other words, circular dichroism is magneto-optic in origin, and the observable $I_R - I_L$ is a spectral consequence of the interaction of $\hat{B}^{(3)}$ with structurally chiral material. From Eq. (20), $I_R - I_L$ is proportional to the real pseudo-scalar quantity $\pm|\mathbf{B}^{(3)}|$ *after it emerges from the chiral material through which the beam has passed*, i.e. after interaction has occurred between the flux quantum $\hat{B}^{(3)}$ and the appropriate molecular property tensor [15].

For a beam consisting of one photon, the observable $(I_R - I_L)$ provides an experimental measure of the transmitted elementary $|\mathbf{B}^{(3)}|$ at each frequency of that beam. Although $|\mathbf{B}^{(3)}|$ is itself independent of frequency, the interacting molecular property tensor is not. Semi-classical perturbation theory [15] gives, for linear optical activity,

$$|\mathbf{B}^{(3)}| = \left(\frac{2I_0}{\epsilon_0 c^3}\right)^{\frac{1}{2}} \tanh\left(\omega\mu_0 c l N|\zeta''_{XYZ}(g)|\right), \tag{21}$$

where μ_0 is the permeability in vacuo, ω the angular frequency of the beam, l the sample length through which the beam has passed, and ζ''_{XYZ} an appropriately averaged molecular property tensor [15], a pseudo-scalar negative to P. *Equation (21) shows that* <u>*all*</u> *circularly dichroic spectra are signatures of* $|\mathbf{B}^{(3)}|$ *of the transmitted beam.* For nonlinear optical activity Eq. (21), as shown by Kielich [19], contains additional terms as described in the introduction to this Section.

More generally, it can be shown that any phenomenon in physical optics that involves S_3 must necessarily involve $\hat{B}^{(3)}$ in its quantized or classical form, whichever is appropriate to the situation being considered. It is well known throughout the literature that there are many of these, and one example at random is the measuring beam ellipticity developed in the well known Kerr effect [15]; in which a static, uniform, electric field is applied to a fluid perpendicular

to the propagating direction, and at 45° to the azimuth, of an incident linearly polarized light beam. At transparent frequencies the development of ellipticity in the transmitted beam does not depend on the initial ellipticity.

For the emerging beam in the Kerr effect it can be shown that

$$|\mathbf{B}^{(3)}| = \left(\frac{2I_0}{\epsilon_0 c^3}\right)^{\frac{1}{2}} \sin(2\eta), \tag{22}$$

where I_0 is the transmitted beam intensity and η is its ellipticity. Equation (22) shows that the Kerr effect's beam ellipticity is governed *fundamentally* by the flux quantum $\hat{B}^{(3)}$ in its classical limiting equivalent. Note that for the incident, linearly polarized, beam, $\mathbf{B}^{(3)}$ is zero, and Eq. (22) shows clearly that the development of ellipticity is another magneto-optic phenomenon whose origin is the transmitted elementary $\hat{B}^{(3)}$.

4. Discussion

Rayleigh refringent scattering theory (chapter 3 of Ref. [15]) shows that the third Stokes parameter S_3 is associated with a change $d\eta/dz$ in ellipticity in a beam passing through a sample of thickness z. It is immediately possible to say therefore that $d\eta/dz$ measures changes in the flux quantum $\hat{B}^{(3)}$ as it passes through the sample. We arrive at the generally valid conclusion that ellipticity in the electromagnetic plane wave is directly related to $\hat{B}^{(3)}$ and that the origin of ellipticity in general is the flux quantum $\hat{B}^{(3)}$ or its classical limiting form $\mathbf{B}^{(3)}$. Ellipticity is therefore magneto-optic in fundamental origin. Note that the scalar amplitude of $\mathbf{B}^{(3)}$ is simply $B^{(0)}$, which corresponds to the time-like polarization (0) of the relativistic quantum field, and which is well known to be the scalar amplitude of the transverse, oscillating, fields $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ of the plane wave. Circular dichroism for example can be interpreted, as we have shown, through changes in $\mathbf{B}^{(3)}$ as it traverses a chiral medium. Equation (21) shows clearly that these changes are related to the Barron [15] molecular property tensor ζ''_{ijk}, which involves the well known Rosenfeld (magnetic-electric dipole) tensor, G_{ij} and the electric quadrupole tensor A_{ijk}. Note carefully, however, that although the latter is an electric molecular property tensor of the *medium*, and the former is a mixed electric/magnetic property tensor, the definition of $\mathbf{B}^{(3)}$ in terms of the third Stokes parameter in *free space* results immediately in Eq. (17). The link between $\mathbf{B}^{(3)}$ and the Barron zeta tensor is given in Eq. (21), which is another fundamental [15] result of Rayleigh refringent scattering theory. In Eq. (21) we have used the result that the magnitude of the longitudinal solution $\mathbf{B}^{(3)}$ of Maxwell's equation is directly proportional to the Stokes parameter S_3 in free space and have replaced the Stokes parameter by a term proportional to $\boldsymbol{B}^{(3)}$. This fundamental free space relation is clearly independent of *any* molecular property tensor of the medium.

Conclusions

It has been shown for the first time that the photon's elementary longitudinal magnetostatic flux quantum $\hat{B}^{(3)}$ is the origin of a) circular dichroism; b) ellipticity.

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Chapter 17

THE MAXWELLIAN LIMIT OF THE EINSTEIN-DE BROGLIE THEORY OF ELECTROMAGNETIC RADIATION

M. W. Evans

Abstract

In the Einstein-de Broglie theory of electromagnetic radiation, the photon has a rest mass of 10^{-68} *kg* (the Einstein photon mass). Within the framework of this theory, it is shown that there exist longitudinal electromagnetic fields which are analytically related to the corresponding transverse components, implying that electric and magnetic fields in vacuo are four-vectors, as first proposed by Einstein and de Broglie. The observation of these longitudinal fields would support the Einstein-de Broglie theory, and experimental arrangements are proposed.

1. Introduction

It is almost universally asserted in the contemporary literature that the photon is massless, and that the range of the electromagnetic field is infinite. However, Einstein [1–5] has proposed that the mass of the photon is about 10^{-68} *kg*, an estimate based on the Hubble constant. In consequence, the range of electromagnetic radiation is about 10^{26} *m*, (several tens of thousands of millions of light years, but finite). The finite photon mass is the basis of the Einstein-de Broglie theory of light [6] in which the Copenhagen interpretation of Bohr and others is rejected in favor of light being constituted by real Maxwellian waves coexisting with photons in Minkowski space-time. This is a causal, stochastic, model of electromagnetic radiation. In the Copenhagen interpretation, on the other hand, light is made up of waves of probability, which can never co-exist with photons in space-time. A recent experiment by Mizobuchi and Ohtake [7] contradicts the Copenhagen interpretation but can be interpreted straightforwardly [8] with the Einstein-de Broglie theory. The experiment demonstrates that electromagnetic waves and photons co-exist.

In this Letter, it is shown that the Einstein-de Broglie theory of light allows longitudinal electromagnetic fields in vacuo, fields which are related analytically to the corresponding transverse components through the equation recently derived by Evans [9–11],

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}}{E^{(0)} c i} = \frac{\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}}{B^{(0)} i} = B^{(0)} \boldsymbol{k}, \tag{1}$$

Here $\boldsymbol{B}^{(3)}$ is the longitudinal magnetic field, $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ are transverse components of the electric field, $\boldsymbol{E}^{(0)}/\sqrt{2}$ is the electric field's scalar amplitude, c the speed of light in vacuo, $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are transverse magnetic field components, and $\boldsymbol{B}^{(0)}/\sqrt{2}$ their scalar amplitude. Farahi and Evans [12] have shown that a non-zero real $\boldsymbol{B}^{(3)}$ implies a non-zero imaginary $i\boldsymbol{E}^{(3)}$, i.e. a longitudinal electric field travelling with the photon in vacuo.

