THE ROLE OF GRAVITATIONAL TORSION IN GENERAL RELATIVITY: THE S TENSOR

by

Myron W. Evans,

Alpha Foundation's Institute for Advanced Study (AIAS)

(www.aias.us, www.atomicprecision.com, emyrone@aol.com)

ABSTRACT

The conventional definition of the Riemann tensor is shown to be incomplete because the torsional component is missing. The commutator of covariant derivatives acting on the four vector is shown to produce a tensor that is conventionally antisymmetric in its first two indices (the conventional curvature or Riemann tensor) stemming from the use of the Christoffel connection. More generally both the Riemann and torsional tensors are asymmetric in their first two indices because there is no torsion free condition in general. The complete tensor is the sum of these two tensors and is named the S tensor, and the generalized Einstein Hilbert field equation deduced for the S tensor. In this way spin or torsion is introduced into general relativity in a novel and fundamental manner, and the ramifications of this modification work through into all areas of dynamics.

Keywords: Riemann tensor, torsion, commutator of covariant derivatives, round trip with covariant derivatives, general relativity.

42nd Paper of the Unified Field Theory

1. INTRODUCTION

The theory of relativity {1} is based conventionally on Riemann geometry and the use of the Christoffel connection {2}, which is symmetric in its lower two indices. The Einstein Hilbert field equation is deduced from the second Bianchi identity with the torsion free condition stemming from the Christoffel connection. In consequence all the information given from considerations of gravitational torsion is lost. Recently {3-25} it has been realized that the electromagnetic field tensor is spacetime torsion within a C negative vector potential magnitude $A^{(0)}$. This is electromagnetic torsion as distinct from the novel gravitational torsion considered in this paper. Therefore torsion is fundamentally important in relativity theory and cannot be neglected. The role of torsion is seen most clearly through the Cartan structure equations and the Bianchi identities of Cartan geometry.

In Section 2 the commutator of covariant derivatives acting on the four vector V^μ in the four dimensions of spacetime is shown to produce in general a sum of two tensors, a sum that premultiplies the vector itself. In addition there are four other terms which premultiply the four derivative of the vector. One of the terms that premultiply the four vector itself has the same structure as the conventional Riemann tensor, but in general the connections within this tensor are asymmetric in their lower two indices, and are not torsion free and are not Christoffel connections in general. The second tensor premultiplying the vector itself is novel to this work, and is named the torsional tensor. The symmetries of the various connections within the torsional tensor are determined by the commutator itself. The sum of these two tensors is named the S tensor in order to distinguish it from the conventional Riemann tensor. The S tensor is therefore defined as the sum of the two tensors that premultiply the vector itself. The S tensor is always needed for a complete description of gravitation in a spacetime with both curvature and torsion present - the Evans spacetime of unified field theory {3-25}.

In Section 3 the generalization of the Einstein Hilbert field equation is deduced for the S tensor, showing the presence of novel terms due to gravitational torsion. In general gravitational torsion affects cosmological observations, but gravitational torsion is neglected in conventional general relativity. The latter appears to be very accurate for the solar system {26} but in other contexts appears to be very inaccurate {27}. Therefore the presence of gravitational torsion is indicated experimentally by data which cannot be explained with the conventional Riemann tensor. This is unsurprising in retrospect because the Riemann tensor is always predicated on the assumption that the connection is the Christoffel connection. This assumption is equivalent to assuming that there is no torsion in the universe, and there is no a priori reason why torsion should be absent. In unified field theory, torsion is the fundamental electromagnetic field itself.

2. DERIVATION OF THE S TENSOR.

The S tensor is derived straightforwardly by operating on the four-vector V^ρ with the commutator of covariant derivatives. This is how the Riemann tensor is derived conventionally {2}, but with the torsion free condition always assumed. There are four terms missing from the derivation by Carroll {2}, a derivation which is corrected as follows to produce the S tensor.

