THE ORIGINAL WHITTAKER PAPERS

Mathematische Annalen, Vol. 57, 1903, p. 333-355.
(Retyped for readability, with corrections)

ON THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS.

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§ 1.

INTRODUCTION

The object of this paper is the solution of Laplace's potential equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

and of the general differential equation of wave-motions

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=k^2\frac{\partial^2 V}{\partial t^2},$$

and of other equations derived from these.

In § 2, the general solution of the potential equation is found.

In § 3, a number of results are deduced from this, chiefly relating to particular solutions of the equation, and expansions of the general solution in terms of them.

In § 4, the general solution of the differential equation of wave-motions is given.

In § 5, a number of deductions from this general solution is given, including a theorem to the effect that any solution of this equation can be compounded from simple uniform plane waves, and an undulatory explanation of the propagation of gravitation.

§ 2.

THE GENERAL SOLUTION OF THE POTENTIAL EQUATION.

We shall first consider the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

which was originally given by Laplace[1]).

This equation is satisfied by the potential of any distribution of matter which attracts according to the Newtonian Law. We shall first obtain a general form for potential-functions, and then shall shew that this

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[1] *Mémoire sur la théorie de l'anneau de Saturne, 1787.*

form constitutes the general solution of Laplace's equation. From the identity

$$\frac{1}{\sqrt{\left\{(x-a)^2+(y-b)^2+(z-c)^2\right\}}}=\frac{1}{2\pi}\int_0^{2\pi}\frac{du}{(z-c)+i(x-a)\cos u+i(y-b)\sin u},$$

we see that the potential at any point (x, y, z) of a particle of mass m, situated at the point (a, b, c), is

$$\frac{m}{2\pi}\int_0^{2\pi}\frac{du}{(z+ix\cos u+iy\sin u)-(c+ia\cos u+ib\sin u)}$$

which, considered as a function of x,y,z, is an expression of the type

$$\int_0^{2\pi} f(z+ix\cos u+iy\sin u,\, u)\,du,$$

where f denotes some function of the two arguments

$$z+ix\cos u+iy\sin u \quad \text{and} \quad u.$$

It follows that the potential of any number of particles $m_1, m_2, \ldots, m_k$, situated at the points $(a_1b_1c_1)$, $(a_2b_2c_2)$, $(a_3b_3c_3)$, ..., $(a_kb_kc_k)$, is an expression of the type

$$\int_0^{2\pi}\{f_1(z+ix\cos u+iy\sin u,\, u)+f_2(z+ix\cos u+iy\sin u,\, u) \\ +f_k(z+ix\cos u+iy\sin u,\, u)\}du$$

or

$$\int_0^{2\pi} f(z+ix\cos u+iy\sin u,\, u)\,du,$$

where f is a new function of the two arguments

$$z+ix\cos u+iy\sin u \quad \text{and} \quad u.$$

In this way we see that *the potential of any distribution of matter which attracts according to the Newtonian Law can be represented by an expression of the type*

$$\int_0^{2\pi} f(z+ix\cos u+iy\sin u,\, u)\,du.$$

The question now naturally suggests itself, whether the most general solution of Laplace's equation can be represented by an expression of this type. We shall shew that the answer to this is in the affirmative.

For let $V(x, y, z)$ be any solution (single-valued or many-valued) of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

Let (x_o, y_o, z_o) be some point at which some branch of the function $V(x, y, z)$ is regular. Then if we write

$$x = x_0 + X, \qquad y = y_0 + Y, \qquad z = z_0 + Z$$

it follows that for all points situated within a finite domain surrounding the point (x_o, y_o, z_o), this branch of the function $V(x, y, z)$ can be expanded in an absolutely and uniformly convergent series of the form

$$V = a_0 + a_1X + b_1Y + c_1Z + a_2X^2 + b_2Y^2 + c_2Z^2 + d_2YZ$$
$$+e_2ZX + f_2XY + a_3X^3 + \ldots.$$

Substituting this expansion in Laplace's equation, which can be written

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

and equating to zero the coefficients of the various powers of X, Y, and Z, we may obtain an infinite number of linear relations, namely

$$a_2 + b_2 + c_2 = 0, \text{ etc.}$$

between the constants in the expansion.

There are $\frac{1}{2}n(n-1)$ of these relations between the $\frac{1}{2}(n+1)(n+2)$ coefficients of the terms of any degree n in the expansion of V; so that only $\left\{\frac{1}{2}(n+1)(n+2)-\frac{1}{2}n(n-1)\right\}$ or $(2n+1)$ of the coefficients of terms of degree n in the expansion of V are really independent. It follows that the terms of degree n in V must be a linear combination of $(2n+1)$ linearly independent particular, solutions of Laplace's equation, which are of degree n in X, Y, Z.

To find these solutions, consider the expansion of the quantity

$$\left(Z + iX\cos u + iY\sin u\right)^n$$

as a sum of sines and cosines of multiples of u, in the form

$$\left(Z + iX\cos u + iY\sin u\right)^n = g_0\left(X,Y,Z\right) + g_1\left(X,Y,Z\right)\cos u$$
$$+g_2\left(X,Y,Z\right)\cos 2u + \ldots + g_n\left(X,Y,Z\right)\cos nu$$
$$+h_1\left(X,Y,Z\right)\sin u + h_2\left(X,Y,Z\right)\sin 2u + \ldots$$
$$+h_n\left(X,Y,Z\right)\sin nu.$$

Now $g_m(X, Y, Z)$ and $h_m(X, Y, Z)$ are together characterised by the fact that the highest power of Z contained in them is Z^{n-m}; moreover $g_m(X, Y, Z)$ is an even function of Y, whereas $h_m(X, Y, Z)$ is an odd function of Y; and hence the $(2n+1)$ quantities

$$g_0\left(X,Y,Z\right), g_1\left(X,Y,Z\right), \ldots, h_n\left(X,Y,Z\right)$$

are linearly independent of each other; and they are clearly homogeneous polynomials of degree n in X, Y, Z; and each of them satisfies Laplace's equation, since the quantity $(Z + iX \cos u + iY \sin u)^n$ does so. They may, therefore be taken as the $(2n + 1)$ linearly independent solutions of degree n of Laplace's equation.

Now since by Fourier's Theorem we have the relations

$$g_m(X,Y,Z) = \frac{1}{\pi}\int_0^{2\pi} (Z + iX\cos u + iY\sin u)^n \cos mu \, du,$$

$$h_m(X,Y,Z) = \frac{1}{\pi}\int_0^{2\pi} (Z + iX\cos u + iY\sin u)^n \cos mu \, du,$$

it follows that each of these $(2n + 1)$ solutions can be expressed in the form

$$\int_0^{2\pi} f(Z + iX\cos u + iY\sin u, u)\, du$$

and therefore any linear combination of these $(2n + 1)$ solutions can be expressed in this form. That is, the terms of any degree n in the expansion of V can be expressed in this form; and therefore V itself can be expressed in the form

$$\int_0^{2\pi} F(Z + iX\cos u + iY\sin u, u)\, du,$$

or

$$\int_0^{2\pi} F(z + ix\cos u + iy\sin u - z_0 - ix_0\cos u - iy_0\sin u, u)\, du,$$

or

$$\int_0^{2\pi} f(z + ix\cos u + iy\sin u, u)\, du,$$

since the $z_0 + ix_o \cos u + iy \sin u$ can be absorbed into the second argument u.

Now V was taken to be any solution of Laplace's equation, with no restriction beyond the assumption that some branch of it was at some point a regular function -- an assumption which is always tacitly made in the solution of differential equations; and thus we have the result, that *the general solution of Laplace's equation*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

is

$$V = \int_0^{2\pi} f(z + ix\cos u + iy\sin u, u)\, du,$$

where f is an arbitrary function of the two arguments

$$z + ix\cos u + iy\sin u \quad and \quad u.$$

Moreover, it is clear from the proof that no generality is lost by supposing that *f* is a periodic function of *u*.

This Theorem is the three-dimensional analogue of the theorem that the general solution of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

is

$$V=f(x+iy)+g(x-iy).$$

§ 3.

DEDUCTIONS FROM THE THEOREM OF § 2; PARTICULAR SOLUTIONS; EXPANSIONS OF THE GENERAL SOLUTION.

1°. *Interpretation of the solution.* We may give to the general solution just obtained a concrete interpretation, as follows.

Since a definite integral can be regarded as the limit of a sum, we can regard V as the sum of an infinite number of terms, each of the type

$$V_r=f_r\left(z+ix\cos u_r+iy\sin u_r\right)$$

each term corresponding to some value of u_r,

But this term is a solution of the equation

$$\frac{\partial^2 V_r}{\partial X_r^2}+\frac{\partial^2 V_r}{\partial Z_r^2}=0,$$

where

$$X_r=x\cos u_r+y\sin u_r,$$
$$Y_r=-x\sin u_r+y\cos u_r,$$
$$Z_r=z,$$

so that (X_r, Y_r, Z_r) represent coordinates derived from (x, y, z) by a rotation of the axes through and angle u_r round the axis of z.

Thus we see that *the general solution of Laplace's equation can be regarded as the sum of an infinite number of elementary constituents* V_r, *each constituent being the solution of an equation*

$$\frac{\partial^2 V_r}{\partial X_r^2}+\frac{\partial^2 V_r}{\partial Z_r^2}=0,$$

and the axes (X_r, Y_r, Z_r) being derived from the axes (x,y,z) by a simple rotation round the axis of z.

2°. *The particular solutions in terms of Legendre functions.* It is interesting to see how the well-known particular solutions of Laplace's equation in terms of Legendre functions can be obtained as a case of the solution given in § 2.

The particular solutions in question are of the form

$$r^n P_n^m(\cos\theta)\cos m\varphi \quad \text{and} \quad r^n P_n^m(\cos\theta)\sin m\varphi$$
$$(n = 0,1,2,...,\infty; \quad m = 0,1,2,...,n),$$

where (r, θ, φ) are the polar coordinates corresponding to the rectangular coordinates (x, y, z), and where

$$P_n^m(\cos\theta) = \frac{(-1)^m \sin^m\theta}{2^n n!} \frac{d^{n+m}(\sin^{2n}\theta)}{d(\cos\theta)^{n+m}}.$$

Now the function $P_n^m(\cos\theta)$ can be expressed by the integral

$$P_n^m(\cos\theta) = \frac{(n+m)(n+m-1)...(n+1)}{\pi}(-1)^{\frac{m}{2}} \int_0^{2\pi} (\cos\theta + i\sin\theta\cos\psi)^n \cos m\psi \, d\psi$$

and thus we have

$$r^n P_n^m(\cos\theta)\cos m\varphi = \frac{(n+m)(n+m-1)...(n+1)}{\pi}(-1)^{\frac{m}{2}}$$
$$\int_0^{2\pi} \left(z + i\sqrt{(x^2+y^2)}\cos\psi\right)^n \cos m\psi \cos m\varphi \, d\psi$$

$$= \frac{(n+m)(n+m-1)...(n+1)}{2\pi}(-1)^{\frac{m}{2}} \int_0^{2\pi} \left(z + i\sqrt{x' + y^2}\cos\psi\right)^n \cos m(\psi - \varphi) \, d\psi$$

$$= \frac{(n+m)(n+m-1)...(n+1)}{2\pi}(-1)^{\frac{m}{2}} \int_0^{2\pi} (z + ix\cos u + iy\sin u)^n \cos mu \, du.$$

We see therefore that the *solution* $r^n P_n^m(\cos\theta)\cos m\varphi$ *is a numerical multiple of*

$$\int_0^{2\pi} (z + ix\cos u + iy\sin u)^n \cos mu \, du.$$

Similarly the solution $r^n P_n^m(\cos\theta)\sin m\varphi$ is a numerical multiple of

$$\int_0^{2\pi} (z + ix\cos u + iy\sin u)^n \sin mu \, du.$$

From this it is clear that *in order to express any solution*

$$\int_0^{2\pi} f(z + ix\cos u + iy\sin u, u) \, du$$

of Laplace's equation, as a series of. harmonic terms of the form

$$r^n P_n^m(\cos\theta)\cos m\varphi \quad \text{and} \quad r^n P_n^m(\cos\theta)\sin m\varphi,$$

it is only necessary to expand the function f as a Taylor series with respect to the first argument $z + ix\cos u + iy\sin u$, and as a Fourier series with respect to the second argument u.