A simple demonstration is given of the existence of $\boldsymbol{B}^{(3)}$ for a finite photon mass, and of the fact that Eq. (1) is the Maxwellian (zero photon mass) limit of the Einstein-de Broglie theory. Experimental observation of $\boldsymbol{B}^{(3)}$ would therefore provide support for this theory, which implies [13] that electric and magnetic fields in vacuo are four-vectors, with physically meaningful space-like and time-like components. This deduction is also the foundation for manifestly covariant electrodynamics, recently proposed by Evans [14] on the basis of Eq. (1).

In the conventional contemporary theory of electromagnetic radiation [15] the longitudinal space-like and time-like polarizations are rejected as unphysical, an arbitrary and self-contradictory procedure [16], because the d'Alembert equation, and its quantized counterpart, the Gupta Bleuler condition [17] produce four polarizations. Recent work [14] has shown that the existence of four *physically meaningful* polarizations can be reconciled straightforwardly with two helicities, coming from the theory of the Poincaré group. Even in the massless limit, therefore, the existence of four field (photon) polarizations is rigorously supported by fundamental considerations. Equation (1) shows clearly that the notion (which has gained acceptance) of arbitrarily rejecting the longitudinal fields as meaningless is untenable, because the longitudinal (3) component is directly proportional to the vector product of the transverse (1) and (2) components, the time-like component (0) being associated with the scalar field amplitudes in vacuo.

Equation (1) is therefore the fundamental link between physically meaningful longitudinal and transverse components of electromagnetic radiation, and provides new insight into the Einstein-de Broglie theory. The equation was first derived [9–11] using the Maxwell equations, equivalent to zero photon mass, but it is shown in this Letter to be valid for finite photon mass. The most important consequence of Eq. (1), however, is that it implies four physically meaningful electromagnetic field polarizations, and this is also implied [13] by the Einstein-de Broglie theory. In the contemporary theory of electrodynamics, however, the notion of abandoning two polarizations (either of the classical field or the photon) has been accepted uncritically. This notion must be questioned in view of equation (1), which is consistent with the Einstein-de Broglie theory. The question arises immediately of whether or not Eq. (1) is consistent or inconsistent with the Copenhagen interpretation, and the best way of answering this is by reference to the Mizobuchi-Ohtake experiment [7] as interpreted by Vigier [7]. Thus, even if the Copenhagen interpretation can be made to satisfy Eq. (1) theoretically, it would still be in contradiction with experimental data, implying that it is better from the outset to work within the framework of the Einstein-de Broglie theory. Significantly, it was shown by de

Broglie [18] and by Schrödinger [19] that this theory allows longitudinal as well as transverse waves in vacuo, and thus longitudinal and transverse photon polarizations, which coexist with the waves, but neither author appears to have realized the existence of equation (1). The latter rigorously links together transverse and longitudinal polarizations, and shows that the longitudinal polarization is independent of the phase of the wave, and thus satisfies [8] the Gauss Theorem in vacuo.

2. Equation (1) for Finite Photon Mass

One of the fundamental equations of the Einstein-de Broglie theory of light is

$$\Box\psi_\mu = 2\mu^2\psi_\mu, \tag{2}$$

where ψ_μ is a complex vector wave [7]. As we have mentioned, this equation was shown by de Broglie and Schrödinger to have longitudinal and transverse components. In a theory structured in Maxwell's framework, Eq. (2) can be written as a d'Alembert equation with a finite right hand side term in vacuo:

$$\Box A = -\xi^2 \boldsymbol{A}, \tag{3}$$

where ξ is a constant, and the d'Alembertian, as usual, is

$$\Box \equiv -\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{4}$$

We use the transformations,

$$\nabla \rightarrow \frac{i}{\hbar}\boldsymbol{p}, \qquad \frac{\partial}{\partial t} \rightarrow -\frac{i}{\hbar}E, \tag{5}$$

into quantum mechanics [20], where, as usual, $\boldsymbol{p}$ and E denote momentum and energy respectively, of a particle associated with the wave equation (3). This particle is the photon with finite Einstein mass

$$\frac{1}{\hbar^2}\left(-p^2 + \frac{E^2}{c^2}\right)\boldsymbol{A} = \xi^2\boldsymbol{A}, \tag{6}$$

and since $\boldsymbol{A}$ is a wave quantity which loses significance [20] in a particulate context,

$$E^2 = p^2c^2 + m^2c^4 . \tag{7}$$

This is Einstein's relativistic equation linking mass and energy, with a photon mass of

$$m = \frac{\hbar}{c}\xi . \tag{8}$$

This mass is 10^{-68} *kg* [7], calculated from the Hubble constant. Equation (8) gives the constant

$$\xi = \frac{mc}{\hbar} \sim 10^{-26}\ m^{-1} , \tag{9}$$

and the finite range of the electromagnetic field,

$$\frac{1}{\xi} = \frac{\hbar}{mc} \sim 10^{26}\ m \tag{10}$$

(some tens of thousands of millions of light years, a cosmic but finite dimension of the order of the radius of the universe). The d'Alembert equation (3) is therefore

$$\Box\boldsymbol{A} \sim 10^{-52}\ \boldsymbol{A} . \tag{11}$$

For practical purposes the right hand side is so small as to be essentially zero, and Eq. (11) reduces to the standard d'Alembert equation in vacuo,

$$\Box\boldsymbol{A} = 0 . \tag{12}$$

It is clear therefore that the Einstein-de Broglie theory of light approximates closely the Maxwell equations in the classical regime described by the wave equation (11). It is also clear that the solutions of Eq. (11) coexist with those of Eq. (7), for photons of finite mass. Equation (1) also holds to an excellent approximation in the Maxwellian description of the Einstein-de Broglie

theory, (Eq. (11)), because Eq. (1) is consistent with the d'Alembert equation (12).

Furthermore, Eq. (3) implies the following equation in magnetic flux density in vacuo,

$$\nabla^2\boldsymbol{B} = \xi^2\boldsymbol{B}, \tag{13}$$

whose physical solution [3] is an exponentially decaying longitudinal field in vacuo,

$$\boldsymbol{B}^{(3)} = B^{(0)}\exp(-\xi Z)\boldsymbol{k} . \tag{14}$$

(Using the relation

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \tag{15}$$

in Eq. (13) implies

$$\nabla^2(\nabla\times\boldsymbol{A}) = \xi^2\nabla\times\boldsymbol{A}, \qquad i.e., \ \nabla^2\boldsymbol{A} = \xi^2\boldsymbol{A}, \tag{16}$$

whose Lorentz covariant form is obtained [20] by replacing the Laplacian by the d'Alembertian. Thus Eq. (3) implies Eq. (13) and vice versa.)

For all practical purposes, Eq. (14) is

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}, \tag{17}$$

which is the left hand side of Eq. (1). It is therefore straightforward to show that Eq. (1) corresponds to the zero photon mass, Maxwellian, form of the Einstein-de Broglie theory of light.

3. Discussion and Experimental Consequences

It is important to note that Eq. (1) is implied in the work of Moles and Vigier [21] and that of Bass and Schrödinger [22]. This has become clear through the following comments by Vigier [23]. The $\boldsymbol{E}^L$ field of Moles and Vigier [21] is parallel to the $\boldsymbol{B}^{(3)}$ field of equation (1). The three vectors $\boldsymbol{E}_1^T$ and $\boldsymbol{B}_1^T$ are then defined by

$$E_i^T = \frac{1}{2}\epsilon_{ij4}F^{j4}, \qquad B_i^T = \frac{1}{2}\epsilon_{ijk}F^{jk}, \tag{18}$$

in the notation of Moles and Vigier [21]. Equation (1) of this paper can then be written in that notation as

$$B_k^{(3)} = \frac{1}{2}\epsilon_{kij}\frac{E_i^T E_j^{T*}}{(B_0 ci)} = \frac{1}{2}\epsilon_{kij}\frac{B_i^T B_j^{T*}}{(B_0 i)}, \tag{19}$$

so that B_k is parallel to k_k and $B_0^2 \propto (J_0)^{\frac{1}{2}}$ keeping to the notation of Ref. [21]. Since all field amplitudes are multiplied in that notation by $\exp(-(k\boldsymbol{X} - \omega t))$, i.e., $k_\mu = -i\hbar(A_\alpha^*\partial_\mu A_\alpha - c.c.)$ and $\boldsymbol{E}^T$, $\boldsymbol{H}^T$ and $\boldsymbol{k}$ are mutually perpendicular, we have [21],

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k} \text{ and } \boldsymbol{E}^L \| \boldsymbol{k}, \tag{20}$$

which is equation (1) of this paper.

