METRIC COMPATIBILITY CONDITION AND TETRAD POSTULATE

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ABSTRACT

The metric compatibility condition of Riemann geometry and the tetrad postulate of differential geometry are cornerstones of general relativity in respectively its Einstein Hilbert and Palatini variations. In the latter the tetrad tensor is the fundamental field, in the former the metric tensor is the fundamental field. In the Evans unified field theory the tetrad becomes the fundamental field for all types of matter and radiation, and the tetrad postulate leads to the Evans Lemma, the Evans wave equation, and to all the fundamental wave equations of physics in various well defined limits. The tetrad postulate is a fundamental requirement of differential geometry, and this is proven in this paper in ~~4~~ seven ways. For centrally directed gravitation therefore both the metric compatibility condition and the tetrad postulate are accurate experimentally to one part in one hundred thousand.

Keywords: Metric compatibility; tetrad postulate; Einstein Hilbert variation of general relativity; Palatini variation of general relativity; Evans unified field theory.

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1. INTRODUCTION

The theory of general relativity was formulated originally in 1915 by Einstein and independently by Hilbert. It was developed for centrally directed gravitation, and was first verified by the Eddington experiment {1}. Recently {2} the precision of the Eddington experiment has been improved to one part in one hundred thousand. Therefore the basic geometrical assumptions used by Einstein and Hilbert have also been verified experimentally to one part in one hundred thousand. One of these is the metric compatibility condition {3-5} of Riemann geometry, a condition which asserts that the covariant derivative of the metric tensor vanishes. The metric tensor is the fundamental field in the Einstein Hilbert variation of general relativity. It is defined by:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(1)$$

where q^a_μ is the tetrad {3-5}, a mixed index rank two tensor. The Latin superscript of the tetrad tensor refers to the spacetime of the tangent bundle at a point P of the base manifold indexed by the Greek subscript of the tetrad. In eqn. (1) η_{ab} is the Minkowski metric:

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad -(2)$$

The metric compatibility condition is then {3-5}, for any spacetime:

$$D_\rho g^{\mu\nu} = D_\rho g_{\mu\nu} = 0. \quad -(3)$$

Using the Leibnitz Theorem {3-5}Eq. (1) and (3) imply:

$$q^b_\nu D_\rho q^a_\mu + q^a_\mu D_\rho q^b_\nu = 0 \quad -(4)$$

one possible solution of which is:

$$D_\rho q^a_\mu = D_\rho q^b_\nu = 0. \quad -(5)$$

Eq. (5) is the tetrad postulate of the Palatini variation {3-8} of general relativity. In Section 2 it will be shown in various complementary ways that Eq. (5) is the unique solution of Eq. (4). It follows that for central gravitation, the tetrad postulate has been verified experimentally {2} to one part in one hundred thousand.

In Section 3 a brief discussion is given of the physical meaning of the metric compatibility condition used by Einstein and Hilbert in 1915 to describe centrally directed gravitation. In 1915 the original metric compatibility condition was supplemented by the additional assumption that the spacetime of gravitational general relativity is free of torsion:

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} = 0 \quad -(6)$$

where $T^\kappa_{\mu\nu}$ is the torsion tensor and where $\Gamma^\kappa_{\mu\nu}$ is the Christoffel symbol. The latter is symmetric in its lower two indices and is also known as the Levi-Civita or Riemann connection {3-5}. For the centrally directed gravitation of the sun these assumptions hold to one part in one hundred thousand {2}. However, the Evans unified field theory {9-15} has recently recognised that electromagnetism is the torsion form of differential geometry {3-5}, gravitation being the Riemann form, and has shown how electromagnetism interacts with gravitation in a spacetime in which the torsion tensor is not in general zero. Therefore in Section 3 we discuss the implications for the metric compatibility condition of the 1915 theory, and summarize the conditions needed for the interaction of gravitation and electromagnetism.

2. SEVEN PROOFS OF THE TETRAD POSTULATE.

It has been shown in the introduction that for any spacetime (whether torsion free or not) the tetrad postulate is a possible solution of the metric compatibility condition. In this section it is shown in seven ways that it is the unique solution.

1) Proof from Fundamental Matrix Invertibility.

Consider the following basic properties of the tetrad tensor {3-5}:

$$q^b_\nu \, q^\nu_b = 1 \qquad -(7)$$

$$q^a_\mu \, q^\mu_a = 1 \qquad -(8)$$

$$q^\mu_a \, q^a_\nu = \delta^\mu_\nu \qquad -(9)$$

$$q^a_\mu \, q^\mu_b = \delta^a_b \qquad -(10)$$

where δ^μ_ν and δ^a_b are Kronecker delta functions. Differentiate Eqs. (7) to (10) covariantly with the Leibnitz Theorem:

$$q^\nu_b \, D_\rho q^b_\nu + q^b_\nu \, D_\rho q^\nu_b = 0 \qquad -(11)$$

$$q^a_\mu \, D_\rho q^\mu_a + q^\mu_a \, D_\rho q^a_\mu = 0 \qquad -(12)$$

$$q^\mu_a \, D_\rho q^a_\nu + q^a_\nu \, D_\rho q^\mu_a = 0 \qquad -(13)$$

$$q^a_\mu \, D_\rho q^\mu_b + q^\mu_b \, D_\rho q^a_\mu = 0. \qquad -(14)$$

Rearranging dummy indices in Eq (11)($a \rightarrow b, \mu \rightarrow \nu$):

$$q^\mu_a \, D_\rho q^a_\mu + q^b_\nu \, D_\rho q^\nu_b = 0. \qquad -(15)$$

Rearranging dummy indices in Eq. (14)($\mu \rightarrow \nu$):