As an example of this procedure, we shall suppose it required to find the potential of a prolate spheroid in the form

$$\int_0^{2\pi} f(z + ix\cos u + iy\sin u,\, u)\,du,$$

and to expand this potential as a series of harmonies. Let

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=0$$

be the equation of the surface of the spheroid; and suppose that it is a homogeneous attracting body of mass M. To find its potential, we can make use of the theorem that the potential at external points is the same as that of a rod joining the foci, of line-density $\dfrac{3M\left(c^2-a^2-z^2\right)}{4\left(c^2-a^2\right)^{\frac{3}{2}}}$; that is, it is

$$\frac{3M}{8\pi\left(c^2-a^2\right)^{\frac{3}{2}}}\int_0^{2\pi} du \int_{-\sqrt{c^2-a^2}}^{\sqrt{c^2-a^2}} \frac{\left(c^2-a^2-\zeta^2\right)d\zeta}{z-\zeta+ix\cos u+iy\sin u}$$

or

$$\frac{3M}{8\pi\left(c-a\right)^{\frac{3}{2}}}\int_0^{2\pi}\left\{\left(c^2-a^2-B^2\right)\log\frac{B+\sqrt{c^2-a^2}}{B-\sqrt{c^2-a^2}}+2\sqrt{c^2-a^2}\,B\right\}du,$$

where B is written for $z + ix\cos u + iy\sin u$.

Expanding the integrand in ascending powers of $\dfrac{1}{B}$, we have the potential in the form

$$\frac{3M}{2\pi}\int_0^{2\pi}\left\{\frac{1}{1.3.B}+\frac{c^2-a^2}{3.5.B^3}+\frac{\left(c^2-a^2\right)^2}{5.7.B^5}+\ldots\right\}du.$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{du}{B^{n+1}}=\frac{P_n(\cos\theta)}{r^{n+1}},$$

this gives *the required expansion of the potential* of the spheroid in Legendre functions, namely the series

$$3M\left\{\frac{1}{1.3r}+\frac{\left(c^2-a^2\right)P_2(\cos\theta)}{3.5.r^3}+\frac{\left(c^2-a^2\right)^2P_4(\cos\theta)}{5.7.r^5}+\ldots\right\}.$$

This result may be extended to the case of the potential, of an ellipsoid with three unequal axes, by using a formula for the potential of an ellipsoid given by Laguerre[1])

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[1] C. R., 1878.

3°. *The particular solutions of Laplace's equation which involve Bessel functions.* We shall next shew how the well-known particular solutions of Laplace's equation in terms of Bessel, functions can be obtained as a case of the general solution. The particular solutions in question are of the form

$$e^{kz}J_m(k\varrho)\cos m\varphi \quad \text{and} \quad e^{kz}J_m(k\varrho)\sin m\varphi,$$

where k and m are constants, and z, ϱ, φ are the cylindrical co-ordinates corresponding to the rectangular co-ordinates x, y, z, so that

$$x = \varrho\cos\varphi, \qquad y = \varrho\sin\varphi.$$

Now if the solution

$$e^{kz}J_m(k\varrho)\cos m\varphi$$

we replace $J_m(k\varrho)$ by its value

$$J_m(k\varrho) = \frac{1}{\pi}\int_0^\pi \cos(m\theta - k\varrho\sin\theta)\,d\theta,$$

we find after a few simple transformations that

$$e^{kz}J_m(k\varrho)\cos m\varphi = \frac{(-1)^{\frac{m}{2}}}{2\pi}\int_0^{2\pi} e^{k(z+ix\cos u+iy\sin u)}\cos mu\,du.$$

The other solutions which involve $\sin m\varphi$ can be similarly expressed:
we see therefore that *the solutions*

$$e^{kz}J_m(k\varrho)\cos m\varphi \quad \text{and} \quad e^{kz}J_m(k\varrho)\sin m\varphi$$

are numerical multiples of

$$\int_0^{2\pi} e^{k(z+ix\cos u+iy\sin u)}\cos mu\,du$$

and

$$\int_0^{2\pi} e^{k(z+ix\cos u+iy\sin u)}\sin mu\,du$$

respectively. It follows from this that in *order to express any solution*

$$\int_0^{2\pi} f(z+ix\cos u+iy\sin u, u)\,du$$

of Laplace's equation as a sum of terms of the form

$$e^{kz}J_m(k\varrho)\cos m\varphi \quad \text{and} \quad e^{kz}J_m(k\varrho)\sin m\varphi,$$

it is only necessary to expand the function f in terms of the exponentials of its first argument $z + ix \cos u + iy \sin u$, *and as a Fourier series with respect to. its second argument u.*

As an example of the use which may be made of these results, we shall suppose it required to express the potential-function

$$V = 1 + e^{-z}J_0(\varrho) + e^{-2z}J_0(2\varrho) + e^{-3z}J_0(3\varrho) + \ldots$$

(where z is supposed positive) as a series of harmonic terms of the type involving Legendre functions: and also to find a distribution of attracting matter of which this in the potential. This can be done in the following way. We have

$$\begin{aligned} V &= 1 + e^{-z}J_0(\varrho) + e^{-2z}J_0(2\varrho) + e^{-3z}J_0(3\varrho) + \ldots \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\{1 + e^{-z-ix\cos u-iy\sin u} + e^{-2(z+ix\cos u+iy\sin u)} + \ldots\right\}du \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{du}{1-e^{-(z+ix\cos u+iy\sin u)}}. \end{aligned}$$

But if t be any variable different from zero, and such that $|t| < 2\pi$ we have

$$\frac{1}{1-e^{t}} = -\frac{1}{t} + \frac{1}{2} - B_1\frac{t}{2!} + B_2\frac{t^3}{4!} - B_3\frac{t^5}{6!} + \ldots,$$

where B_1, B_2, are Bernoulli's numbers. Therefore, so long as z is positive and $|z + ix \cos u + iy \sin u| < 2\pi$ i.e., so long as z is positive and $x^2 + y^2 + z^2 < 4\pi^2$ we have

$$V = \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{1}{z+ix\cos u+iy\sin u} + \frac{1}{2} + \frac{B_1}{2!}\left(z+ix\cos u+iy\sin u\right) + \ldots\right\}du$$

or

$$V = \frac{1}{r} + \frac{1}{2} - \frac{B_1}{2!}rP_1(\cos\theta) - \frac{B_2}{4!}r^3P_2(\cos\theta) + \frac{B_3}{6!}r^5P_5(\cos\theta) + \ldots$$

and *this is the required expansion of V as a series* of *harmonics involving Legendre functions.*

Next, since

$$\frac{1}{1-e^{-z}} = \frac{1}{2} + \frac{1}{z} + \sum_{n=1}^{\infty}\frac{1}{z+2ni\pi} + \frac{1}{z-2ni\pi},$$

we have

$$V=\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2}+\frac{1}{z+ix\cos u+iy\sin u}+\sum_{n=1}^{\infty}\left\{\begin{array}{l}\dfrac{1}{z+ix\cos u+iy\sin u+2ni\pi}\\+\dfrac{1}{z+ix\cos u+iy\sin u-2ni\pi}\end{array}\right\}\right]du,$$

or

$$V=\frac{1}{2}+\frac{1}{\sqrt{x^2+y^2+z^2}}+\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{x^2+y^2+(z+2ni\pi)^2}}+\frac{1}{\sqrt{x^2+y^2+(z-2ni\pi)^2}}\right\},$$

and therefore *V can be regarded as the potential due to a set of attracting masses placed at equal imaginary intervals $2i\pi$ along the axis of z.*

§ 4.

THE DIFFERENTIAL EQUATION $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=k^2\frac{\partial^2 V}{\partial t^2}$.

We shall next consider the general differential equation of wave-motions,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=k^2\frac{\partial^2 V}{\partial t^2},$$

where k is a constant.

Writing kt for t, this become

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial t^2},$$

which we shall take for the present as the standard form of the equation.

In order to find the general solution of this equation, we follow a procedure analogous to that of § 2. Let $V(x, y, z, t)$ be any solution (single-valued or many-valued) of the equation; and let (x_o, y_o, z_o, t_o) be a place at which some branch of the function V is regular. Then if we write $x=x_0+X$, $y=y_0+Y$, $z=z_0+Z$, $t=t_0+T$, it will be possible to expand this branch of the function V as a power-series of the form

$$V=a_0+a_1X+b_1Y+c_1Z+d_1T+a_2X^2+b_2Y^2+c_2Z^2+d_2T^2+e_2XY$$
$$+f_2XZ+g_2XT+h_2YZ+k_2YT+l_2ZT+a_3X^3+\ldots,$$

which will be absolutely and uniformly convergent for a certain finite domain of values of X, Y, Z, T. Substituting this expansion in the differential equation, which may be written

$$\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}+\frac{\partial^2 V}{\partial Z^2}=\frac{\partial^2 V}{\partial T^2},$$

and equating to zero the coefficients of various powers of X, Y and Z, we obtain an infinite number of linear relations, namely

$$a_2 + b_2 + c_2 = d_2, \quad \text{etc.},$$

between the constants in the expansion. There are $\frac{1}{6}(n-1)n(n+1)$ of these relations between the $\frac{1}{6}(n+1)(n+2)(n+3)$ coefficients of terms of any degree n in the expansion of V; so that only

$$\frac{1}{6}\{(n+1)(n+2)(n+3)-(n-1)n(n+1)\}$$

or

$$(n+1)^2$$

of the coefficients of terms of degree n in the expansion of V are really independent. It follows that the terms of degree n in V must be a linear combination of $(n + 1)^2$ linearly, independent particular solutions of degree n in X, Y, Z, T.

To find these solutions, consider the expansion of the quantity

$$(X \sin u \cos v + Y \sin u \sin v + Z \cos u + T)^n .$$

If we first take the expansion in the form

$$g_0 + g_1 \cos v + g_2 \cos 2v + \ldots + g_n \cos nv$$
$$+h_1 \sin v + h_2 \sin 2v + \ldots + h_n \sin nv,$$

we have seen in § 2 that $g_0, g_1, \ldots, g_n, h_1, \ldots, h_n$, are linearly independent functions of X, Y, Z, and T. Moreover, g_m and h_m are of the form $\sin^m uXa$ polynomial of degree $(n - m)$ in cos u, and therefore each of them contains $(n - m + 1)$ independent polynomials in X, Y, Z, T. Thus the total number of independent polynomials in X, Y, Z, T, in the expansion of

$$(X \sin u \cos v + Y \sin u \sin v + Z \cos u + T)^n$$

in sines and cosines of multiples of u and v, is

$$(n+1) + 2n + 2(n-1) + 2(n-2) + \ldots + 2$$

or

$$(n+1)^2.$$

Now each of these polynomials must satisfy the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial t^2},$$