Therefore equation (1) is straightforwardly implied by the equations of Moles and Vigier [21] in a paper which discusses the possible physical consequences of the existence of a non zero photon mass in the interaction of light and matter. It becomes clear that the $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ fields are the Maxwellian limiting forms of the longitudinal fields implied by the Einstein-de Broglie theory of light, and for all practical purposes, are indistinguishable (because ξ in Eq. (14) is of the order $10^{-26}\ m^{-1}$). Thus, experimental evidence for $\boldsymbol{B}^{(3)}$ would be evidence for finite photon mass, because it would be consistent both with the Einstein-de Broglie theory and with *other* experimental evidence for finite photon mass reviewed by Vigier [8]. There is no experimental evidence for zero photon mass.

The following proposed experiments, if positive, can therefore be considered [23] as evidence for finite photon mass as well as for the longitudinal field $\boldsymbol{B}^{(3)}$; the exchange of longitudinal photons; the action of longitudinal photons on matter, and for the fact that sources emit longitudinal as well as transverse photons in *three* polarization states, not two as in the conventional interpretation of light.

The major experimental consequence of Eq. (1), (or Eq. (17)), is that there exist longitudinal fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ which are proportional to the square root of light intensity in vacuo [9–11]. In the context of $\boldsymbol{B}^{(3)}$, for example, the field has all the attributes of a magnetic flux density [9–11, 14] and should

therefore produce optical effects in analogy with effects due to a conventional magnetic field. Examples have been proposed and discussed in some detail in the literature [9–11, 14] and include the following, all proportional to the square root of light intensity (watt m^{-2}) of a circularly polarized laser pulse:

(a) An inverse Faraday effect (magnetization).
(b) An optical Faraday effect (azimuth rotation).
(c) An optical Zeeman effect (spectral splitting).
(d) Optically induced shifts in ESR and NMR.
(e) Optically induced Cotton Mouton and Majorana effects.
(f) Optically induced forward backward birefringence effects.
(g) Extra effects in Compton scattering.
(h) Other magnetic effects.

Furthermore, it has been shown [9–11, 14] that conventional interpretations of such well known phenomena as simple absorption, ellipticity, circular dichroism, the Kerr effect, antisymmetric scattering, and well known parameters such as those of Stokes [15], can be developed in terms of $\boldsymbol{B}^{(3)}$ with equal validity as the conventional interpretation in terms of $\boldsymbol{E}^{(1)}$, $\boldsymbol{E}^{(2)}$, $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$, the oscillating, transverse fields.

We can therefore conclude that the Einstein-de Broglie theory, which has been shown to be experimentally verifiable by recent work [7] produces physically meaningful longitudinal magnetic and electric fields which are linked to the transverse fields by Eq. (1), and which are expected to produce new effects [9–11, 14] as summarized in this Section. It is interesting to note that there is a growing literature [23] on new, physically meaningful, solutions of Maxwell's equations in vacuo. Risset, for example, has shown [24] that there is a class of non-diffracting solutions which emerge as a superposition of circularly polarized evanescent waves, called "lip waves". They appear as right and left circularly polarized, non-diffracting, TEM waves, and two alternative interpretations have been proposed by Risset [24], namely that in these solutions, the electromagnetic field progresses, as a whole, along its axis with a phase velocity c, or that the field spins, without apparent propagation, around this axis, with angular velocity $\epsilon(\omega/2)$, where $\epsilon = \pm 1$, and ω is the angular frequency. Applying Eq. (1) to these solutions, it becomes clear that lip waves also imply a longitudinal photon polarization, given in the notation of Risset [24] by

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{B}\times\boldsymbol{B}^*}{B_0 i} = 2B_0\delta^4\frac{(x^2 - y^2 + 4xy)^2}{(x^2+y^2)^4}k, \tag{21}$$

where δ is a normalization constant for length [24]. A schematic of the lip waves is given by Risset on page 1057 of Ref. [24], to which should be added the longitudinal fields generated as above by his novel solutions of Maxwell's equations.

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Appendix 1: THE EFFECT OF $\boldsymbol{B}^{(3)}$ AND $i\boldsymbol{E}^{(3)}$ ON THE FUNDAMENTALS OF THE OLD QUANTUM THEORY

M. W. Evans

The core logic of the foregoing papers in this book asserts that there exists a cyclically symmetric algebra,

$$\left.\begin{array}{l} \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \\ \boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} = iB^{(0)}\boldsymbol{B}^{(1)*}, \\ \boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} = iB^{(0)}\boldsymbol{B}^{(2)*}, \end{array}\right\} \tag{A1}$$

which implies that the real field $\boldsymbol{B}^{(3)}$ is non-zero in free space. Furthermore, this conclusion is reinforced by the experimental observation of the conjugate product in, for example, the inverse Faraday effect. This deduction changes fundamentally the current appreciation of electrodynamics and therefore changes the principles on which the old quantum theory was derived, for example the Planck law, and the quantum hypothesis. In this Appendix we carefully examine the original papers which led to the emergence of the fundamental ideas of the old quantum theory, and demonstrate in each case that the existence of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ is rigorously consistent with the foundations of these theories. In so doing we rely heavily on the monograph by Pais [1] which carefully analyses the key papers of the old quantum theory, for example papers by Planck, Einstein, and Bose. In so doing it is shown that the Planck radiation law, and its classical equivalents, the Rayleigh-Jeans, Steffan-Boltzmann and Wien laws, can be derived with, and are not affected by, the knowledge that there exists the cyclical algebra (A1) which signals the existence of the longitudinal fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ in free space. This conclusion is a radical departure from the conventional one that the solutions of Maxwell's equations are transverse, as described in the text. Indeed, Eqs. (A1) mean that the very existence of transverse solutions implies that of longitudinal solutions. In the quantum theory, the conventional notion that the photon, regarded as a particle, have only two transverse degrees of polarization is changed to a photon with a third, axial, dimension which is relativistically invariant, being the longitudinal unit axial vector $\boldsymbol{e}^{(3)}$ which defines the field $\boldsymbol{B}^{(3)}$. This is a conclusion which is consistent with the fact that the photon may have mass, and that classical electrodynamics may be more rigorously described by the Proca equation rather than the d'Alembert equation in free space. In the text we have also seen that this deduction means that the eigenvalues of the photon's intrinsic, or spin, angular momentum are $-\hbar$, 0 and $\hbar$; and not $-\hbar$ and $\hbar$ as in the conventional theory. It has also been shown that the quantum mechanical equivalent of $\boldsymbol{B}^{(3)}$ is the operator,

$$\hat{B}^{(3)} = B^{(0)}\frac{\hat{J}}{\hbar}, \tag{A2}$$

and that the electromagnetic energy density associated with $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ is zero,

$$U^{(3)} = \frac{1}{\mu_0}\boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)} + \epsilon_0 i\boldsymbol{E}^{(3)}\cdot i\boldsymbol{E}^{(3)} = 0. \tag{A3}$$

The longitudinal Poynting vector is zero, because $\boldsymbol{B}^{(3)}$ is parallel to $i\boldsymbol{E}^{(3)}$, and so the contribution of both of these vectors to light intensity (watts per square meter) is zero. The field $\boldsymbol{B}^{(3)}$ therefore has no Planck energy because the longitudinal classical electromagnetic energy density and flux density is zero. For this reason, we have seen that $\boldsymbol{B}^{(3)}$ is not absorbed or re-emitted at any frequency, but can be detected by its "elastic" interaction with matter, an interaction in which angular momentum is exchanged between field and matter, but no energy.

