STANDARD TENSOR FORMULATION OF THE INHOMOGENEOUS AND HOMOGENEOUS MAXWELL HEAVISIDE FIELD EQUATIONS.

Inhomogeneous Field Equation

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 c J^\nu} \quad -(1)$$

where:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} \quad -(2)$$

Coulomb Law $(\nu = 0)$

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \mu_0 c J^0 \quad -(3)$$

Ampère Maxwell Law $(\nu = 1, 2, 3)$

$$\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = \mu_0 c J^1 \quad -(4)$$

$$\partial_0 F^{02} + \partial_1 F^{12} + \partial_3 F^{32} = \mu_0 c J^2 \quad -(5)$$

$$\partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} = \mu_0 c J^3 \quad -(6)$$

Coulomb Law

From eqn. (3):

$$\boxed{\underline{\nabla} \cdot \underline{E} = \mu_0 c J^0 = \mu_0 c^2 \rho = \frac{\rho}{\epsilon_0}} \quad -(7)$$

where:

$$J^\mu = (c\rho, \underline{J}) \quad -(8)$$

$$\mu_0 \epsilon_0 = 1 / c^2 . \quad -(9)$$

2)

Ampère Maxwell Law

$$-\partial_0 E^1 + c(\partial_2 B^3 - \partial_3 B^2) = \mu_0 J^1 \quad -(10)$$

$$-\partial_0 E^2 - c(\partial_1 B^3 + \partial_3 B^1) = \mu_0 J^2 \quad -(11)$$

$$-\partial_0 E^3 + c(\partial_1 B^2 - \partial_2 B^1) = \mu_0 J^3 \quad -(12)$$

e.g. from eqn (12):

$$-\frac{1}{c^2}\frac{\partial E_Z}{\partial t} + \frac{\partial B_Y}{\partial X} - \frac{\partial B_X}{\partial Y} = \mu_0 J_Z \quad -(13)$$

Therefore:

$$\boxed{\underline{\nabla} \times \underline{B} = \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} + \mu_0 \underline{J}} \quad -(14)$$

This is the Ampère Maxwell Law.

Homogeneous Field Equation

The dual tensor is:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \quad -(15)$$

where:

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \quad -(16)$$

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(17)$$

3)

Therefore, for example:

$$F^{23} = g^{22} g^{33} F_{23} = F_{23}, \quad -(18)$$

$$F^{23} = -B^1 \quad -(19)$$

$$F_{23} = B_1 \quad -(20)$$

$$B^1 = -B_1. \quad -(21)$$

So:

$$\tilde{F}^{01} = \epsilon^{0123} F_{23} = \epsilon^{0123} F^{23} = cB_1 = -cB^1 \quad -(22)$$

$$\tilde{F}^{12} = \epsilon^{1230} F_{30} = \epsilon^{1230} F^{30} = E^3 \quad -(23)$$

and so on.

In eqn (23) we have used:

$$\epsilon^{1230} = -\epsilon^{1203} = \epsilon^{1023} = -\epsilon^{0123} = -1, \quad -(24)$$

$$F^{30} = g^{33} g^{00} F_{30} = -F_{30}. \quad -(25)$$

The homogeneous Maxwell Heaviside field equation is therefore:

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0} \quad -(26)$$

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{bmatrix}. \quad -(27)$$

<u>The Gauss Law</u> <u>($\nu = 0$)</u>

$$\partial_1 \tilde{F}^{10} + \partial_2 \tilde{F}^{20} + \partial_3 \tilde{F}^{30} = 0 \quad -(28)$$

Therefore:

$$\boxed{\underline{\nabla} \cdot \underline{B} = 0} \qquad (29)$$

<u>The Faraday Law of Induction</u> ($\nu = 1, 2, 3$)

$$\partial_0 \tilde{F}^{01} + \partial_2 \tilde{F}^{21} + \partial_3 \tilde{F}^{31} = 0 \qquad (30)$$

$$\partial_0 \tilde{F}^{02} + \partial_1 \tilde{F}^{12} + \partial_3 \tilde{F}^{32} = 0 \qquad (31)$$

$$\partial_0 \tilde{F}^{03} + \partial_1 \tilde{F}^{13} + \partial_2 \tilde{F}^{23} = 0 \qquad (32)$$

From eqns (30) to (32):

$$\boxed{\frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times \underline{E} = \underline{0}} \qquad (33)$$

<u>Overall Result</u>

$$\begin{tikzcd}
\boxed{\begin{array}{l} \partial_\mu \tilde{F}^{\mu\nu} = 0 \\ \partial_\mu F^{\mu\nu} = \mu_0 J^\nu \end{array}} \arrow[r] & \boxed{\begin{array}{l} \underline{\nabla} \cdot \underline{B} = 0 \\ \dfrac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times \underline{E} = \underline{0} \\ \underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \\ \underline{\nabla} \times \underline{B} = \dfrac{1}{c^2} \dfrac{\partial \underline{E}}{\partial t} + \mu_0 \underline{J} \end{array}} \\
\boxed{\begin{array}{l} d \wedge F = 0 \\ d \wedge \tilde{F} = \mu_0 J \end{array}} \arrow[u] &
\end{tikzcd}$$

Differential Form Notation

$$\boxed{\begin{aligned} d \wedge F &= 0 \\ d \wedge \tilde{F} &= \mu_0 J \end{aligned}}$$

— (34)

— (35)

F = field two-form

$\tilde{F}$ = dual of F

J = charge-current density three-form.

Evans Unified Field Theory

$$\boxed{\begin{aligned} d \wedge F^a &= \mu_0 j^a \\ d \wedge \tilde{F}^a &= \mu_0 J^a \end{aligned}}$$

— (36)

— (37)

where:

$$j^a = \frac{A^{(0)}}{\mu_0}\left(R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b\right) \sim 0 \quad \text{— (38)}$$

$$J^a = \frac{A^{(0)}}{\mu_0}\left(\tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b\right) > 0 \quad \text{— (39)}$$

Units and Constants

$$\epsilon_0 = 8.854188 \times 10^{-12}\ J^{-1} C^2 m^{-1}$$

$$\mu_0 = 4\pi \times 10^{-7}\ J s^2 C^{-2} m^{-1}$$

$$\underline{E} = V m^{-1} = J C^{-1} m^{-1}$$

$$\rho = C m^{-3}$$

$$\underline{B} = T = Wb\, m^{-2} = V s m^{-2} = J s C^{-1} m^{-2}$$

$$\underline{J} = C s^{-1} m^{-2}$$

These are ~~the~~ S.I. system of units

Notes

If we include polarization and magnetization ~~then~~:

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}$$

$$\underline{B} = \mu_0 (\underline{H} + \underline{M})$$

and

$$\underline{\nabla} \cdot \underline{D} = \rho$$

$$\underline{\nabla} \times \underline{H} = \underline{J} + \frac{\partial D}{\partial t}$$

$$\underline{D} = C m^{-2} = \text{electric displacement}$$

$$\underline{H} = A m^{-1} = \text{magnetic field strength}.$$