since the quantity

$$(X \sin u \cos v + Y \sin u \sin v + Z \cos u + T)^n$$

does so: and therefore they may be taken as the $(n+1)^2$ linearly independent solutions of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial t^2}$$

which are homogeneous of degree n in X, Y, Z, T. Now by Fourier's theorem we have

$$g_m=\frac{1}{\pi}\int_0^{2\pi}(X\sin u\cos v+Y\sin u\sin v+Z\cos u+T)^n\cos mv\,dv;$$

and since g_m is of the form

$$\sum_{r=0}^{n-m}u_r\sin^m u\cos^r u,$$

where u_r is one of the polynomials in question, it is clear that g_m can be expressed as a sum of sines or cosines of multiples of u, according as m is even or odd; and the coefficient of one of these sines or cosines, say of $\cos su$, is

$$\frac{2}{\pi}\int_0^{\pi}g_m\cos su\,du.$$

It follows that each of the polynomials u_r can be expressed in the form

$$\int_0^{\pi}g_m f(u)du,$$

where $f(u)$ denotes some periodic function of u; that is, it can be expressed in the form

$$\int_0^{2\pi}\int_0^{\pi}(X\sin u\cos v+Y\sin u\sin v+Z\cos u+T)^n f(u)\cos mv\,du\,dv.$$

It follows from this that each of the $(n+1)^2$ polynomial solution of degree n can be expressed in the form

$$\int_0^{2\pi}\int_0^{\pi}(X\sin u\cos v+Y\sin u\sin v+Z\cos u+T)^n f(u,v)du\,dv,$$

where $f(u, v)$ denotes some periodic function of u and v; and therefore the terms of degree n in V can be expressed in this form.

The function V itself can therefore be expressed in the form

$$\int_0^{2\pi}\int_0^{\pi}f(X\sin u\cos v+Y\sin u\sin v+Z\cos u+T,u,v)du\,dv,$$

where f denotes some function of the three arguments

$$X\sin u\cos v+Y\sin u\sin v+Z\cos u+T,\ u,\ \text{and}\ v;$$

and f may without loss of generality be supposed to be periodic in u and v.

Now

$$X\sin u\cos v + Y\sin u\sin v + Z\cos u + T$$
$$= (x\sin u\cos v + y\sin u\sin v + z\cos u + t)$$
$$-(x_0\sin u\cos v + y_0\sin u\sin v + z_0\cos u + t_0);$$

and the term

$$(x_0\sin u\cos v + y_0\sin u\sin v + z_0\cos u + t_0)$$

can be absorbed into the arguments u and v; moreover V was taken to be any solution of the partial differential equation; we have, therefore, on writing $\frac{t}{k}$ for t, the result that the *general solution of the partial differential equation of wave-motions,*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = k^2\frac{\partial^2 V}{\partial t^2},$$

is

$$V = \int_0^{2\pi}\int_0^{\pi} f\left(x\sin u\cos v + y\sin u\sin v + z\cos u + \frac{t}{k}, u, v\right)du\,dv,$$

where f is an arbitrary function of the three arguments

$$x\sin u\cos v + y\sin u\sin v + z\cos u + \frac{t}{k}, u, v.$$

§ 5.

DEDUCTIONS FROM THE GENERAL SOLUTION OF § 4.

1°. *The analysis of wave-motions.* We shall now deduce from the general solution thus obtained a result relating the analysis of those phenomena which are represented by solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = k^2\frac{\partial^2 V}{\partial t^2}.$$

If we revert to the fundamental idea of the definite integral as the limit of a sum of an infinite number of terms, we see that the general solution

$$V = \int_0^{2\pi}\int_0^{\pi} f\left(x\sin u\cos v + y\sin u\sin v + z\cos u + \frac{t}{k}, u, v\right)du\,dv$$

can be interpreted as meaning that V is the sum of an infinite number of terms of the type

$$f\left(x\sin u\cos v + y\sin u\sin v + z\cos u + \frac{t}{k}, u, v\right),$$

there being one of these terms corresponding to every direction in space given by the direction-cosines

$$\sin u \cos v, \qquad \sin u \sin v, \qquad \cos u.$$

The solution V can therefore be regarded as the sum of constituent solutions, each of the type

$$F\left(x \sin u \cos v + y \sin u \sin v + z \cos u + \frac{t}{k}\right)$$

where the function F varies from one direction (u, v) to another.

Now let us fix our attention on one of these constituent solutions F. If for some range of values of the quantity

$$x \sin u \cos v + y \sin u \sin v + z \cos u + \frac{t}{k},$$

the function F *is* finite and continuous, we can for this range of values express F by Fourier's integral formula in the form

$$\frac{1}{\pi}\int_0^\infty d\lambda \int_a^b F(\alpha) \cos\left\{\lambda\left(x \sin u \cos v + y \sin u \sin v + z \cos u + \frac{t}{k}\right) - \lambda\alpha\right\} d\lambda\, d\alpha,$$

where a and b are the terminals of this range of values; or supposing the integration with respect to α to be performed,

$$\int_0^\infty g(\lambda) {}_{\sin}^{\cos}\left\{\lambda\left(x \sin u \cos v + y \sin u \sin v + z \cos u + \frac{t}{k}\right)\right\} d\lambda,$$

where $g(\lambda)$ denotes some function of λ.

Now let us again revert to the idea of the definite integral as the limit of a sum. Then this latter integral can be regarded as the sum of an infinite number of terms of the type

$${}_{\sin}^{\cos}\left\{\lambda\left(x \sin u \cos v + y \sin u \sin v + z \cos u + \frac{t}{k}\right)\right\},$$

each term being multiplied by some factor depending on λ.