These conclusions have several implications for the very foundations of the old quantum theory, which emerged [2] following the derivation in November 1900 by Planck of his law of radiation intensity, a derivation which required the assumption that there exist energy elements which are proportional to frequency, ν, through the Planck constant h,

$$\epsilon = h\nu. \tag{A4}$$

Einstein [3] later considered the meaning of Planck's derivation, and concluded that energy is absorbed and re-emitted by matter in integral units of $h\nu$. His later [4] introduction of the A and B coefficients fused together this concept with the Bohr atom, resulting in the expression,

$$E_m - E_n = h\nu, \tag{A5}$$

where E_n and E_m are energy levels of states n and m respectively of an atom. Thus absorption of light takes place only if there is a transfer of $h\nu$ of energy from radiation to atom. Finally, we are concerned in this Appendix with the derivation by Bose [4] of the Planck law without classical electrodynamics, a derivation which assumed that the photon has only two degrees of transverse polarization in free space, corresponding classically to right and left circular polarization. This necessitated the introduction by Bose of the idea of a particle with only two degrees of polarization in three dimensional space.

The existence of Eqs. (A1), which emerged in 1992, sixty eight years after Bose's derivation, means that the basic assumptions used by him, and also by his predecessors, must be carefully re-examined. It is the purpose of this Appendix to do this by reference to the commentary given by Pais [1] on the original papers, papers which are well known to lead to the accepted contemporary picture, but which contain errors and assumptions which in the light of years of research, have become untenable. For example, the statistical methods used by Planck and Bose are incorrect, and some of the assumptions used by Einstein are no longer tenable. In each case we indicate specifically the effect of a non-zero $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ on the original derivations in the original papers. For example, we discuss the effect of a third, axial, degree of polarization on the derivation by Bose in 1924 of the Planck law.

The Effect of the Ghost Field on Planck's law

As described by Pais [1], Planck based his quantization of energy on a classical relation that he derived to describe the joint equilibrium of matter and radiation,

$$\rho(\nu, T) = \frac{8\pi\nu^2}{c^3} U(\nu, T), \tag{A6}$$

where U is oscillator equilibrium energy and where $\rho(\nu, T)$ is the spectral density, i.e. the energy density per unit volume for frequency ν. Here T denotes temperature. Planck's quantization of energy rests fundamentally on an assumption equivalent to [1],

$$U = \frac{P}{N} h\nu, \tag{A7}$$

where P is the number of indistinguishable energy elements, and N the number of distinguishable oscillators. It was assumed by Planck that $U_N = P\epsilon$, i.e. that the total energy is made up of finite elements ϵ, and he derived a relation which asserts $\epsilon = h\nu$. As Pais [1] carefully describes, the statistical procedure used by Planck is erroneous, but the result $\epsilon = h\nu$ is accepted by contemporary science. The universal Planck constant h is therefore

$$h = \frac{\epsilon}{\nu}, \tag{A8}$$

i.e. is a finite energy element divided by frequency.

Our purpose here is to investigate the effect of the ghost field $\boldsymbol{B}^{(3)}$ on the Planck relation (A8), and on the Planck radiation law,

$$\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3}(e^{h\nu/kT} - 1)^{-1}, \tag{A9}$$

for the spectral density $\rho(\nu, T) = I(\nu)/c$ where $I(\nu)$ is the intensity of radiation in watts per unit area, the time average of the Poynting vector as discussed in the text. This question is answered straightforwardly, because

$$\boldsymbol{N}^{(3)} = \frac{1}{\mu_0}\boldsymbol{B}^{(3)} \times i\boldsymbol{E}^{(3)} = \boldsymbol{0}, \tag{A10}$$

and there is no effect on $\rho(\nu, T)$ of $\boldsymbol{B}^{(3)}$ (and $i\boldsymbol{E}^{(3)}$). Since $\boldsymbol{B}^{(3)}$ is independent of phase, it is associated with zero frequency, ν, and therefore the oscillator energy U associated with $\boldsymbol{B}^{(3)}$ is zero,

$$U = \frac{P}{N} h\nu \xrightarrow{\nu \to 0} 0. \tag{A11}$$

This means also that the finite energy element ϵ associated with $\boldsymbol{B}^{(3)}$ is zero. These deductions are consistent with the fact that the classical electromagnetic energy density is formally zero,

$$U^{(3)} = 0, \tag{A12}$$

a result which, as we have seen in the text, is also consistent with Poynting's theorem. These considerations can be summarized by the limiting procedures

$$\left(\frac{\epsilon}{\nu}\right)_{\nu \to 0} = h, \tag{A13a}$$

$$\left(\frac{U}{\nu}\right)_{\nu\to 0} = h\frac{P}{N}, \tag{A13b}$$

i.e. the limit of ϵ/ν as ν goes to zero is h. Similarly the ratio $U/(h\nu)$ as frequency goes to zero approaches the limit P/N. Therefore in the limit of zero frequency, the entropy of the linear oscillator, derived *classically* by Planck [1] remains finite,

$$S = k\left[\left(1+\frac{U}{h\nu}\right)\log_e\left(1+\frac{U}{h\nu}\right) - \frac{U}{h\nu}\log_e\frac{U}{h\nu}\right]. \tag{A14}$$

Therefore, despite the fact that $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are associated with zero U, ν, and ϵ, they are states of electromagnetism, and for this reason are associated with finite classical entropy S.

As described by Pais [1], Planck, in 1900, incorrectly forced Eq. (A14) to be identical with

$$S = k\left[\left(1+\frac{U}{\epsilon}\right)\log_e\left(1+\frac{U}{\epsilon}\right) - \frac{U}{\epsilon}\log_e\frac{U}{\epsilon}\right], \tag{A15}$$

using an incorrect statistical procedure.

In March, 1905, Einstein investigated [3] the meaning of Planck's Eq. (A9) with an independent quantization hypothesis, using as starting point the classical equi-partition theorem of Maxwell and Boltzmann. The equilibrium energy of a one dimensional material harmonic oscillator is

$$U(\nu, T) = kT, \tag{A16}$$

showing that if the frequency of the oscillator vanishes, there is no contribution to kT. The limit

$$\left(\frac{kT}{\nu}\right)_{\nu\to 0} = h\frac{P}{N}, \tag{A17}$$

is finite, showing that a zero frequency oscillator has no temperature. The longitudinal $\boldsymbol{B}^{(3)}$ adds nothing to kT. When there is energy, it is equally partitioned in light among two transverse modes, but since the longitudinal $\boldsymbol{B}^{(3)}$ (with $i\boldsymbol{E}^{(3)}$) generates no energy, there is no longitudinal energy to partition. The existence of these ghost fields does not violate the Maxwell-Boltzmann law

of equi-partition of energy. The Maxwell Theorem, i.e. that the numerical value of the radiation pressure equals one third of the energy per unit volume, is also not affected by $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$, because the latter generate no electromagnetic energy per unit volume, and thus no radiation pressure. This is consistent with the fact that $\boldsymbol{B}^{(3)}$ is generated from photon angular momentum, and since it is phase free, contributes nothing to photon linear momentum. However, $\boldsymbol{B}^{(3)}$ does produce magnetization, and thus produces the non-zero radiation torque,

$$T_q = -\boldsymbol{m}\times\boldsymbol{B}^{(3)}, \tag{A18}$$

when $\boldsymbol{B}^{(3)}$ interacts with matter in which there is a net magnetic dipole moment $\boldsymbol{m}$. The integral over this radiation torque is the angular momentum imparted elastically by $\boldsymbol{B}^{(3)}$ to matter without exchange of energy. As discussed by Pais [1] the equi-partition theorem applied to the classical equation (A6) results in the Rayleigh-Einstein-Jeans law:

$$\rho(\nu, T) = \frac{8\pi\nu^2}{c^3}kT, \tag{A19}$$

from which the spectral density of states vanishes at zero ν. Thus $\boldsymbol{B}^{(3)}$, having no ν, produces no spectral density, a deduction which is consistent with the fact that the Poynting vector generated by $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ vanishes, Eq. (A10), with

$$I^{(3)}(\nu) = \langle|\boldsymbol{N}^{(3)}|\rangle = 0. \tag{A20}$$

We are now ready to discuss the effect of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ on the light quantum hypothesis of Einstein, proposed in March, 1905, which states that monochromatic radiation of low density behaves in thermodynamic respect as if it consists of mutually independent energy quanta of magnitude $h\nu$. These energy quanta were named "photons" by Lewis in 1926. As discussed by Pais [1] the basis of Einstein's quantization is that Eq. (A6), derived classically by Planck, is also valid in the old quantum theory. For our purposes we deduce immediately that as $\nu\to 0$, the spectral density ρ and the oscillator equilibrium energy U both vanish in the quantum theory. *Therefore* $\hat{B}^{(3)}$, *the longitudinal photomagneton, has no Planck energy.* Einstein's quantization [3] is related to the spectral density $\rho(\nu, T)$, whereas we recall that Planck's quantization is related to U. Einstein deduced in his March 1905 paper that the basis of Planck's theory is that *the energy of a Planck oscillator can take on only those values that are integral multiples of* $h\nu$; in absorption or emission the energy changes by integral multiples of $h\nu$, so that if $\nu = 0$ there is no absorption or emission. We conclude that $\boldsymbol{B}^{(3)}$ cannot be absorbed or emitted by an atom or molecule as in the text. This is a deduction that is wholly consistent with the light quantum hypothesis of Einstein, the basis of quantum theory. The interaction of $\boldsymbol{B}^{(3)}$ with matter is elastic, and involves no changes in energy $h\nu$. There is no exchange of photons $h\nu$ from radiation to matter in this context. Thus, the inverse Faraday effect can occur far from resonance, as observed experimentally as recounted in the text.