$$q^\mu_b \, D_\rho q^a_\mu + q^a_\nu \, D_\rho q^\nu_b = 0. \qquad -(16)$$

Multiply Eq. (15) by q_μ^a :

$$D_\rho q_\mu^a + q_\mu^a q_\nu^b D_\rho q_b^\nu = 0. \quad -(17)$$

Multiply Eq. (16) by q_μ^b :

$$D_\rho q_\mu^a + q_\mu^b q_\nu^a D_\rho q_b^\nu = 0. \quad -(18)$$

It is seen that Eq. (17) is of the form:

$$x + ay = 0 \quad -(19)$$

and Eq. (18) is of the form:

$$x + by = 0 \quad -(20)$$

where

$$a \neq b. \quad -(21)$$

The only possible solution is:

$$x = y = 0. \quad -(22)$$

This gives the tetrad postulate, Q.E.D.:

$$D_\rho q_\mu^a = D_\rho q_b^\nu = 0, \quad -(23)$$

which is therefore the unique solution of Eq. (4). Note the tetrad postulate is true for any connection, whether torsion free or not.

2) Proof from Coordinate Independence of Tensors.

A tensor of any kind is independent of the way it is written {3-5}. Consider the covariant derivative of any tensor X in two different bases 1 and 2. It follows that:

$$(DX)_1 = (DX)_2 . \quad -(24)$$

In the coordinate basis {3}:

$$(DX)_1 = (D_\mu X^\nu)\, dx^\mu \otimes \partial_\nu$$
$$= (\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda)\, dx^\mu \otimes \partial_\nu . \quad -(25)$$

In the mixed basis:

$$(DX)_2 = (D_\mu X^a)\, dx^\mu \otimes \hat{e}_{(a)}$$
$$= (\partial_\mu X^a + \omega^a_{\mu b} X^b)\, dx^\mu \otimes \hat{e}_{(a)}$$
$$= q^\sigma_a \left(q^a_\nu \partial_\mu X^\nu + X^\nu \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\lambda X^\lambda \right) dx^\mu \otimes \partial_\sigma \quad -(26)$$

where we have used the commutation rule for tensors. Now switch dummy indices σ to μ and use:

$$q^\nu_a q^a_\nu = 1 \quad -(27)$$

to obtain:

$$(DX)_1 = \left(\partial_\mu X^\nu + q^\nu_a \partial_\mu q^a_\lambda X^\lambda + q^\nu_a q^b_\lambda \omega^a_{\mu b} X^\lambda \right) dx^\mu \otimes \partial_\nu \quad -(28)$$

Now compare Eq. (25) and Eq. (28) to give:

$$\Gamma^\nu_{\mu\lambda} = q^\nu_a \partial_\mu q^a_\lambda + q^\nu_a q^b_\lambda \omega^a_{\mu b} \quad -(29)$$

Multiply both sides of Eq. (29) by q^a_ν :

$$q^a_\nu \Gamma^\nu_{\mu\lambda} = \partial_\mu q^a_\lambda + q^b_\lambda \omega^a_{\mu b} \quad -(30)$$

to obtain the tetrad postulate, Q. E. D.:

$$D_\mu q^a_\lambda = \partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - \Gamma^\nu_{\mu\lambda} q^a_\nu = 0. \quad -(31)$$

3) Proof from Basic Definition.

For any vector V^a {3}:

$$V^a = q^a_\nu V^\nu \quad -(32)$$

and using the Leibnitz Theorem:

$$D_\mu V^a = q^a_\nu D_\mu V^\nu + V^\nu D_\mu q^a_\nu . \quad -(33)$$

Using the result:

$$D_\mu q^a_\nu = 0 \quad -(34)$$

obtained in proofs (1) and (2), it is proven here that Eqs. (32) and (34) imply:

$$D_\mu q^a_\lambda = \partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - \Gamma^\nu_{\mu\lambda} q^a_\nu . \quad -(35)$$

From Eqs. (33) and (34):

$$\partial_\mu V^a + \omega^a_{\mu b} V^b = q^a_\nu \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \right). \quad -(36)$$

From Eq. (32):

$$\partial_\mu V^a = V^\nu \partial_\mu q^a_\nu + q^a_\nu \partial_\mu V^\nu \quad -(37)$$

and

$$\omega^a_{\mu b} V^b = \omega^a_{\mu b} q^b_\nu V^\nu . \quad -(38)$$

Add Eqs. (37) and (38):

$$\partial_\mu V^a + \omega^a_{\mu b} V^b = q^a_\nu \partial_\mu V^\nu + V^\nu \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu V^\nu \quad -(39)$$

Comparing Eqs. (36) and (39):

$$q^a_\nu \Gamma^\nu_{\mu\lambda} V^\lambda = V^\nu \left(\partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu \right) \quad -(40)$$

and switching dummy indices $\nu \rightarrow \lambda$, we obtain:

$$\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - q^a_\nu \Gamma^\nu_{\mu\lambda} = 0 . \quad -(41)$$

This equation has been obtained from the assumption (34), so it follows that:

$$D_\mu q^a_\nu = \partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - q^a_\nu \Gamma^\nu_{\mu\lambda} = 0 \quad -(42)$$

Q.E.D.

4) Proof from the First Cartan Structure Equation { 9 }.

This proof has been given in all detail in ref. { 9 } and is summarized here for convenience. Similarly for Proofs (5) to (7). The first Cartan structure equation {3-8} is a fundamental equation of differential geometry first derived by Cartan. It defines the torsion form as the covariant exterior derivative of the tetrad form: