THE COMPLEX CIRCULAR BASIS

The complex circular basis is well known, and is an o(3) symmetry basis for 3-D Euclidean space. First consider the Cartesian basis:

$$\underline{i} \times \underline{j} = \underline{k} \qquad -(1)$$

$$\underline{k} \times \underline{i} = \underline{j} \qquad -(2)$$

$$\underline{j} \times \underline{k} = \underline{i}. \qquad -(3)$$

It can be seen ~~that~~ this has a cyclic symmetry.

Now define the complex circular basis using the following unit vectors:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) \qquad -(4)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) \qquad -(5)$$

$$\underline{e}^{(3)} = \underline{k}. \qquad -(6)$$

It can be seen that:

$$\underline{e}^{(1)} = \underline{e}^{(2)*} \qquad -(7)$$

where $*$ denotes complex conjugation.

Next form the vector cross product of $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$ as follows:

$$\underline{e}^{(1)} \times \underline{e}^{(2)} = \frac{1}{2}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -i & 0 \\ 1 & i & 0 \end{vmatrix} = i\underline{k} \qquad -(8)$$

2)

Therefore:

$$\underline{e}^{(1)} \times \underline{e}^{(2)} = i \underline{e}^{(3)} . \quad -(9)$$

It is convenient to write this as:

$$\underline{e}^{(1)} \times \underline{e}^{(2)} = i \underline{e}^{(3)*} . \quad -(10)$$

Now form the vector cross product of $\underline{e}^{(2)}$ and $\underline{e}^{(3)}$:

$$\underline{e}^{(2)} \times \underline{e}^{(3)} = \frac{1}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & i & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{i}{\sqrt{2}} \left(\underline{i} + i \underline{j} \right)$$

$$= i \underline{e}^{(1)*} . \quad -(11)$$

Finally form the vector cross product of $\underline{e}^{(3)}$ and $\underline{e}^{(1)}$:

$$\underline{e}^{(3)} \times \underline{e}^{(1)} = \frac{1}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1 \\ 1 & -i & 0 \end{vmatrix}$$

$$= \frac{i}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right)$$

$$= i \underline{e}^{(2)*} \quad -(12)$$

We therefore obtain the result that:

3)

$$\boxed{\begin{aligned} \underline{e}^{(1)} \times \underline{e}^{(2)} &= i\,\underline{e}^{(3)*} \\ \underline{e}^{(2)} \times \underline{e}^{(3)} &= i\,\underline{e}^{(1)*} \\ \underline{e}^{(3)} \times \underline{e}^{(1)} &= i\,\underline{e}^{(2)*} \end{aligned}} \quad \begin{aligned} &-(13) \\ &-(14) \\ &-(15) \end{aligned}$$

Quod erat demonstrandum.

Eqns (13) to (15) are those of the complex circular basis. This has the same type of O(3) symmetry as the Cartesian basis $\underline{i}$, $\underline{j}$, $\underline{k}$ of eqns (1) to (3).

O(3) Electrodynamics

The transverse plane waves of the radiated magnetic field are defined as follows:

$$\underline{B}^{(1)} = B^{(0)}\,\underline{e}^{(1)}\,e^{i\phi} \quad -(16)$$

$$\underline{B}^{(2)} = B^{(0)}\,\underline{e}^{(2)}\,e^{-i\phi} \quad -(17)$$

The Evans spin field is:

$$\underline{B}^{(3)} = B^{(0)}\,\underline{e}^{(3)} \quad -(18)$$

The B Cyclic Theorem is therefore:

$$\underline{B}^{(1)} \times \underline{B}^{(2)} = iB^{(0)}\underline{B}^{(3)*} \quad -(19)$$
$$\underline{B}^{(2)} \times \underline{B}^{(3)} = iB^{(0)}\underline{B}^{(1)*} \quad -(20)$$
$$\underline{B}^{(3)} \times \underline{B}^{(1)} = iB^{(0)}\underline{B}^{(2)*} \quad -(21)$$

Here ϕ is the phase of the wave. Multiply both sides of eqns (13) to (15) by $B^{(0)2}$ to give eqns (19) - (21).

Notes and References

1) The complex circular basis is well known and is described for example in B.L.Silver, "Irreducible Tensorial Sets" (Academic, New York, 1976).

2) The complex circular basis becomes the B Cyclic theorem through equations (16) to (18), and so the complex circular basis describes circular polarization, as is well known.

3) For considerable development of these notes see: M.W.Evans, J.-P. Vigier et al., "The Enigmatic Photon" (Kluwer, Dordrecht, 1994 - 2002, hardback and softback).

MWE.