A Four-Dimensional Continuum Theory of Space-Time and the Classical Physical Fields

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Abstract

In this work, we attempt to describe the classical physical fields of gravity, electromagnetism, and the so-called intrinsic spin (chirality) in terms of a set of fully geometrized constitutive equations. In our formalism, we treat the four-dimensional space-time continuum as a deformable medium and the classical fields as intrinsic stress and spin fields generated by infinitesimal displacements and rotations in the space-time continuum itself. In itself, the unifying continuum approach employed herein may suggest a possible unified field theory of the known classical physical fields.

1. Introduction

Many attempts have been made to incorporate the so-called standard (Hookean) linear elasticity theory into general relativity in the hope to describe the dynamics of material bodies in a fully covariant four-dimensional manner. As we know, many of these attempts have concentrated solely on the treatment of material bodies as linearly elastic continua and not quite generally on the treatment of space-time itself as a linearly elastic, deformable continuum. In the former case, taking into account the gravitational field as the only intrinsic field in the space-time continuum, it is therefore true that the linearity attributed to the material bodies means that the general consideration is limited to weakly gravitating objects only. This is because the curvature tensor is in general quadratic in the the so-called connection which can be said to represent the displacement field in the space-time manifold. However, in most cases, it is enough to consider an infinitesimal displacement field only such that the linear theory works perfectly well. However, for the sake of generality, we need not assume only the linear behavior of the properly-stressed space-time continuum (and material bodies) such that the possible limiting consequences of the linear theory can be readily overcome whenever it becomes necessary. Therefore, in the present work, we shall both consider both the linear and non-linear formulations in terms of the response of the space-time geometry to infinitesimal deformations and rotations with intrinsic generators.

A few past attempts at the full description of the elastic behavior of the space-time geometry in the presence of physical fields in the language of general relativity have been quite significant. However, as standard general relativity describes only the field of gravity in a purely geometric fashion, these past attempts have generally never gone beyond the simple reformulation of the classical laws of elasticity in the presence of gravity which means that these classical laws of elasticity have merely been referred to the general four-dimensional curvilinear coordinates of Riemannian geometry, nothing more. As such, any possible interaction between the physical fields (e.g., the interaction between gravity and electromagnetism) has not been investigated in detail.

In the present work, we develop a fully geometrized continuum theory of space-time and the classical physical fields in which the actions of these physical fields contribute directly to the dynamics of the space-time geometry itself. In this model, we therefore assume that a physical field is directly associated with each and every point in the region of space-time occupied by the field (or, a material body in the case of gravity). This allows us to describe the dynamics of the space-time geometry solely in terms of the translational and rotational behavior of points within the occupied region. Consequently, the geometric quantities (objects) of the space-time continuum (e.g., curvature) are directly describable in terms of purely kinematic variables such as displacement, spin, velocity, acceleration, and the particle symmetries themselves.

As we have said above, at present, for the sake of simplicity, we shall assume the inherently elastic behavior of the space-time continuum. This, I believe, is adequate especially in most cosmological cases. Such an assumption is nothing but intuitive, especially when considering the fact that we do not fully know the reality of the constituents of the fabric of the Universe yet. As such, the possible limitations of the present theory, if any, can be neglected considerably until we fully understand how the fabric of the space-time continuum is actually formed and how the properties of individual elementary particles might contribute to this formation.

2. The Fundamental Geometric Properties of a Curved Manifold

Let us present the fundamental geometric objects of an $n-$dimensional curved manifold. Let $\omega_a = \frac{\partial X^i}{\partial x^a} E_i = \partial_a X^i E_i$ (the Einstein summation convention is assumed throughout this work) be the covariant (*frame*) basis spanning the $n-$dimensional base manifold C^∞ with local coordinates $x^a = x^a\left(X^k\right)$. The contravariant (*coframe*) basis θ^b is then given via the orthogonal projection $\left\langle \theta^b, \omega_a \right\rangle = \delta^b_a$, where δ^b_a are the components of the Kronecker delta (whose value is unity if the indices coincide or null otherwise). The set of linearly independent local directional derivatives $E_i = \frac{\partial}{\partial X^i} = \partial_i$ gives the coordinate basis of the locally flat tangent space $T_x(M)$ at a point $x \in C^\infty$. Here M denotes the topological space of the so-called $n-$tuples $h(x) = h\left(x^1, ..., x^n\right)$ such that relative to a given chart $\left(U, h(x)\right)$ on a neighborhood U of a local coordinate point x, our $C^\infty-$differentiable manifold itself is a topological space. The dual basis to E_i spanning the locally flat cotangent space $T^*_x(M)$ will then be given by the differential elements dX^k via the relation $\left\langle dX^k, \partial_i \right\rangle = \delta^k_i$. In fact and in general, the *one-forms* dX^k indeed act as a linear map $T_x(M) \to IR$ when applied to an arbitrary vector field

$F \in T_x(M)$ of the explicit form $F = F^i \dfrac{\partial}{\partial X^i} = f^a \dfrac{\partial}{\partial x^a}$. Then it is easy to see that $F^i = F\ X^i$ and $f^a = F\ x^a$, from which we obtain the usual transformation laws for the contravariant components of a vector field, i.e., $F^i = \partial_a X^i\ f^a$ and $f^i = \partial_i x^a\ F^i$, relating the localized components of F to the general ones and vice versa. In addition, we also see that $\langle dX^k, F \rangle = F X^k = F^k$.

