1)

Example of Partial Simultaneous Differential Equations

The example is the homogeneous Maxwell Heaviside field equation:

$$d \wedge F = 0 \qquad -(1)$$

In tensor notation this is:

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0. \qquad -(2)$$

The Hodge dual in Minkowski spacetime is:

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \qquad -(3)$$

and it follows that eqn (2) is equivalent to:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad -(4)$$

Now develop eqn (4) as a system of simultaneous partial differential equations.

For $\nu = 0$

$$\partial_0 \tilde{F}^{00} + \partial_1 \tilde{F}^{10} + \partial_2 \tilde{F}^{20} + \partial_3 \tilde{F}^{30} = 0$$

Using $\tilde{F}^{00} = 0$:

$$\partial_1 \tilde{F}^{10} + \partial_2 \tilde{F}^{20} + \partial_3 \tilde{F}^{30} = 0 \qquad -(5)$$

For $\nu = 1$

$$\partial_0 \tilde{F}^{01} + \partial_2 \tilde{F}^{21} + \partial_3 \tilde{F}^{31} = 0 \qquad -(6)$$

For $\nu = 2$

$$\partial_0 \tilde{F}^{02} + \partial_1 \tilde{F}^{12} + \partial_3 \tilde{F}^{32} = 0 \qquad -(7)$$

2) For $\underline{\nu = 3}$:

$$\partial_0 \tilde{F}^{03} + \partial_1 \tilde{F}^{13} + \partial_2 \tilde{F}^{23} = 0 \quad -(8)$$

Therefore eqn (1) is equivalent to the following simultaneous partial differential equations:

$$\partial_1 \tilde{F}^{10} + \partial_2 \tilde{F}^{20} + \partial_3 \tilde{F}^{30} = 0 \quad -(5)$$
$$\partial_0 \tilde{F}^{01} + \partial_2 \tilde{F}^{21} + \partial_3 \tilde{F}^{31} = 0 \quad -(6)$$
$$\partial_0 \tilde{F}^{02} + \partial_1 \tilde{F}^{12} + \partial_3 \tilde{F}^{32} = 0 \quad -(7)$$
$$\partial_0 \tilde{F}^{03} + \partial_1 \tilde{F}^{13} + \partial_2 \tilde{F}^{23} = 0 \quad -(8)$$

Use:
$$\tilde{F}^{10} = -\tilde{F}^{01} \quad -(9)$$
$$\tilde{F}^{20} = -\tilde{F}^{20} \quad -(10)$$
$$\tilde{F}^{30} = -\tilde{F}^{03} \quad -(11)$$
$$\tilde{F}^{12} = -\tilde{F}^{21} \quad -(12)$$
$$\tilde{F}^{23} = -\tilde{F}^{32} \quad -(13)$$
$$\tilde{F}^{13} = -\tilde{F}^{31} \quad -(14)$$

There are six unknowns and four simultaneous equations. The usual plane wave solutions are found by assuming:

$$\tilde{F}^{30} = -\tilde{F}^{03} = 0 \quad -(15)$$
$$\tilde{F}^{21} = -\tilde{F}^{12} = 0 \quad -(16)$$

and:

$$\partial_1 \tilde{F}^{10} = \partial_2 \tilde{F}^{20} = \partial_2 \tilde{F}^{21} = \partial_1 \tilde{F}^{12} = 0 \quad -(17)$$

giving:

$$\boxed{\begin{aligned} \partial_0 \tilde{F}^{01} + \partial_3 \tilde{F}^{31} &= 0 \quad -(18) \\ \partial_0 \tilde{F}^{02} + \partial_3 \tilde{F}^{32} &= 0 \quad -(19) \end{aligned}}$$

The solutions are:

$$\tilde{F}^{01} = -B^1 = -B_X = -i\frac{B^{(0)}}{\sqrt{2}} e^{i(\omega t - \kappa z)} \quad -(20)$$

$$\tilde{F}^{02} = -B^2 = -B_Y = -\frac{B^{(0)}}{\sqrt{2}} e^{i(\omega t - \kappa z)} \quad -(21)$$

$$\tilde{F}^{31} = \frac{E^2}{c} = \frac{E_Y}{c} = -\frac{iE^{(0)}}{\sqrt{2}\,c} e^{i(\omega t - \kappa z)} \quad -(22)$$

$$\tilde{F}^{32} = -\frac{E^1}{c} = -\frac{E_X}{c} = -\frac{E^{(0)}}{\sqrt{2}\,c} e^{i(\omega t - \kappa z)} \quad -(23)$$

Check

$$\partial_0 \tilde{F}^{01} + \partial_3 \tilde{F}^{31} = \frac{1}{c}\frac{\partial}{\partial t}\left(-i\frac{B^{(0)}}{\sqrt{2}} e^{i(\omega t - \kappa z)}\right) + \frac{\partial}{\partial z}\left(-\frac{iE^{(0)}}{\sqrt{2}\,c} e^{i(\omega t - \kappa z)}\right)$$

$$= \frac{B^{(0)}}{\sqrt{2}} e^{i(\omega t - \kappa z)}\left(+\frac{\omega}{c} - \kappa\right) \quad -(24)$$

$$= 0 \quad \text{if} \quad \kappa = \frac{\omega}{c}$$

4)

$$\partial_0 \tilde{F}^{02} + \partial_3 \tilde{F}^{32}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{B^{(0)}}{\sqrt{2}} e^{i(\omega t - kz)}\right) + \frac{\partial}{\partial Z}\left(-\frac{E^{(0)}}{\sqrt{2}\,c} e^{i(\omega t - kz)}\right)$$

$$= \frac{B^{(0)}}{\sqrt{2}} e^{i(\omega t - kz)}\left(-\frac{i\omega}{c} + ik\right) = 0 \quad \checkmark \quad -(25)$$

<u>Vector Notation</u>

Eqs (18) and (19) are ~~to~~ Faraday Law of induction:

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(26)$$

with:

$$\underline{E} = \frac{E^{(0)}}{\sqrt{2}}(\underline{i} - i\underline{j})\, e^{i(\omega t - kz)} \quad -(27)$$

$$\underline{B} = \frac{B^{(0)}}{\sqrt{2}}(i\underline{i} + \underline{j})\, e^{i(\omega t - kz)} \quad -(28)$$

Eqs (1), (2) and (4) are equivalent to:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= \underline{0} \end{aligned}} \quad -(29)$$

5)

The Complete Free Space Problem

This is:

$$\boxed{\begin{aligned} d \wedge F &= 0 \\ d \wedge \tilde{F} &= 0 \end{aligned}}$$

— (30)

— (31)

i.e.:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(32)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0 \quad -(33)$$

$$\underline{\nabla} \cdot \underline{E} = 0 \quad -(34)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \underline{0} \quad -(35)$$

The simultaneous partial differential problem then reduces to:

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= \underline{0} \\ \underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} &= \underline{0} \end{aligned}}$$

— (36)

— (37)

Eqs (36) & (37) is a system of six eqns. in six unknowns: $E_X, E_Y, E_Z, B_X, B_Y, B_Z$