INTRODUCTION OF TORSION INTO GENERAL RELATIVITY.

Conventional general relativity depends on the use of a Christoffel connection. Once this assumption is dropped the basics of the subject are changed. Basic definitions such as that of the Riemann tensor are affected by the presence of torsion. In general, the Riemann tensor is defined by a round trip with covariant derivatives (Carroll Eq. (3.65)):

$$[D_\mu, D_\nu] V^\rho = D_\mu D_\nu V^\rho - D_\nu D_\mu V^\rho \quad - (1)$$

where D_μ is the covariant derivative and V^ρ is a four vector. If eq. (1) is investigated carefully we obtain:

$$[D_\mu, D_\nu] V^\rho = \partial_\mu (D_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} D_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} D_\nu V^\sigma$$
$$- \partial_\nu (D_\mu V^\rho) + \Gamma^\lambda_{\nu\mu} D_\lambda V^\rho - \Gamma^\rho_{\nu\sigma} D_\mu V^\sigma$$

$$= \partial_\mu (\partial_\nu V^\rho + \Gamma^\rho_{\nu\sigma} V^\sigma) - \Gamma^\lambda_{\mu\nu} (\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma)$$
$$+ \Gamma^\rho_{\mu\sigma} (\partial_\nu V^\sigma + \Gamma^\sigma_{\nu\lambda} V^\lambda) - (\mu \leftrightarrow \nu)$$

2.

$$
\begin{aligned}
&= \partial_\mu \partial_\nu \bar{V}^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) \bar{V}^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu \bar{V}^\sigma \\
&\quad - \Gamma^\lambda_{\mu\nu} \partial_\lambda \bar{V}^\rho - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} \bar{V}^\sigma \\
&\quad + \Gamma^\rho_{\mu\sigma} \partial_\nu \bar{V}^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} \bar{V}^\lambda \\
&\quad - \partial_\nu \partial_\mu \bar{V}^\rho - (\partial_\nu \Gamma^\rho_{\mu\sigma}) \bar{V}^\sigma - \Gamma^\rho_{\mu\sigma} \partial_\nu \bar{V}^\sigma \\
&\quad + \Gamma^\lambda_{\nu\mu} \partial_\lambda \bar{V}^\rho + \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\sigma} \bar{V}^\sigma \\
&\quad - \Gamma^\rho_{\nu\sigma} \partial_\mu \bar{V}^\sigma - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} \bar{V}^\lambda
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \\
&\qquad + \left(\Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} \right) \Gamma^\rho_{\lambda\sigma} \Big) \bar{V}^\sigma \\
&\qquad - \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) \partial_\lambda \bar{V}^\rho \\
&\qquad + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma - \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma
\end{aligned}
\qquad -(2)
$$

In the presence of torsion :

$$
T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \qquad -(3)
$$

the round trip with covariant derivatives gives :

$$[D_\mu, D_\nu] \bar{V}^\rho = \left(R^\rho{}_{\sigma\mu\nu} - T^\lambda{}_{\mu\nu} \Gamma^\rho_{\lambda\sigma} \right) \bar{V}^\sigma$$

$$- T^\lambda{}_{\mu\nu} \partial_\lambda \bar{V}^\rho \qquad -(4$$

$$+ \Gamma^\rho_{\mu\sigma} \partial_\nu \bar{V}^\sigma - \Gamma^\rho_{\nu\sigma} \partial_\mu \bar{V}^\sigma$$

where: (curvature)

$$R^\rho{}_{\sigma\mu\nu} \Big] = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \qquad -(5)$$

is the conventional definition of the Riemann tensor.

In the presence of torsion however, the Riemann tensor becomes:

$$R^\rho{}_{\sigma\mu\nu} (\text{torsion + curvature}) \qquad -(6)$$

$$= R^\rho{}_{\sigma\mu\nu} (\text{curvature}) - T^\lambda{}_{\mu\nu} \Gamma^\rho_{\lambda\sigma}$$

The <u>torsional</u> <u>Riemann</u> <u>tensor</u> is:

$$\boxed{R^\rho{}_{\sigma\mu\nu} (\text{torsion}) = - T^\lambda{}_{\mu\nu} \Gamma^\rho_{\lambda\sigma}} \qquad -(7)$$

and the complete Riemann tensor is:

$$R^\rho{}_{\sigma\mu\nu} (\text{torsion + curvature}) = R^\rho{}_{\sigma\mu\nu} (\text{curvature})$$

$$+ R^\rho{}_{\sigma\mu\nu} (\text{torsion})$$

4.

Symmetries

1) The torsional Riemann tensor is in general <u>asymmetric</u> in μ and ν because the gamma connection is in general asymmetric in μ and ν.

2) The complete Riemann tensor is in general asymmetric in μ and ν.

3) In the presence of torsion the first Bianchi identity is no longer true, because the complete Riemann tensor is no longer antisymmetric in μ and ν.

4) The torsional Riemann tensor with lowered indices is:

$$R_{\rho\sigma\mu\nu}\,(\text{torsion}) = -g_{\rho\tau}\,\Gamma^{\tau}{}_{\lambda\sigma}\,T^{\lambda}{}_{\mu\nu} \quad -(8)$$

and $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} \neq 0 \quad -(9)$

in general.

Modification to the Second Cartan Equation

There is an additional term:

$$R^a{}_{b\mu\nu} = -q^a_\rho \, q^\sigma_b \, q^\lambda_a \, \Gamma^\rho_{\lambda\sigma} \, T^a{}_{\mu\nu} \quad -(10)$$

and so:

$$\boxed{R^a{}_b = D \wedge \omega^a{}_b - q^a_\rho \, q^\sigma_b \, q^\lambda_a \, \Gamma^\rho_{\lambda\sigma} \, T^a} \quad -(11)$$

Modification to the Second Bianchi Identity

This is very important because the Einstein field equation is based on the second Bianchi identity. The second Bianchi identity must be interpreted in the presence of torsion as:

$$[[D_\lambda, D_\rho], D_\sigma] + [[D_\rho, D_\sigma], D_\lambda] + [[D_\sigma, D_\lambda], D_\rho] := 0 \quad -(12)$$

In the absence of torsion this is:

$$D^\mu G_{\mu\nu} := 0 \quad -(13)$$

where:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad -(14)$$

but in ~~the~~ presence of torsion ~~the~~ Einstein field equation itself contains additional terms. Therefore torsion has a fundamental influence on gravitational general relativity.