The solution V can therefore be regarded as constituted by the superposition of terms of this last type. But a term of this type represents a *simple uniform plane wave;* for on transforming the axes so that the new axis of x is the line whose direction-cosines are

$$\sin u \cos v, \qquad \sin u \sin v, \qquad \cos u,$$

the term becomes

$${}_{\sin}^{\cos}\lambda\left(x + \frac{t}{k}\right),$$

which represents a simple plane wave, whose direction of propagation is the new axis of x. We see therefore that *the general finite solution of the differential equation of, wave-motions,*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = k^2 \frac{\partial^2 V}{\partial t^2}.$$

can be analysed into simple plane waves, represented by terms of the type

$$F(\lambda,u,v)_{\sin}^{\cos}\left\{\lambda\left(x\sin u\cos v+y\sin u\sin v+z\cos u+\frac{t}{k}\right)\right\}.$$

It is interesting to observe that Dr. Johnstone Stoney in 1897[1]) shewed by physical reasoning, and without any reference to the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=k^2\frac{\partial^2 V}{\partial t^2}.$$

that all the disturbance of the luminiferous ether arising from sources of certain kinds can be resolved into trains of plane waves.

2°. *Solution of the equation*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0.$$

If a solution W of the equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}=\frac{\partial^2 W}{\partial t^2}$$

be of the form Ve^{it}, where V is a function of x, y, z only, which does not involve t, then V clearly satisfies the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0,$$

and therefore, on reference to the general solution of the warve-motion equation found in § 4, we see that *the general solution of the equation*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0$$

is

$$V=\int_0^{2\pi}\int_0^{\pi}e^{i(x\sin u\cos v+y\sin u\sin v+z\cos v)}f(u,v)\,du\,dv.$$

3°. *Deduction of the known particular solutions of the equation*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0.$$

It is know that the particular solution of the equation

[1] Philosoph. Magazine, (V) XLIII.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0$$

exist, which are of the form

$$V=r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(r)P_n^m(\cos\theta)_{\sin}^{\cos} m\varphi$$

$$(n=0,1,2,...;\qquad m=0,1,2,...n),$$

where (r, θ, φ) are the polar coordinates corresponding to x, y, z. We shall now shew, how these may be derived from the general solution of the equation which has just been found.

For let the general. solution be written in the form

$$V=\int_0^{2\pi}\int_0^{\pi} e^{i(x\sin u\cos v+y\sin u\sin v+z\cos v)}f(u,v)\,du\,dv,$$

where $f(u, v)$ is an arbitrary function of the two arguments u and v, which may without loss of generality be taken to be periodic in u and v.

Now let the function $f(u, v)$ be expanded in surface-harmonics of u and v, so that

$$V=\sum_{n=0}^{\infty}\int_0^{2\pi}\int_0^{\pi} e^{i(x\sin u\cos v+y\sin u\sin v+z\cos v)}Y_n(u,v)\sin u\,du\,dv$$

where Y_n is a surface-harmonic of order n, i.e., if

$$\xi=\varrho\sin u\cos v,\qquad \eta=\varrho\sin u\sin v,\qquad \zeta=\varrho\cos u,$$

are regarded as the co-ordinates of a point in space, then $\varrho^n Y_n(u,v)$ is a homogeneous polynomial of degree n in ξ, η, ζ satisfying Laplace's equation

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}+\frac{\partial^2 V}{\partial \zeta^2}=0.$$

Next, let the variables be changed by the substitution

$$\cos u=\cos\theta\cos\omega+\sin\theta\sin\omega\cos v',$$

$$\sin u\sin(\varphi-v)=\sin\omega\sin v',$$

$$\sin u\cos(\varphi-v)=\cos\omega\sin\theta-\sin\omega\cos v'\cos\theta,$$

so that ($\varrho\sin\omega\cos v'$, $\varrho\sin\omega\sin v'$, $\varrho\cos\omega$) are the co-ordinates of the point (ξ, η, ζ) referred to new axes, the line whose direction-cosines are ($\sin\theta\cos\varphi$, $\sin\theta\sin\varphi$, $\cos\theta$) being taken as the new axis of z.

Thus

$$V=\sum_{n=0}^{\infty}\int_0^{2\pi}\int_0^{\pi} e^{ir\cos\omega}Y_n(u,v)\sin\omega\,d\omega\,dv'.$$

But a surface-harmonic of any order n remains a surface-harmonic of order n under any transformation of axes in which the origin is unchanged, and therefore $Y_n(u, v)$ is a surface harmonic of order n in ω and v'; and consequently it can be expanded in the form

$$A_n(\theta,\varphi)P_n(\cos\omega)+A_n^1(\theta,\varphi)P_n^1(\cos\omega)\cos v'+A_n^2(\theta,\varphi)P_n^2(\cos\omega)\cos 2v'$$
$$+\ldots+A_n^n(\theta,\varphi)P_n^n(\cos\omega)\cos nv'$$
$$+B_n^1(\theta,\varphi)P_n^1(\cos\omega)\sin v'+\ldots+B_n^n(\theta,\varphi)P_n^1(\cos\omega)\sin nv',$$

where $A_n(\theta,\varphi),\ldots,B_n^n(\theta,\varphi)$ are functions of θ and φ. Substituting this value for $Y_n(u, v)$ in the integral, and performing the integration with respect to v', we have

$$V=\sum_{n=0}^{\infty}A_n(\theta,\varphi)\int_0^{\pi}e^{ir\cos\omega}P_n(\cos\omega)\sin\omega\,d\omega;$$

and in virtue of the relation[2])

$$\int_0^{\pi}e^{ir\cos\omega}P_n(\cos\omega)\sin\omega\,d\omega=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{i^nJ_{n+\frac{1}{2}}(r)}{\sqrt{r}},$$

this can be written in the form

$$V=\sum_{n=0}^{\infty}r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(r)f_n(\theta,\varphi)$$

where $f_n(\theta, \varphi)$ denotes some function of θ and φ.

