O(3) ELECTRODYNAMICS FROM GENERAL RELATIVITY AND UNIFIED FIELD THEORY

In generally covariant or objective unified field theory the field tensor is defined by the first Maurer Cartan structure relation of differential geometry:

$$\boxed{F^a = d \wedge A^a + \omega^a{}_b \wedge A^b} \quad - (1)$$

(where the base manifold indices have been suppressed for clarity of presentation). In eqn (1) F^a is the field two-form, A^a is the potential one form, and $\omega^a{}_b$ is the spin connection.

The homogeneous and inhomogeneous field equations (HE and IE respectively) are found from the first Bianchi identity of differential geometry:

Homogeneous Field Equation (HE)

$$\boxed{d \wedge F^a = \mu_0 j^a = -A^{(0)} \left(q^b \wedge R^a{}_b + \omega^a{}_b \wedge T^b \right)} \quad - (2)$$

Inhomogeneous Field Equation (IE)

$$\boxed{d \wedge \tilde{F}^a = \mu_0 J^a = -A^{(0)} \left(q^b \wedge \tilde{R}^a{}_b + \omega^a{}_b \wedge \tilde{T}^b \right)} \quad - (3)$$

2) Eqns (1) to (3) are the correctly objective equations of electrodynamics. They are the direct logical consequence of the basic principle of objectivity in physics. This is the principle of <u>general relativity</u>, all equations of physics must retain their form under any type of coordinate transformation - the equations must be <u>generally</u> <u>covariant</u>.

The Maxwell Heaviside field theory (MH) does not obey this basic principle of objectivity because it is a theory of special relativity. It is well known that special relativity is <u>Lorentz</u> <u>covariant</u> but not generally covariant. The Maxwell Heaviside equations corresponding to eqns (1) to (3) are well known to be:

$$F = d \wedge A \qquad - (4)$$

$$d \wedge F = 0 \qquad - (5)$$

$$d \wedge \tilde{F} = \mu_0 J. \qquad - (6)$$

In eqns (3) and (6) the tilde denotes the Hodge dual. In eqn (2) T^a is the torsion form of differential geometry, $R^a{}_b$ is the curvature or Riemann form, q^b is the tetrad form. The number of independent variables on the right hand side

3) of eqn (2) is reduced by the first and second Maurer Cartan equations of differential geometry:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad -(7)$$

$$R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad -(8)$$

Therefore the independent variables are q^a and $\omega^a{}_b$.

Eqn (7) is transformed into eqn (1) using the fundamental rules:

$$A^a = A^{(0)} q^a \quad -(9)$$

$$F^a = A^{(0)} T^a. \quad -(10)$$

The Hodge duals in eqn (3) are defined by the rules of general relativity (see Carroll):

$$\tilde{T}^b_{\mu\nu} = \frac{1}{2} |g|^{1/2} \epsilon_{\mu\nu}{}^{\rho\sigma} T_{\rho\sigma} \quad -(11)$$

$$\tilde{R}^a{}_{b\mu\nu} = \frac{1}{2} |g|^{1/2} \epsilon_{\mu\nu}{}^{\rho\sigma} R^a{}_{b\rho\sigma}. \quad -(12)$$

Here

$$g = |g_{\mu\nu}| \quad -(13)$$

is the determinant of the metric and $\epsilon_{\mu\nu}{}^{\rho\sigma}$ the Levi Civita symbol in the general 4-D manifold (Evans spacetime).

4) It is seen that q^a and $\omega^a{}_b$ are the same in eqns (2) and (3), but duals are used of the torsion and curvature forms.

Experimentally, we know that:

$$j^a \sim 0 \qquad -(14)$$

$$J^a > 0 . \qquad -(15)$$

Therefore:

$$d \wedge F^a = 0 \qquad -(16)$$

within contemporary instrumental precision. For each index a eqn (16) is eqn (5). The latter is a combination of the Gauss law of magnetism:

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad -(17)$$

and the Faraday law of induction:

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} . \qquad -(18)$$

Both laws are tested to high precision, so eqn (14) follows experimentally. In general however j^a is not zero when gravitation influences electromagnetism. The latter influence is missing completely from MH theory because MH is a theory only of special relativity.

5) From eqn. (2) and (16) we obtain the free space condition:

$$\boxed{\omega^a{}_b \wedge T^b = R^a{}_b \wedge q^b} \quad -(19)$$

which is a condition or constraint on eqns. (1) to (3) imposed by the laws (17) and (18). Eqn (19) is therefore an experimental constraint. Using the Maurer Cartan structure equations (7) and (8):

$$T^a = D \wedge q^a \quad -(20)$$

$$R^a{}_b = D \wedge \omega^a{}_b, \quad -(21)$$

where $D\wedge$ is the covariant exterior derivative, eqn (19) becomes a relation between q^a and $\omega^a{}_b$:

$$\boxed{\omega^a{}_b \wedge (D \wedge q^b) = (D \wedge \omega^a{}_b) \wedge q^b} \quad -(20)$$

Therefore there is only one independent variable on the right hand sides of eqns. (2) and (3).