Consideration some years later of the Bohr atom produced the Einstein A_{mn} and B_{mn} coefficients, giving the conventional contemporary picture of absorption of a photon,

$$E_m - E_n = h\nu, \qquad \frac{A_{mn}}{B_{mn}} = \rho(\nu, T)\left(e^{(E_m - E_n)/kT} - 1\right), \tag{A21}$$

where E_m and E_n are the atomic energy levels of states m and n respectively. The existence of $\boldsymbol{B}^{(3)}$ therefore does not affect the theory of the Einstein A and B coefficients.

In 1924, Bose [4] derived the Planck radiation law, Eq. (A9), without recourse to classical electrodynamics. In so doing, Bose was forced to assert that there exists a particle with two states of polarization in order to obtain the correct premultiplier. This procedure, as explained by Pais [1] amounts to replacing the counting of standing waves in a cavity of volume V with the counting of cells in a one particle, position-momentum, phase space dxdp. For our purposes we see immediately that if there is a "wave without frequency", such as $\boldsymbol{B}^{(3)}$, then there are no additional cells to count. We arrive at the important result that $\boldsymbol{B}^{(3)}$ is compatible with Bose's derivation of Planck's law. As recounted by Pais: "When Bose introduced his polarization factor of two, he noted that "it seemed required to do so" "...and ".....who in 1924 had heard of a particle with two states of polarization?".

The essence of Bose's method, as described by Pais [1] is therefore to integrate the one particle phase space element dxdp over V and over all momenta between $\boldsymbol{p}$ and $\boldsymbol{p} + d\boldsymbol{p}$; *then supply a factor of two to count polarizations*. It was not known by Bose [4] why this factor was two, and it was used to obtain the correct premultiplier in Planck's radiation law. Pais recounts further [1] that it was not clear to Einstein why the photon spin appeared only to have two values, a deduction which Pais bases on a reply by Einstein to a letter from Ehrenfest asking for the relativistically correct evaluation of spin for a massless particle, the photon, shortly after the Stern Gerlach experiment had been explained in terms of electron spin. Indeed Einstein at that point in time thought that angular momentum conservation could not be sustained in a relativistically correct manner for a massless particle.

It continues to be asserted conventionally that the photon, being a massless boson, has two degrees of polarization. Analysis of this statement reveals

however, that it must refer to polar vectors. A *polar* vector in the direction (Z) of propagation of the massless boson must be zero in the observer frame by special relativity (usually referred to as the Fitzgerald-Lorentz contraction [1]). An *axial* vector in Z, however, is relativistically invariant. Thus,

$$\boldsymbol{k}^{(A)} = \boldsymbol{k}_0^{(A)}, \qquad \boldsymbol{k}^{(P)} = \boldsymbol{k}_0^{(P)}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}, \tag{A22}$$

where v is the velocity of the rest frame with respect to the observer frame. If $v = c$, then a polar unit vector $\boldsymbol{k}^{(P)}$ is zero in the observer frame. An axial unit vector $\boldsymbol{k}^{(A)}$ is NOT zero in the observer frame. Thus, the pure real field $\boldsymbol{B}^{(3)}$, as discussed in the text, is non-zero and relativistically invariant. (It may also be shown using special relativity that $i\boldsymbol{E}^{(3)}$ is relativistically invariant.) Since Maxwell's equations are also invariant in free space, it follows that the wave fields $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are similarly invariant. Equation (A1) shows that this must be the case, and we conclude that *the photon has three degrees of polarization*. We have seen that this is entirely consistent with Planck's radiation law, because the axial $\boldsymbol{B}^{(3)}$ has no energy.

This is therefore the origin of Bose's assumption that a factor two is sufficient.

In fundamental special relativity [5] the angular momentum of a particle behaves as follows. Define a frame K in which the particle moves with velocity v in Z. K is the frame of the *observer*. Define a frame K_0 in which the body does not translate. K_0 is the *rest frame*. It may then be shown that [5]:

$$J_Z = J_Z^{(0)}, \qquad J_Y = \gamma J_Y^{(0)}, \qquad J_X = \gamma J_X^{(0)}, \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}. \tag{A23}$$

If $v = c$, then $J_Z = J_Z^{(0)}$; $J_Y^{(0)} = J_X^{(0)} = 0$ for all J_Y and J_X of the observer frame. Thus the transverse angular momenta in the observer frame become *indeterminate*. The longitudinal angular momentum *remains invariant*.

In the quantum theory, the projection $\hat{J}_Z$ of the angular momentum operator in Z is relativistically invariant. From Eq. (A2), the projection of $\hat{B}^{(3)}$ in axis Z also remains relativistically invariant and well defined at the speed of light.

The eigenvalues of $\hat{J}_Z$ for a spin one boson are well known to be $-\hbar$, 0, $\hbar$, and if the photon mass is finite there is no reason for the conventional assertion that the eigenvalue 0 is not present. Since $\hat{B}^{(1)}$, $\hat{B}^{(2)}$ and $\hat{B}^{(3)}$ are all three non-zero at the speed of light, the photon has three polarizations in the basis (1), (2), (3) defined in the text, and therefore has three degrees of polarization in any convenient basis such as the Cartesian X, Y, Z. The replacement of the eigenvalues $(-\hbar, 0, \hbar)$ by $(-\hbar, \hbar)$ for the massless photon is *not* equivalent to asserting that $\hat{B}^{(3)} =? \hat{0}$. This would contradict the inverse Faraday effect, i.e. contradict experimental data, and destroy the structure (A1). Recognizing the existence of $\hat{B}^{(3)}$ restores self-consistency to the theory. For example, $\hat{B}^{(1)}$, $\hat{B}^{(2)}$ and $\hat{B}^{(3)}$ are proportional to rotation generators and spherical harmonic components which are all non-zero. In a conventional text, however, such as the excellent monograph by Atkins [6], we find an argument of the following type on p. 211 of the second edition. "In the case of a particle travelling at the speed of light (like photons, and perhaps also neutrinos and gravitons) relativity forbids any component (of angular momentum) perpendicular to the propagation direction, and so, whatever the spin, only two components are allowed." However, since $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$, the standard transverse fields, are proportional as in the text to rotation operators which become angular momenta in quantum theory, these angular momenta are clearly allowed. Since $\hat{B}^{(3)}$ is proportional to a relativistically invariant rotation generator, which becomes a longitudinal angular momentum in quantum theory, this too is allowed, and $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are tied together ineluctably as in Eq. (A1).

Atkins' argument for the non-existence of the eigenvalue 0 for the massless photon is therefore perfectly consistent with the existence of the Lie algebra, Eq. (A1). On the other hand, the existence of the eigenvalue 0 is consistent with the notion of finite photon mass.