The components of the symmetric metric tensor $g = g_{ab}\ \theta^a \otimes \theta^b$ of the base manifold C^∞ are readily given by

$$g_{ab} = \langle \omega_a, \omega_b \rangle$$

satisfying

$$g_{ac}\ g^{bc} = \delta_a^b$$

where $g^{ab} = \langle \theta^a, \theta^b \rangle$. It is to be understood that the covariant and contravariant components of the metric tensor will be used to raise and the (component) indices of vectors and tensors.

The components of the metric tensor $g(x_N) = \eta_{ik}\ dX^i \otimes dX^k$ describing the locally flat tangent space $T_x(M)$ of rigid frames at a point $x_N = x_N(x^a)$ are given by

$$\eta_{ik} = \langle E_i, E_k \rangle = diag(\pm 1, \pm 1, ..., \pm 1)$$

In four dimensions, the above may be taken to be the components of the Minkowski metric tensor, i.e., $\eta_{ik} = \langle E_i, E_k \rangle = diag(1, -1, -1, -1)$.

Then we have the expression

$$g_{ab} = \eta_{ik}\ \partial_a X^i\ \partial_b X^k$$

The line-element of C^∞ is then given by

$$ds^2 = g = g_{ab}\ (\partial_i x^a\ \partial_k x^b)\, dX^i \otimes dX^k$$

where $\theta^a = \partial_i x^a\ dX^i$.

Given the existence of a local coordinate transformation via $x^i = x^i(\bar{x}^\alpha)$ in C^∞, the components of an arbitrary tensor field $T \in C^\infty$ of rank (p, q) transform according to

$$T^{ab\ldots g}_{cd\ldots h} = T^{\alpha\beta\ldots\lambda}_{\mu\nu\ldots\eta}\ \partial_\alpha x^a\ \partial_\beta x^b\ \ldots \partial_\lambda x^g\ \partial_c \bar{x}^\mu\ \partial_d \bar{x}^\nu\ \ldots \partial_h \bar{x}^\eta$$

Let $\delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p}$ be the components of the generalized Kronecker delta. They are given by

$$\delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p} = \in_{j_1 j_2 \ldots j_p}\ \in^{i_1 \ldots i_p} = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \delta^{i_2}_{j_1} & \ldots & \delta^{i_p}_{j_1} \\ \delta^{i_1}_{j_2} & \delta^{i_2}_{j_2} & \ldots & \delta^{i_p}_{j_2} \\ \ldots & \ldots & \ldots & \ldots \\ \delta^{i_1}_{j_p} & \delta^{i_2}_{j_p} & \ldots & \delta^{i_p}_{j_p} \end{pmatrix}$$

where $\in_{j_1 j_2 \ldots j_p} = \sqrt{\det(g)}\ \varepsilon_{j_1 j_2 \ldots j_p}$ and $\in^{i_1 i_2 \ldots i_p} = \dfrac{1}{\sqrt{\det(g)}}\ \varepsilon^{i_1 i_2 \ldots i_p}$ are the covariant and contravariant components of the completely anti-symmetric Levi-Civita permutation tensor, respectively, with the ordinary permutation symbols being given as usual by $\varepsilon_{j_1 j_2 \ldots j_q}$ and $\varepsilon^{i_1 i_2 \ldots i_p}$. Again, if ω is an arbitrary tensor, then the object represented by

$${}^*\omega_{j_1 j_2 \ldots j_p} = \frac{1}{p!}\ \delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p}\ \omega_{i_1 i_2 \ldots i_p}$$

is completely anti-symmetric.