Consider the commutator of covariant derivatives D_μ, acting on the four - vector V^ρ in Evans spacetime:

$$[D_\mu, D_\nu] V^\rho = (D_\mu D_\nu - D_\nu D_\mu) V^\rho \quad -(1)$$

The covariant derivative is defined by:

$$D_\nu V^\rho = \partial_\nu V^\rho + \Gamma^\rho_{\nu\sigma} V^\sigma \quad -(2)$$

and in general the connection $\Gamma^{\rho}_{\nu\sigma}$ is asymmetric in its lower two indices, indicating the simultaneous presence of curving and spinning:

$$\Gamma^{\rho}_{\nu\sigma} \neq \Gamma^{\rho}_{\sigma\nu} . \qquad -(2)$$

The Christoffel connection is symmetric in its lower two indices:

$$\Gamma^{\rho}_{\nu\sigma} = \Gamma^{\rho}_{\sigma\nu} \qquad -(3)$$

indicating the absence of spinning or torsion, but the presence of curving.

From Eqs. (1) and (2):

$$\begin{aligned}[D_\mu, D_\nu] V^\rho = {} & \partial_\mu \left(\partial_\nu V^\rho + \Gamma^{\rho}_{\nu\sigma} V^\sigma \right) \\ & - \Gamma^{\lambda}_{\mu\nu} \left(\partial_\lambda V^\rho + \Gamma^{\rho}_{\lambda\sigma} V^\sigma \right) \qquad -(4) \\ & + \Gamma^{\rho}_{\mu\sigma} \left(\partial_\nu V^\sigma + \Gamma^{\sigma}_{\nu\lambda} V^\lambda \right) - (\mu \leftrightarrow \nu) .\end{aligned}$$

Now use the Leibnitz Theorem to obtain:

$$\begin{aligned}[D_\mu, D_\nu] V^\rho = {} & \partial_\mu \partial_\nu V^\rho + \left(\partial_\mu \Gamma^{\rho}_{\nu\sigma} \right) V^\sigma + \Gamma^{\rho}_{\nu\sigma} \partial_\mu V^\sigma \\ & - \Gamma^{\lambda}_{\mu\nu} \partial_\lambda V^\rho - \Gamma^{\lambda}_{\mu\nu} \Gamma^{\rho}_{\lambda\sigma} V^\sigma \\ & + \Gamma^{\rho}_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^{\rho}_{\mu\sigma} \Gamma^{\sigma}_{\nu\lambda} V^\lambda \\ & - \partial_\nu \partial_\mu V^\rho - \left(\partial_\nu \Gamma^{\rho}_{\mu\sigma} \right) V^\sigma - \Gamma^{\rho}_{\mu\sigma} \partial_\nu V^\sigma \\ & + \Gamma^{\lambda}_{\nu\mu} \partial_\lambda V^\rho + \Gamma^{\lambda}_{\nu\mu} \Gamma^{\rho}_{\lambda\sigma} V^\sigma \\ & - \Gamma^{\rho}_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^{\rho}_{\nu\sigma} \Gamma^{\sigma}_{\mu\lambda} V^\lambda .\end{aligned}$$

$$-(5)$$

Finally rearrange terms and dummy indices to obtain:

$$
\begin{aligned}
[D_\mu, D_\nu] V^\rho = \; & \Big(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \\
& + \left(\Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} \right) \Gamma^\rho_{\lambda\sigma} \Big) V^\sigma \\
& - \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) \partial_\lambda V^\rho \qquad -(6) \\
& + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma - \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma \\
:= \; & \left(R^\rho{}_{\sigma\mu\nu} - \Gamma^\rho{}_{\lambda\sigma} T^\lambda{}_{\mu\nu} \right) V^\sigma - T^\lambda{}_{\mu\nu} \partial_\lambda V^\rho \\
& + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma - \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma .
\end{aligned}
$$