Finally, the most rigorous treatment of the eigenvalues of the massless photon was given by Wigner in 1939 [7], in terms of the ratio of the Pauli Lubansky pseudo-vector to the Wigner generator of space-time translations. This conventionally leads to helicities +1 and -1, corresponding to eigenvalues $+\hbar$ and $-\hbar$ in the quantum theory. However, the Wigner theory also allows the value 0 for helicity when there is photon mass, however small, because the Pauli Lubansky pseudo-vector is directly proportional to the rotation generator of the Lorentz group, of which there are *three* space-like components, as in the text. As we have seen in the text, the conventional assertion $\hat{B}^{(3)} =? \hat{0}$ in Minkowski space incorrectly assumes that the rotation generator $\hat{J}^{(3)}$ vanishes.

We arrive at the fundamental conclusion that the eigenvalues of the massless photon in quantum field theory are $-\hbar$, 0, and $\hbar$. The absurdity of having to assert that a particle have two states of polarization in three dimensional space is therefore removed. This is compatible with the experimentally verified existence of $\hat{B}^{(3)}$.

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Appendix 2: LONGITUDINAL FIELDS IN FREE SPACE AND THE DUAL TRANSFORM OF SPECIAL RELATIVITY

M. W. Evans

The dual transform of special relativity provides an elegant and concise method for obtaining several significant results in the text. The dual transform is based on a general theorem of tensor algebra which asserts that the dual of a four dimensional second rank tensor $\boldsymbol{C}$ may be defined by $\boldsymbol{C}^{(D)}$, whose components are given by,

$$C^{D}_{ij} = \frac{1}{2!}\delta_{ijkl}C_{kl}, \tag{B1}$$

where δ_{ijkl} is the four dimensional Levi-Civita symbol. Applying the result (B1) to the four-tensor of electromagnetism, we obtain the standard result, in S.I. units:

$$\epsilon_0\begin{bmatrix} 0 & cB_Z & -cB_Y & -iE_X \\ -cB_Z & 0 & cB_X & -iE_Y \\ cB_Y & -cB_X & 0 & -iE_Z \\ iE_X & iE_Y & iE_Z & 0 \end{bmatrix} \underset{dual}{\rightarrow} \epsilon_0\begin{bmatrix} 0 & -iE_Z & iE_Y & cB_X \\ iE_Z & 0 & -iE_X & cB_Y \\ -iE_Y & iE_X & 0 & cB_Z \\ -cB_X & -cB_Y & -cB_Z & 0 \end{bmatrix}, \tag{B2}$$

i.e.

$$c\boldsymbol{B} \rightarrow i\boldsymbol{E}, \qquad -i\boldsymbol{E} \rightarrow c\boldsymbol{B}. \tag{B3}$$

These dual transformations leave the Maxwell equations invariant in vacuo, and are consistent with the fact that

$$U^{(3)} \equiv c^2B^{(3)2} - E^{(3)2} = 0, \tag{B4}$$

is a Lorentz invariant in vacuo, along with $2ic\boldsymbol{B}^{(3)}\cdot\boldsymbol{E}^{(3)}$.

Several important results flow immediately from these standard results.

1. From the inverse Faraday effect there exists experimentally a pure real $c\boldsymbol{B}^{(3)}$ which is dual to the pure imaginary $-i\boldsymbol{E}^{(3)}$ from Eq. (B3). There is no real, physical, longitudinal electric field in vacuo. This deduction is consistent with the Fitzgerald-Lorentz contraction and with $\hat{P}$, and $\hat{T}$ symmetry considerations, as in the text.

2. The longitudinal, electromagnetic energy density $U^{(3)}$ is a Lorentz invariant, and vanishes in free space. This is consistent with the quantum theory, as discussed in Appendix 1.

3. The various possible contributions to the Poynting vector of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ vanish in vacuo. For example,

$$(\boldsymbol{B}^{(1)} + \boldsymbol{B}^{(3)})\times(\boldsymbol{E}^{(1)} - i\boldsymbol{E}^{(3)}) = \boldsymbol{B}^{(1)}\times\boldsymbol{E}^{(1)}, \tag{B5}$$

because $\boldsymbol{B}^{(3)}\times\boldsymbol{E}^{(1)}$ is dual with and equal to $-i\boldsymbol{E}^{(3)}\times\boldsymbol{B}^{(1)}$. Similarly,

$$(\boldsymbol{B}^{(1)}\times\boldsymbol{E}^{(1)})^* = (\boldsymbol{B}^{(2)}\times\boldsymbol{E}^{(2)}) = (\boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)})\times(\boldsymbol{E}^{(2)} + i\boldsymbol{E}^{(3)}), \tag{B6}$$

because $\boldsymbol{B}^{(3)}\times\boldsymbol{E}^{(2)}$ is dual with and equal to $i\boldsymbol{E}^{(3)}\times\boldsymbol{B}^{(2)}$.

4. In vacuo:

$$\boldsymbol{B}^{(1)} \rightarrow \frac{\boldsymbol{E}^{(1)}}{c}, \qquad \boldsymbol{B}^{(2)} \rightarrow \frac{-\boldsymbol{E}^{(2)}}{c}, \qquad \boldsymbol{B}^{(3)} \rightarrow \frac{-i\boldsymbol{E}^{(3)}}{c}, \tag{B7}$$

therefore

$$\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \tag{B8}$$

is dual with

$$\boldsymbol{E}^{(1)}\times\boldsymbol{E}^{(2)} = cB^{(0)}(i\boldsymbol{E}^{(3)}) = ic^2B^{(0)}\boldsymbol{B}^{(3)}. \tag{B9}$$

5. Equations (B2) and (B8) emphasize the fact that the pure imaginary $-i\boldsymbol{E}^{(3)}$ is dual with the pure real $c\boldsymbol{B}^{(3)}$, and has the same axial symmetry, as discussed in the text. Thus, Lorentz invariants are formed from the

square of the complex vector $c\mathbf{B}^{(3)} - i\mathbf{E}^{(3)}$. In writing this vector it is assumed implicitly in standard special relativity that the discrete symmetries of $c\mathbf{B}^{(3)}$ and $-i\mathbf{E}^{(3)}$ are the same. (It is not possible to form the sum of two quantities whose discrete symmetries differ.)

6. The vector potential for $\mathbf{B}^{(3)}$, defined by the pure real $\nabla\times\mathbf{A}_{(3)}$ is dual with that for $-i\mathbf{E}^{(3)}/c$ i.e.,

$$(\nabla\times\mathbf{A}_{(3)}) \to -\frac{i}{c}(\nabla\times\mathbf{A}_{E(3)}), \tag{B10}$$

where $\mathbf{A}_{(3)}$ and $\mathbf{A}_{E(3)}$ are both pure real quantities, Hertz vector potentials. The existence of $\mathbf{A}_{(3)}$ signals the existence of an optical Bohm-Aharonov effect as in the text. The Hertz potentials are well known to be defined in general by,

$$\mathbf{B} = \nabla\times\mathbf{A}, \qquad \mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \tag{B11a}$$

$$\mathbf{E} = \nabla\times\mathbf{A}_E, \qquad \mathbf{B} = -\nabla\phi_E - \frac{\partial\mathbf{A}_E}{\partial t}, \tag{B11b}$$

Since $\mathbf{B}^{(3)}$ is pure real, solenoidal, and phase-free, then there is no real part of $i\mathbf{E}^{(3)}$ from Eq. (B11a) and since $i\mathbf{E}^{(3)}$ is pure imaginary there is no real imaginary part of $\mathbf{B}^{(3)}$ from Eq. (B11b). This is, of course, consistent with Eq. (B3) and with the text.