Introducing a generally asymmetric connection Γ via the covariant derivative

$$\partial_b\, \omega_a = \Gamma^c_{ab}\ \omega_c$$

i.e.,

$$\Gamma^c_{ab} = \left\langle \theta^c, \partial_b\, \omega_a \right\rangle = \Gamma^c_{(ab)} + \Gamma^c_{[ab]}$$

where the round index brackets indicate symmetrization and the square ones indicate anti-symmetrization, we have, by means of the local coordinate transformation given by $x^a = x^a\left(\bar{x}^\alpha\right)$ in C^∞

$$\partial_b\, e^\alpha_a = \Gamma^c_{ab}\ e^\alpha_c - \bar{\Gamma}^\alpha_{\beta\lambda}\ e^\beta_a\ e^\lambda_b$$

where the tetrads of the *moving frames* are given by $e^\alpha_a = \partial_a \bar{x}^\alpha$ and $e^a_\alpha = \partial_\alpha x^a$. They satisfy $e^a_\alpha\ e^\alpha_b = \delta^a_b$ and $e^\alpha_a\ e^a_\beta = \delta^\alpha_\beta$. In addition, it can also be verified that

$$\partial_\beta e^a_\alpha = \overline{\Gamma}^\lambda_{\alpha\beta}\, e^a_\lambda - \Gamma^a_{bc}\, e^b_\alpha\, e^c_\beta$$
$$\partial_b e^a_\alpha = e^a_\lambda\, \overline{\Gamma}^\lambda_{\alpha\beta}\, e^\beta_b - \Gamma^a_{cb}\, e^c_\alpha$$

We know that Γ is a non-tensorial object, since its components transform as

$$\Gamma^c_{ab} = e^c_\alpha\, \partial_b e^\alpha_a + e^c_\alpha\, \overline{\Gamma}^\alpha_{\beta\lambda}\, e^\beta_a\, e^\lambda_b$$

However, it can be described as a kind of displacement field since it is what makes possible a comparison of vectors from point to point in C^∞. In fact the relation $\partial_b \omega_a = \Gamma^c_{ab}\, \omega_c$ defines the so-called metricity condition, i.e., the change (during a displacement) in the basis can be measured by the basis itself. This immediately translates into

$$\nabla_c g_{ab} = 0$$

where we have just applied the notion of a covariant derivative to an arbitrary tensor field T:

$$\nabla_m T^{ab\ldots g}_{cd\ldots h} = \partial_m T^{ab\ldots g}_{cd\ldots h} + \Gamma^a_{pm}\, T^{pb\ldots g}_{cd\ldots h} + \Gamma^b_{pm}\, T^{ap\ldots g}_{cd\ldots h} + \ldots + \Gamma^g_{pm}\, T^{ab\ldots p}_{cd\ldots h}$$
$$- \Gamma^p_{cm}\, T^{ab\ldots g}_{pd\ldots h} - \Gamma^p_{dm}\, T^{ab\ldots g}_{cp\ldots h} - \ldots - \Gamma^p_{hm}\, T^{ab\ldots g}_{cd\ldots p}$$

such that $\left(\partial_m T\right)^{ab\ldots g}_{cd\ldots h} = \nabla_m T^{ab\ldots g}_{cd\ldots h}$.

The condition $\nabla_c g_{ab} = 0$ can be solved to give

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\left(\partial_b g_{da} - \partial_d g_{ab} + \partial_a g_{bd}\right) + \Gamma^c_{[ab]} - g^{cd}\left(g_{ae}\, \Gamma^e_{[db]} + g_{be}\, \Gamma^e_{[da]}\right)$$

from which it is customary to define

$$\Delta^c_{ab} = \frac{1}{2} g^{cd}\left(\partial_b g_{da} - \partial_d g_{ab} + \partial_a g_{bd}\right)$$

as the Christoffel symbols (symmetric in their two lower indices) and

$$K^c_{ab} = \Gamma^c_{[ab]} - g^{cd}\left(g_{ae}\, \Gamma^e_{[db]} + g_{be}\, \Gamma^e_{[da]}\right)$$

as the components of the so-called contorsion tensor (anti-symmetric in the first two mixed indices).

Note that the components of the torsion tensor are given by

$$\Gamma^{a}_{[bc]} = \frac{1}{2}\, e^{a}_{\alpha}\left(\partial_c\, e^{\alpha}_{b} - \partial_b\, e^{\alpha}_{c} + e^{\beta}_{b}\, \overline{\Gamma}^{\alpha}_{\beta c} - e^{\beta}_{c}\, \overline{\Gamma}^{\alpha}_{\beta b}\right)$$

where we have set $\overline{\Gamma}^{\alpha}_{\beta c} = \overline{\Gamma}^{\alpha}_{\beta\lambda}\, e^{\lambda}_{c}$, such that for an arbitrary scalar field Φ we have

$$\left(\nabla_a \nabla_b - \nabla_b \nabla_a\right)\Phi = 2\, \Gamma^{c}_{[ab]}\, \nabla_c \Phi$$