The S tensor is defined as the sum:

$$S^\rho{}_{\sigma\mu\nu} := R^\rho{}_{\sigma\mu\nu} - \Gamma^\rho{}_{\lambda\sigma} T^\lambda{}_{\mu\nu} \qquad -(7)$$

of the general Riemann tensor $R^\rho{}_{\sigma\mu\nu}$ (denoted henceforth as the R tensor) and the general torsional tensor $T^\rho{}_{\sigma\mu\nu}$ (denoted henceforth as the T tensor):

$$T^\rho{}_{\sigma\mu\nu} := - \Gamma^\rho{}_{\lambda\sigma} T^\lambda{}_{\mu\nu} . \qquad -(8)$$

Note carefully that the T tensor is different from the conventional torsion tensor used in Cartan geometry {2}:

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} . \qquad -(9)$$

The T tensor is one of the terms that premultiplies V^σ in Eq. (6) while the conventional torsion tensor (9) premultiplies $\partial_\lambda V^\rho$ in Eq. (6). There are two other terms present in Eq. (6) which are incorrectly omitted by Carroll {2}. These terms, which premultiply $\partial_\nu V^\sigma$ and $\partial_\mu V^\sigma$, also have a fundamental physical significance in relativity theory but will be considered in future work. The S, T and R tensors are by definition all antisymmetric in their last two indices μ and ν, but in general are asymmetric in their first two indices ρ and σ. The conventional Riemann tensor is antisymmetric in its first two indices ρ and σ because of the torsion free condition used in deriving it {2}. The same torsion free condition means that the conventional Ricci and metric tensors {2} are symmetric. More generally they are asymmetric {3-25} and in general there is no unique Ricci type tensor definable from the S tensor by index contraction. Therefore the conventional Einstein Hilbert field equation is a special case of many possible field equations of relativity and unified field theory {3-25}.

Due to the antisymmetry in μ and ν the S tensor obeys the identities:

$$S_{\rho\sigma\mu\nu} + S_{\rho\mu\nu\sigma} + S_{\rho\nu\sigma\mu} := 0 \quad -(10)$$

and

$$D_\lambda S_{\rho\sigma\mu\nu} + D_\rho S_{\sigma\lambda\mu\nu} + D_\sigma S_{\lambda\rho\mu\nu} := 0 \quad -(11)$$

which are generalizations of the first and second Bianchi identities obeyed by the conventional Riemann tensor. The Bianchi identities are examples of the Jacobi identity:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$$

$$=$$

$$ABC - BCA - ACB + CBA$$
$$+ BCA - CAB - BAC + ACB$$
$$+ CAB - ABC - CBA + BAC \qquad - (12)$$
$$:= 0 .$$

The second Bianchi identity is most generally a relation between covariant derivatives:

$$[[D_\lambda, D_\rho], D_\sigma] + [[D_\rho, D_\sigma], D_\lambda] + [[D_\sigma, D_\lambda], D_\rho] := 0 \qquad - (13)$$

and this is true for any connection. As first shown by Feynman { 28 } the Jacobi identity can be derived by a round trip of covariant derivatives around a cube. In the condensed notation of differential geometry {2-25} the identities (10) and (11) become:

$$S^a{}_b \wedge q^b := 0 \qquad - (14)$$
$$D \wedge S^a{}_b := 0 \qquad - (15)$$

where q^b is the tetrad form and where D ^ is the covariant exterior derivative of differential geometry.

3. FIELD EQUATION FOR THE S TENSOR.

The well known historical route to the Einstein Hilbert field equation is adhered to in this section, but the end result is more general, because it considers non-zero torsion. The first step is to define the S tensor with lowered indices:

$$S_{\rho\sigma\mu\nu} = g_{\rho\lambda} S^\lambda{}_{\sigma\mu\nu} \qquad - (16)$$