Since the surface-harmonies $Y_n(\theta, \varphi)$ were independent of each other, the functions $f_n(\theta, \varphi)$, will be independent of each other and therefore each of the quantities

$$r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(r)f_n(\theta,\varphi)$$

will be a solution of the equation

$$\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\frac{\partial^2V}{\partial z^2}+V=0.$$

But on transforming this equation to polar co-ordinates, and substituting the

$$r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(r)f_n(\theta,\varphi)$$

for V, we find that the function $f_n(\theta, \varphi)$ must satisfy the differential equation for a surface-harmonic in θ and φ of order n. It follows that $f_n(\theta, \varphi)$ can be expanded in the form

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[2] A proof of this and several related results will be found in a paper shortly to be published by the author.

$$f_n(\theta,\varphi) = A_n P_n(\cos\theta) + A_n^1 \cos\varphi P_n^1(\cos\theta) + \ldots + A_n \cos n\varphi P_n^n(\cos\theta)$$
$$+ B_n^1 \sin\varphi P_n^2(\cos\theta) + \ldots + B_n^n \sin n\varphi P_n^n(\cos\theta),$$

and thus *the particular solutions*

$$r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(r) P_n^m(\cos\theta)_{\sin}^{\cos} m\varphi$$

are obtained.

Moreover, it is clear from the above proof that *in order to expand any solution*

$$V = \int_0^{2\pi}\int_0^{\pi} e^{i(x\sin u\cos v + y\sin u\sin v + z\cos u)} f(u,v)\sin u\, du\, dv$$

of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + V = 0.$$

as series of the form

$$\sum_{n=0}^{\infty} r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(r) Y_n(\theta,\varphi),$$

where Y_n is a surface-harmonic of order n in θ and φ, it is only necessary to expand the function f(u, v) in surface-harmonics of u and v.

4^0. *Expression of the solution of the equation*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + V = 0$$

as a series of generalised Bessel functions.

Another analysis of the solutions of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + V = 0.$$

entirely different from that given in 3^0, can be found in the following way.

Consider the expression

$$e^{\frac{1}{4}x\left(s-\frac{1}{s}\right)\left(t+\frac{1}{t}\right)-\frac{1}{4}y\left(s-\frac{1}{s}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}z\left(s+\frac{1}{s}\right)};$$

if this expression be regarded as a function of s and t, it can for finite non-zero values of s and t be expanded as a series of (positive and negative) integral powers of s and t, the coefficients in this series being functions of x, y and z. Let the coefficient of the term in $s^m t^n$ be denoted by $J_{m,n}(x, y, z)$: so that we have, the relation

$$e^{\frac{1}{4}x\left(s-\frac{1}{s}\right)\left(t+\frac{1}{t}\right)-\frac{1}{4}y\left(s-\frac{1}{s}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}z\left(s+\frac{1}{s}\right)} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} J_{m,n}(x,y,z)s^m t^n.$$

This equation can be regarded as a generalisation of the equation

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}J_n(z)t^n,$$

which defines the ordinary Bessel functions; and we shall consequently call the functions $J_{m,n}(x, y, z)$ *generalised Bessel functions.*

We now proceed to establish some properties of the functions $J_{m,n}(x, y, z)$; it will be seen that they are very similar to those of the ordinary Bessel functions.

In the first place, since the expression

$$V=e^{\frac{1}{4}x\left(s-\frac{1}{s}\right)\left(t+\frac{1}{t}\right)-\frac{1}{4}y\left(s-\frac{1}{s}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}z\left(s+\frac{1}{s}\right)}$$

satisfies the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0,$$

it follows that each of the functions $J_{m,n}(x, y, z)$ satisfies the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0.$$

In the second place, we shall obtain an expression for $J_{m,n}(x, y, z)$ as a definite integral. By Laurent's theorem, we know that the coefficient of s^m in the expansion of

$$e^{\frac{1}{4}x\left(s-\frac{1}{s}\right)\left(t+\frac{1}{t}\right)-\frac{1}{4}y\left(s-\frac{1}{s}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}z\left(s+\frac{1}{s}\right)}$$

is

$$\frac{1}{2\pi i}\int_C s^{-m-1}e^{\frac{1}{4}x\left(s-\frac{1}{s}\right)\left(t+\frac{1}{t}\right)-\frac{1}{4}y\left(s-\frac{1}{s}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}z\left(s+\frac{1}{s}\right)}ds,$$

where C is any simple contour in the s-plane surrounding the origin; and again applying Laurent's theorem, the coefficient of t^n in this expression is seen to be

$$\frac{1}{2\pi^2}\int_C\int_D s^{-m-1}t^{-n-1}e^{\frac{1}{4}x\left(s-\frac{1}{s}\right)\left(t+\frac{1}{t}\right)-\frac{1}{4}y\left(s-\frac{1}{s}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}z\left(s+\frac{1}{s}\right)}dsdt,$$

where D is any simple contour in the t-plane surrounding the origin.

Now write $s = e^{iu}$, $t = e^{iv}$. Thus we *have the result*

$$J_{m,n}(x,y,z)=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}e^{-miu-niv+ix\sin u\cos v+iy\sin u\sin v+iz\cos u}du\,dv,$$

which may, be regarded as the analogue of Bessel's integral

$$J_n(z)=\frac{1}{\pi}\int_0^\pi \cos(nu - z\sin u)\,du.$$

The functions $J_{m,n}(x, y, z)$ likewise possess an addition theorem: for we have

$$e^{\frac{1}{4}(x+a)\left(s-\frac{1}{s}\right)\left(t+\frac{1}{t}\right)-\frac{1}{4}(y+b)\left(s-\frac{1}{s}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}(z+c)\left(s+\frac{1}{s}\right)}$$
$$=e^{\frac{1}{4}x\left(s-\frac{1}{s}\right)\left(t+\frac{1}{t}\right)-\frac{1}{4}y\left(s-\frac{1}{s}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}z\left(s+\frac{1}{s}\right)}e^{\frac{1}{4}a\left(s-\frac{1}{s}\right)\left(t+\frac{1}{t}\right)-\frac{1}{4}b\left(s-\frac{1}{s}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}c\left(s+\frac{1}{s}\right)}$$

and so

Equating coefficients on both sides of this equation, we *have the result*

$$J_{m,n}(x+a,y+b,z+c)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}J_{p,q}(x,y,z)\,J_{m-p,n-q}(a,b,c),$$

which is the addition-theorem for generalised Bessel functions, and is the analogue of the well-known result

$$J_n(z+c)=\sum_{p=-\infty}^{\infty}J_p(z)\,J_{n-p}(c).$$

We shall now shew how the generalised Bessel functions furnish an analysis of the general solution of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0.$$