A solution of eqn (20) is:

$$\omega^a{}_b = -\frac{\kappa}{2} \epsilon^a{}_{bc}\, q^c \quad -(21)$$

where κ is the wavenumber and $\epsilon^a{}_{bc}$ is the Levi-Civita symbol of the Minkowski spacetime

6) of the tangent bundle. Given the experimental constraint (20), eqn (21) is true both for eqns (2) and (3), and also for eqn (1). The Levi Civita ~~tens~~ symbol is defined by:

$$\epsilon^{a}{}_{bc} = g^{ad}\epsilon_{dbc} \quad -(22)$$

where:

$$g^{ad} = \mathrm{diag}(1, -1, -1, -1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad -(23)$$

Thus:

$$\epsilon^{1}{}_{23} = g^{1d}\epsilon_{d23} = g^{11}\epsilon_{123} = -\epsilon_{123} \quad -(24)$$

$$\epsilon^{3}{}_{12} = g^{3d}\epsilon_{d12} = g^{33}\epsilon_{312} = -\epsilon_{312} \quad -(25)$$

$$\epsilon^{2}{}_{31} = g^{2d}\epsilon_{d31} = g^{22}\epsilon_{231} = -\epsilon_{231} \quad -(26)$$

and:

$$F^1 = d\wedge A^1 + \frac{g}{2}\left(\epsilon^{1}{}_{23}A^3\wedge A^2 + \epsilon^{1}{}_{32}A^2\wedge A^3\right)$$

$$= d\wedge A^1 - \frac{g}{2}\left(\epsilon_{123}A^3\wedge A^2 + \epsilon_{132}A^2\wedge A^3\right)$$

$$\boxed{\begin{aligned} F^1 &= d\wedge A^1 + gA^2\wedge A^3 \\ F^2 &= d\wedge A^2 + gA^3\wedge A^1 \\ F^3 &= d\wedge A^3 + gA^1\wedge A^2 \end{aligned}} \quad -(27)$$

7) Eqns (27) are the fundamental definition of the field tensor of electromagnetism in objective physics. ... these equations:

$$g = \frac{\kappa}{A^{(0)}}, \quad -(28)$$

and should not be confused with the determinant of the metric. In the complex circular basis eqns (27) define o(3) electrodynamics:

$$\boxed{\begin{aligned} F^{(1)*} &= d \wedge A^{(1)*} - ig\, A^{(2)} \wedge A^{(3)} \\ F^{(2)*} &= d \wedge A^{(2)*} - ig\, A^{(3)} \wedge A^{(1)} \\ F^{(3)*} &= d \wedge A^{(3)*} - ig\, A^{(1)} \wedge A^{(2)} \end{aligned}}$$

$$-(29)$$

and the Evans spin field:

$$\boxed{B^{(3)*} = -ig\, A^{(1)} \wedge A^{(2)}} \quad -(30)$$

observed experimentally in the inverse Faraday effect and the generally covariant phase of electromagnetism.

It is seen that the spin field and o(3) electrodynamics originate in general relativity, in the fundamental requirement that physics be objective to any observer. The spin field originates in the spin connection of eqn. (1), and the spin connection originates

8) in the realization that electromagnetism is spinning spacetime. Similarly gravitation is curving spacetime. Spinning is described by the torsion form T^a and curving described by the Riemann form $R^a{}_b$.

To translate from form notation to vector notation proceed as follows:

$$
\boxed{
\begin{aligned}
F^a &= d \wedge A^a + g A^b \wedge A^c \\
F^a_{\mu\nu} &= (d \wedge A^a)_{\mu\nu} + g A^b_\mu \wedge A^c_\nu \\
F^{(3)*}_{12} &= (d \wedge A^{(3)*})_{12} - ig A^{(1)}_1 \wedge A^{(2)}_2 \\
B^{(3)*}_{12} &= -ig A^{(1)}_1 \wedge A^{(2)}_2 \\
B^{(3)*}_3 &= \frac{1}{2}\left(\epsilon_{123} B^{(3)*}_{12} + \epsilon_{213} B^{(3)*}_{21}\right) \\
\underline{B}^{(3)*} &= -ig\, \underline{A}^{(1)} \times \underline{A}^{(2)}
\end{aligned}
}
$$

There is no $\underline{E}^{(3)}$ field because:

$$
\boxed{
\begin{aligned}
F^{(3)*}_{03} &= (d \wedge A^{(3)})^*_{03} - ig A^{(1)}_0 \wedge A^{(2)}_3 \\
&= 0
\end{aligned}
}
$$

as observed experimentally, there being no electric analogue of the inverse Faraday effect.