Appendix 3: **THE VECTOR POTENTIAL OF $\mathbf{B}^{(3)}$**

M. W. Evans

In this appendix it is shown that the pure real vector potential $\mathbf{A}_{(3)}$ of $\mathbf{B}^{(3)}$ can be defined directly from the Maxwell equations in vacuo. This anticipates the existence of an optical Bohm-Aharonov effect as in the text and Appendix 2. Using the notation,

$$\left.\begin{aligned} A_\mu &\equiv (A_1, A_2, A_3, A_4) = c\epsilon_0\left(A_X, A_Y, A_Z, \frac{i\phi}{c}\right), \\ x_\mu &\equiv (x_1, x_2, x_3, x_4) = (X, Y, Z, ict), \end{aligned}\right\} \tag{C1}$$

then, from standard special relativity,

$$B_i = \mu_0 c\left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}\right), \qquad i = 1, 2, 3, \tag{C2}$$

i.e. in Cartesian coordinates,

$$\mathbf{B} = B_X\mathbf{i} + B_Y\mathbf{j} + B_Z\mathbf{k}, \tag{C3}$$

where

$$B_X = \frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z}, \qquad B_Y = \frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X}, \qquad B_Z = \frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y}. \tag{C4}$$

In spherical coordinates (1), (2) and (3),

$$\mathbf{B} = \mathbf{B}^{(1)} + \mathbf{B}^{(2)} + \mathbf{B}^{(3)}, \tag{C5}$$

so that

$$B_X = -\sqrt{2}B^{(0)}\sin\phi, \qquad B_Y = \sqrt{2}B^{(0)}\cos\phi, \qquad B_Z = B^{(0)}, \tag{C6}$$

where ϕ is the phase of the plane wave in vacuo, propagating in Z, i.e.:

$$\phi = \omega t - \kappa Z. \tag{C7}$$

From Eqs. (C2) and (C6)

$$\frac{\partial A_Y}{\partial X} = -\frac{\partial A_X}{\partial Y} = \frac{B^{(0)}}{2}, \tag{C8}$$

giving the result

$$\boldsymbol{A}_{(3)} = \frac{B^{(0)}}{2}(-Y\boldsymbol{i} + X\boldsymbol{j}) + constant \neq \boldsymbol{0}, \qquad \boldsymbol{B}^{(3)} = \nabla \times \boldsymbol{A}_{(3)} \neq \boldsymbol{0}, \tag{C9}$$

as in the text. Therefore $\boldsymbol{A}_{(3)}$ is a non-zero vector field with zero divergence and time derivative,

$$\boldsymbol{A}_{(3)} \neq \boldsymbol{0}, \qquad \nabla \cdot \boldsymbol{A}_{(3)} = 0, \qquad \frac{\partial \boldsymbol{A}_{(3)}}{\partial t} = \boldsymbol{0}. \tag{C10}$$

In spherical coordinates,

$$\boldsymbol{A}_{(3)} = \frac{B^{(0)}}{2\sqrt{2}}((iX - Y)\boldsymbol{e}^{(1)} + (-iX - Y)\boldsymbol{e}^{(2)}) + constant, \tag{C11}$$

i.e. $\boldsymbol{A}_{(3)}$ is a pure real sum of two complex conjugates, or modes.

The complete vector potential of the electromagnetic plane wave in vacuo is given by

$$\boldsymbol{A} = A_X\boldsymbol{i} + A_Y\boldsymbol{j} + A_Z\boldsymbol{k}, \tag{C12}$$

where, $A_X = -\frac{B^{(0)}}{2}\left(\frac{\sqrt{2}}{\kappa}\sin\phi + Y\right) + constant_X$, $A_Y = \frac{B^{(0)}}{2}\left(\frac{\sqrt{2}}{\kappa}\cos\phi + X\right) + constant_Y$, $A_Z = -\frac{\sqrt{2}}{2}B^{(0)}(Y\sin\phi + X\cos\phi) + constant_Z$. It is to be noted that if,

$$\tan\phi = -\frac{X}{Y}, \tag{C13}$$

then A_Z disappears, assuming that the constant of integration for A_Z is also zero.

In a Cartesian basis, the vector potential $\boldsymbol{A}_{(3)}$ forms part of A_X and A_Y in free space, and is missing in the standard theory of electromagnetism. There is no optical Bohm-Aharonov effect due to $\boldsymbol{A}_{(3)}$ in this standard theory. Finally we note that there exists a cyclically symmetric

$$(\nabla \times \boldsymbol{A}_{(1)}) \times (\nabla \times \boldsymbol{A}_{(2)}) = iB^{(0)}(\nabla \times \boldsymbol{A}_{(3)})^*,$$

where,

$$\boldsymbol{B}^{(1)} = \nabla \times \boldsymbol{A}_{(1)}, \qquad \boldsymbol{B}^{(2)} = \nabla \times \boldsymbol{A}_{(2)}, \qquad \boldsymbol{B}^{(3)} = \nabla \times \boldsymbol{A}_{(3)}, \tag{C14}$$

with cyclical permutations of (1), (2) and (3), so that, if

$$\boldsymbol{A}_{(3)} =? \boldsymbol{0}, \qquad \boldsymbol{A}_{(1)} = \boldsymbol{A}_{(2)} =? \boldsymbol{0}, \tag{C15}$$

and electromagnetism disappears in vacuo. Thus, because, in general,

$$\boldsymbol{A}_{(1)} \neq \boldsymbol{0}, \qquad \boldsymbol{A}_{(2)} \neq \boldsymbol{0}, \tag{C16}$$

then $\boldsymbol{A}_{(3)} \neq \boldsymbol{0}$ and $\boldsymbol{B}^{(3)} \neq \boldsymbol{0}$, as in the text.

Appendix 4: THE COMPLETE SET OF CYCLICALLY SYMMETRIC RELATIONS IN VACUO

M. W. Evans

Starting with a complex transverse vector potential,

$$\mathbf{A}^{(1)} = \mathbf{A}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}\kappa}(i\mathbf{i} + \mathbf{j})e^{i\phi}, \tag{D1}$$

it follows that

$$\nabla \times \mathbf{A}^{(1)} = \kappa \mathbf{A}^{(1)} = \mathbf{B}^{(1)}, \tag{D2}$$

and that

$$\mathbf{B}^{(3)} = \mathbf{B}^{(3)*} = -i\kappa^2 \frac{\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}}{B^{(0)}}. \tag{D3}$$

Note that (1), (2), and (3) cannot be permuted in (D3) because $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are polar and $\mathbf{B}^{(3)}$ is axial. Eq. (D2) reveals that there is a *symmetry duality* in the quantity κ, because although a scalar, it mediates a direct proportionality between $\mathbf{B}^{(1)}$ ($\hat{P}$ positive, $\hat{T}$ negative) and $\mathbf{A}^{(1)}$ ($\hat{T}$ positive, $\hat{P}$ negative). Eq. (D2) suggests that κ is $\hat{P}$ and $\hat{T}$ *negative*, a property which is reminiscent of vector linear momentum. The scalar magnitude of the wave vector $\boldsymbol{\kappa}$ is therefore linked with particle momentum. There is wave particle dualism inherent in Maxwell's equations. The fact that κ can be $\hat{T}$ and $\hat{P}$ negative indicates that the classical field is not a complete understanding. From Eq. (D2),

$$\nabla \times (\nabla \times \mathbf{A}^{(1)}) = \kappa \nabla \times \mathbf{A}^{(1)} = \kappa^2 \mathbf{A}^{(1)}, \tag{D4}$$

i.e.

$$\nabla^2 \mathbf{A}^{(1)} = -\kappa^2 \mathbf{A}^{(1)}, \tag{D5}$$

which is an eigenfunction equation reminiscent of the static limit of the Proca equation discussed in the text, i.e. of

$$\nabla^2 \mathbf{A}^{(1)} = \xi^2 \mathbf{A}^{(1)}. \tag{D6}$$

Note that: a) there is a change of sign between Eqs. (D5) and (D6); b) Eq. (D5) now conserves $\hat{P}$ and $\hat{T}$ symmetry, unlike Eq. (D2). This is despite the fact that Eq. (D5) is a direct consequence of Eq. (D2), and so if the former is accepted as physical, so must the latter. The quantity κ has symmetry that cannot be understood in the classical framework of Maxwell's equations.