No assumptions are made concerning the symmetry of the metric tensor $g_{\rho\lambda}$. In general it is a tensor with symmetric and asymmetric components. This can be seen using differential geometry, in which the symmetric metric is the dot product of two tetrads:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(17)$$

where η_{ab} is the Minkowski metric. The asymmetric metric is the wedge product of two tetrads:

$$g^c_{\mu\nu} = -g^c_{\nu\mu} = q^a_\mu \wedge q^b_\nu \quad -(18)$$

and for each index c is an antisymmetric tensor of the base manifold, Q.E.D. The most general metric is the outer or tensor product of two tetrads:

$$g^{ab}_{\mu\nu} = q^a_\mu q^b_\nu \quad -(19)$$

and for index ab is an asymmetric tensor of the base manifold, Q.E.D. Therefore the symmetric metric is a special case (the symmetric part) of the most general metric formed form the tensor or outer product of two tetrads. Since tetrads are always mixed index tensors {2-25}, a dot, wedge and tensor product of two tetrads may always be defined, and so the asymmetric metric may always be defined in the n dimensional manifold using the principles of standard differential geometry. The asymmetric metric $g_{\mu\nu}$ in Riemann geometry is thus defined for a given index ab of the tangent space to the n dimensional base manifold at point P. This tangent space always exists but was not considered in Riemann geometry (which predated differential geometry by many years). This appears to be the root cause of the incorrect assertion sometimes made that the metric must always be a symmetric tensor. Therefore, as in Eq. (16) it is always possible to define the S tensor with lowered indices using a metric of any symmetry. It is understood that Eq. (16) applies in the base manifold for each ab index of the tangent spacetime in general.

Now make a double index contraction on the identity (11):

$$g^{\nu\sigma} g^{\mu\lambda} \left(D_\lambda S_{\rho\sigma\mu\nu} + D_\rho S_{\sigma\lambda\mu\nu} + D_\sigma S_{\lambda\rho\mu\nu} \right) := 0 \quad -(20)$$

and define:

$$D^\mu S_{\rho\mu} := -\left(g^{\mu\lambda} D_\lambda\right)\left(g^{\nu\sigma} S_{\rho\sigma\mu\nu}\right) \quad -(21)$$

$$D^\nu S_{\rho\nu} := -\left(g^{\nu\sigma} D_\sigma\right)\left(g^{\mu\lambda} S_{\lambda\rho\mu\nu}\right) \quad -(22)$$

$$D_\rho S := D_\rho \left(g^{\nu\sigma} g^{\mu\lambda} S_{\sigma\lambda\mu\nu} \right). \quad -(23)$$

The sign difference convention comes from the antisymmetry of the S tensor in μ and ν.

This convention, used by Einstein in 1915, is defined as follows. If indices are in the same order in the metric and in the tensor multiplied by the metric, then the resulting sign is positive. If indices are in the opposite order in the tensor to the index order in the metric, then the sign is negative. Adhering to this convention then:

$$D^\mu S_{\rho\mu} - D_\rho S + D^\nu S_{\rho\nu} := 0 \quad -(24)$$

i.e.

$$D^\mu S_{\rho\mu} - \frac{1}{2} D_\rho S := 0 \quad -(25)$$

or

$$S_{\rho\nu} = \frac{1}{4} S g_{\rho\nu}. \quad -(26)$$

Finally use:

$$D_\rho = g_{\rho\mu} D^\mu \quad -(27)$$

to obtain:

$$D^{\mu}\left(S_{\rho\mu} - \frac{1}{2} S g_{\rho\mu}\right) := 0. \quad -(28)$$

The field equation is obtained by the equation:

$$D^{\mu}\left(S_{\rho\mu} - \frac{1}{2} S g_{\rho\mu}\right) = k D^{\mu} T_{\rho\mu} \quad -(29)$$

where k is the Einstein constant, and $T_{\rho\mu}$ is a more general canonical energy-momentum tensor than used by Einstein and Hilbert. Here $T_{\rho\mu}$ contains angular or torsional energy-momentum as well as energy momentum defined by curvature as in the original Einstein Hilbert field equation. Therefore the field equation of the S tensor is:

$$S_{\rho\mu} - \frac{1}{2} S g_{\rho\mu} = k T_{\rho\mu}. \quad -(30)$$

DISCUSSION