For the general solution is, by 2^0,

$$V=\int_0^{2\pi}\int_0^{\pi}e^{i(x\sin u\cos v+y\sin u\sin v+z\cos u)}f(u,v)\sin u\,du\,dv,$$

where $f(u, v)$ can without loss of generality be taken to be a periodic function of u and v.

Now let the function $f(u, v)$ be expanded by the extended form of Fourier's theorem, in the form

$$f(u,v)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}a_{m,n}e^{imu+inv}.$$

Then we have

$$V=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}a_{m,n}\int_0^{\pi}\int_0^{2\pi}e^{i(x\sin u\cos v+y\sin u\sin v+z\cos u+mu+nv)}du\,dv.$$

$$\sum_{m,n}J_{m,n}(x+a,y+b,z+c)s^m t^n=\sum_{m,n}J_{m,n}(x,y,z)s^m t^n\times\sum_{m,n}J_{m,n}(a,b,c)s^m t^n.$$

Comparing this with the form just found for the generalised Bessel functions, we see that the *general solution of the equation*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+V=0$$

can be written

$$V=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{m,n}J_{m,n}(x,y,z),$$

where the quantities $a_{m,n}$ are arbitrary constants. This furnishes an alternative analysis of the solution to that given in 2°.

5°. *Gravitation and Electrostatic Attraction explained as modes of Wave disturbance.*
The result of 1°, namely that any solution of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=k^2\frac{\partial^2 V}{\partial t^2}$$

can be analysed into simple plane waves, throws a new light on the nature of those forces, such as gravitation and electrostatic attraction, which vary as the inverse square of the distance. For if a system of forces of this character be considered, their potential (or their component in any given direction) satisfies the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

and therefore *à fortiori* it satisfies the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=k^2\frac{\partial^2 V}{\partial t^2}$$

where k is any constant. It follows from 1° that this potential (or forcecomponent) can be analysed into simple plane waves in various directions, each wave being propagated with constant velocity. These waves interfere with each other in such a way that, when the action has once been set up, the disturbance at any point does not vary with the time, and depends only on the coordinates (x, y, z) of the point.

It is not difficult to construct, synthetically, systems of coexistent simple waves, having the property that the total disturbance at any point (due to the sum of all the waves) varies from point to point, but does not vary with the time. A simple example of such a system in the following.

Suppose that a particle is emitting spherical waves, such that the disturbance at a distance r from the origin, at time t, due to those waves whose wave-length lies between $\frac{2\pi}{\mu}$ and $\frac{2\pi}{\mu+d\mu}$, is represented by

$$\frac{2d\mu}{\pi\mu}\frac{\sin(\mu Vt-\mu r)}{r}$$

where V is the velocity of propagation of the waves. Then after the waves have reached the point r, so that $(Vt - r)$ is positive, the total disturbance at the point (due to the sum of all the waves) is

$$\int_0^\infty \frac{2d\mu}{\pi\mu}\frac{\sin(\mu Vt-\mu r)}{r}.$$

Take μVt - $\mu_r = y$, where y is a new variable. Then this disturbance is

$$\frac{2}{\pi r}\int_0^\infty \frac{\sin y}{y}dy;$$

or, since

$$\int_0^\infty \frac{\sin y}{y}dy = \frac{\pi}{2},$$

it is

$$\frac{1}{r}.$$

The total disturbance at any point, due to this system of waves, is therefore independent of the time, and is everywhere proportional. to the gravitational potential due to the particle at the point.

It is clear, from the foregoing that the field of force duo to a gravitating body can be analysed, by a "spectrum analysis" as it were, into an infinite number of constituent fields; and although the whole field of force does not vary with the time, yet *each of the constituent fields, is of an undulatory character, consisting of a simple wave-disturbance propagated with uniform velocity.* This analysis of the field into constituent fields can most easily be accomplished by analysing the potential $\frac{1}{r}$ of each attracting particle into terms of the type

$$\frac{\sin(\mu Vt-\mu r)}{r}$$

as in the example already given. To each of these terms will correspond one of the constituent fields. In each of these constituent fields the potential will be constant along each wave-front, and consequently the gravitational force in each constituent field will be perpendicular to the wave-front, i.e. the waves will be longitudinal.

But these results assimilate the propagation of gravity to that of light: for the undulatory phenomena just described, in which the varying vector is a gravitational force perpendicular to the wave-front, may be compared with the undulatory phenomena made familiar by the electromagnetic theory of light, in which the varying vectors consist of electric and magnetic forces parallel to the wave-front. The waves are in other respects exactly similar, and it seems probable that an identical property of the medium ensures their transmission through space.

This undulatory theory of gravity would require that gravity should be propagated with a finite velocity, which however need not be the same as that of light, and may be enormously greater.

Of course this investigation does not explain the *cause* of gravity; all that is done is to show that in order to account for the propagation across space of forces which vary, as the inverse square of the distance, we have only to suppose that the medium is capable of transmitting, with a definite though large velocity, simple periodic undulatory disturbances, similar to those whose propagation by the medium constitutes, according to the electromagnetic theory, the transmission of light.