The introduction of quantum concepts is one way of resolving the fundamental *symmetry dualism* inherent in Maxwell's field equations. The fact that Eq. (D5) is an eigenfunction equation shows that ∇^2 plays the role of a quantum mechanical operator, whose eigenvalue is $-\kappa^2$. The vector potential $\mathbf{A}^{(1)}$ then plays the role of a wave function. From the fundamental axioms of quantum mechanics,

$$\hat{\nabla} = \frac{i}{\hbar}\hat{p}, \tag{D7}$$

i.e. the del vector becomes a del operator, directly proportional to a linear momentum operator, $\hat{p}$. From Eq. (D7) in (D5),

$$\hat{p}^{(2)} \mathbf{A}^{(1)} = \hbar^2 \kappa^2 \mathbf{A}^{(1)}. \tag{D8}$$

Thus,

$$\langle \hat{p}^2 \rangle = \hbar^2 \kappa^2, \qquad p = \hbar\kappa = \hbar\frac{\omega}{c} = h\frac{\nu}{c}, \tag{D9}$$

which defines the linear momentum of the photon in vacuo. Note that,

$$\hbar = \frac{p}{\kappa}, \tag{D10}$$

so that κ and p are dimensionally the same. Therefore κ is identifiable as the expectation value of the operator $\hat{\kappa} = \hat{p}/\hbar$; i.e. the expectation value of a linear momentum operator of quantized field theory. However, κ is also a component of the classical wave vector, and so is an example of wave particle dualism.

The de Broglie guiding theorem in radiation is derived by noting from Eq.

(D1) that

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}^{(1)} = -\frac{\omega^2}{c^2}\boldsymbol{A}^{(1)} = -\kappa^2\boldsymbol{A}^{(1)}. \tag{D11}$$

The axioms of quantum mechanics imply that

$$\frac{\partial}{\partial t} \rightarrow -\frac{i}{\hbar}En, \tag{D12}$$

where En denotes energy. From Eq. (D12) in Eq. (D11),

$$En = \kappa c\hbar = pc = m_0c^2 = \omega\hbar = \nu h, \tag{D13}$$

where m_0 is a mass defined formally by $p = m_0c$, . As in the text, the de Broglie guiding theorem is usually stated as

$$h\nu = m_0c^2, \tag{D14}$$

for radiation *and* matter. This means that the quantum of energy, $h\nu$, is formally expressible as m_0c^2. If there is finite photon rest mass, m_0, then $h\nu$ becomes the rest energy m_0c^2. As we saw in Appendix 1, the photon was originally introduced as the quantum of energy, $h\nu$. If the photon mass is asserted to be identically zero, then $h\nu$ is pure energy, i.e. is definable in terms only of energy.

From Eqs. (D5) and (D11),

$$\Box\boldsymbol{A}^{(1)} = \left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A}^{(1)} = 0, \tag{D15}$$

which is the space part of the d'Alembert equation,

$$\Box A_\mu = 0. \tag{D16}$$

Equation (D1) is a transverse solution of Eq. (D14), and so Eq. (D3) shows that $\boldsymbol{B}^{(3)}$ is formed from the cross product of complex conjugate solutions of the d'Alembert equation.

The question now arises as to what is the longitudinal solution of Eq. (D15). In other words, is there a cyclically symmetric relation between components of the vector potential in vacuo? If there is one, it is clear that the longitudinal, phase free, component must be formed from the cross product of two polar vectors, the complex conjugates $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, and so must be an *axial* vector, the pure imaginary $i\boldsymbol{A}^{(3)}$. The cyclic relation is therefore,

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = -A^{(0)}(i\boldsymbol{A}^{(3)})^*, \quad \boldsymbol{A}^{(2)} \times (i\boldsymbol{A}^{(3)}) = -A^{(0)}\boldsymbol{A}^{(1)*}, \quad (i\boldsymbol{A}^{(3)}) \times \boldsymbol{A}^{(1)} = -A^{(0)}\boldsymbol{A}^{(2)*}, \tag{D17}$$

in which the three permuted vectors are $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$, and $i\boldsymbol{A}^{(3)}$. These are relations among the space parts of the complex potential four-vector A_μ, and therefore should be relativistically self-consistent. That this is indeed the case can be checked as follows, using the dual transform described in Appendix 2. We first note that the complete set of three cyclically symmetric algebra's is

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = -A^{(0)}(i\boldsymbol{A}^{(3)})^*, \quad \text{and cyclic permutations,} \tag{D18a}$$

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = -iB^{(0)}\boldsymbol{B}^{(3)*}, \quad \text{and cyclic permutations,} \tag{D18b}$$

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = -E^{(0)}(i\boldsymbol{E}^{(3)})^*, \quad \text{and cyclic permutations,} \tag{D18c}$$

in which $\boldsymbol{B}^{(3)}$ is pure real, and $i\boldsymbol{E}^{(3)}$ and $i\boldsymbol{A}^{(3)}$ are pure imaginary. All three quantities are phase free in vacuo. Under the dual transformation of special relativity in vacuo,

$$B^{(0)} \rightarrow -iB^{(0)}, \quad E^{(0)} \rightarrow iE^{(0)}, \quad A^{(0)} \rightarrow -iA^{(0)}, \tag{D19}$$

leaving Maxwell's equations invariant. Applying Eqs. (D19) to Eqs. (D18) the following results are obtained.

1. Equations (D18) are each invariant under dual transformation, and are *self-dual* in vacuo. They are therefore invariant under Lorentz transformation and relativistically invariant and covariant. For example, applying the transform $B^{(0)} \rightarrow -iB^{(0)}$ to Eq. (D18b) produces the same equation, a self-consistent equation of special relativity in vacuo.

2. Using the dual transforms in vacuo,

$$B^{(0)} \rightarrow -\frac{iE^{(0)}}{c}, \quad E^{(0)} \rightarrow icB^{(0)}, \quad A^{(0)} \rightarrow -iA^{(0)} = \frac{-iB^{(0)}}{\kappa} = \frac{-iE^{0}}{\omega}, \tag{D20}$$

it is seen that each equation of the set (D18) is self-dual and also dual with the other two. For example, the transformation $B^{(0)} \rightarrow -iE^{(0)}/c$ converts Eq. (D18b) into Eq. (D18c). The transformation $A^{(0)} \rightarrow -iB^{(0)}/\kappa$ converts Eq. (D18a) into Eq. (D18b), and so on.

3. If an attempt is made to arbitrarily change the structure of any of Eqs. (D18), e.g. if it is asserted arbitrarily that $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$, as in the conventional view of electrodynamics in vacuo, then the Lorentz covariance of Eqs. (D18) is lost. For example, if $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} =? \boldsymbol{0}$ which follows from $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$, then under the dual transform $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} \rightarrow -\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$, there is a change of sign, but the same dual transform results in $\boldsymbol{0} \rightarrow \boldsymbol{0}$, where there is no change of sign. Thus the assertion $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$ is relativistically inconsistent. It is also inconsistent to assert that the imaginary $i\boldsymbol{E}^{(3)}$ and $i\boldsymbol{A}^{(3)}$ are zero, they are rigorously non-zero and imaginary, and are indispensable to maintain the symmetry of Eqs. (D18) in vacuo.

4. The longitudinal electromagnetic energy density defined by,

$$U^{(3)} \equiv \frac{1}{\mu_0}\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} + \epsilon_0(i\boldsymbol{E}^{(3)}) \cdot (i\boldsymbol{E}^{(3)}), \tag{D21}$$

is rigorously self-dual in vacuo,

$$U^{(3)} \rightarrow \epsilon_0(i\boldsymbol{E}^{(3)}) \cdot (i\boldsymbol{E}^{(3)}) + \frac{1}{\mu_0}\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} = U^{(3)}, \tag{D22}$$

and its numerical magnitude is zero. *The fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ make no contribution to electromagnetic energy density in vacuo, as discussed in the text and Appendix 1.* The quantity $U^{(3)}$ is Lorentz invariant and is therefore consistent with the definition of energy in special relativity. In the quantum field theory these classical considerations translate into the finding that $\hat{B}^{(3)}$ generates no Planck energy and is directly proportional to the spin angular momentum of the photon in vacuo. In interpreting Eqs. (D18) it is noted that $\boldsymbol{B}^{(3)}$ is real and physical, but $i\boldsymbol{E}^{(3)}$ and $i\boldsymbol{A}^{(3)}$ are imaginary and unphysical. Pure real quantities in electrodynamics are physical, pure imaginary quantities are unphysical. *Mathematically*, however, the presence of $i\boldsymbol{E}^{(3)}$ and $i\boldsymbol{A}^{(3)}$ is essential if self consistency with special relativity is to be maintained. For example, without $i\boldsymbol{E}^{(3)}$ we obtain the incorrect assertion $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} =? \boldsymbol{0}$.