The components of the curvature tensor R of C^{∞} are then given via the relation

$$\begin{aligned}\left(\nabla_q \nabla_p - \nabla_p \nabla_q\right) T^{ab\ldots s}_{cd\ldots r} &= T^{ab\ldots s}_{wd\ldots r}\, R^{w}{}_{cpq} + T^{ab\ldots s}_{cw\ldots r}\, R^{w}{}_{dpq} + \ldots + T^{ab\ldots s}_{cd\ldots w}\, R^{w}{}_{rpq} \\ &\quad - T^{wb\ldots s}_{cd\ldots r}\, R^{a}{}_{wpq} - T^{aw\ldots s}_{cd\ldots r}\, R^{b}{}_{wpq} - \ldots - T^{ab\ldots w}_{cd\ldots r}\, R^{s}{}_{wpq} \\ &\quad - 2\, \Gamma^{w}_{[pq]}\, \nabla_w T^{ab\ldots s}_{cd\ldots r}\end{aligned}$$

where

$$\begin{aligned}R^{d}{}_{abc} &= \partial_b \Gamma^{d}_{ac} - \partial_c \Gamma^{d}_{ab} + \Gamma^{e}_{ac}\, \Gamma^{d}_{eb} - \Gamma^{e}_{ab}\, \Gamma^{d}_{ec} \\ &= B^{d}{}_{abc}(\Delta) + \hat{\nabla}_b K^{d}_{ac} - \hat{\nabla}_c K^{d}_{ab} + K^{e}_{ac}\, K^{d}_{eb} - K^{e}_{ab}\, K^{d}_{ec}\end{aligned}$$

where $\hat{\nabla}$ denotes covariant differentiation with respect to the Christoffel symbols alone, and where

$$B^{d}{}_{abc}(\Delta) = \partial_b \Delta^{d}_{ac} - \partial_c \Delta^{d}_{ab} + \Delta^{e}_{ac}\, \Delta^{d}_{eb} - \Delta^{e}_{ab}\, \Delta^{d}_{ec}$$

are the components of the Riemann-Christoffel curvature tensor of C^{∞}.

From the components of the curvature tensor, namely, $R^{d}{}_{abc}$, we have (using the metric tensor to raise and lower indices)

$$\begin{aligned}R_{ab} &\equiv R^{c}{}_{acb} = B_{ab}(\Delta) + \hat{\nabla}_c K^{c}_{ab} - K^{c}_{ad}\, K^{d}_{cb} - 2\, \hat{\nabla}_b \Gamma^{c}_{[ac]} + 2\, K^{c}_{ab}\, \Gamma^{d}_{[cd]} \\ R &\equiv R^{a}_{a} = B(\Delta) - 4\, g^{ab}\, \hat{\nabla}_a \Gamma^{c}_{[bc]} - 2\, g^{ac}\, \Gamma^{b}_{[ab]}\, \Gamma^{d}_{[cd]} - K_{abc}\, K^{acb}\end{aligned}$$

where $B_{ab}(\Delta) \equiv B^{c}{}_{acb}(\Delta)$ are the components of the symmetric Ricci tensor and $B(\Delta) \equiv B^{a}{}_{a}(\Delta)$ is the Ricci scalar. Note that $K_{abc} \equiv g_{ad}\, K^{d}_{bc}$ and $K^{acb} \equiv g^{cd}\, g^{be}\, K^{a}_{de}$.

Now since

$$\Gamma^b_{ba} = \Delta^b_{ba} = \Delta^b_{ab} = \partial_a \left(\ln \sqrt{\det(g)} \right)$$
$$\Gamma^b_{ab} = \partial_a \left(\ln \sqrt{\det(g)} \right) + 2\, \Gamma^b_{[ab]}$$

we see that for a continuous metric determinant, the so-called homothetic curvature vanishes:

$$H_{ab} \equiv R^c{}_{cab} = \partial_a \Gamma^c_{cb} - \partial_b \Gamma^c_{ca} = 0$$

Introducing the traceless Weyl tensor W, we have the following decomposition theorem:

$$R^d{}_{abc} = W^d{}_{abc} + \frac{1}{n-2} \left(\delta^d_b\, R_{ac} + g_{ac}\, R^d_b - \delta^d_c\, R_{ab} - g_{ab}\, R^d_c \right)$$
$$+ \frac{1}{(n-1)(n-2)} \left(\delta^d_c\, g_{ab} - \delta^d_b\, g_{ac} \right) R$$

which is valid for $n > 2$. For $n = 2$, we have

$$R^d{}_{abc} = K_G \left(\delta^d_b\, g_{ac} - \delta^d_c\, g_{ab} \right)$$

where

$$K_G = \frac{1}{2} R$$

is the Gaussian curvature of the surface. Note that (in this case) the Weyl tensor vanishes.

Any $n-$dimensional manifold (for which $n > 1$) with constant sectional curvature R and vanishing torsion is called an Einstein space. It is described by the following simple relations:

$$R^d{}_{abc} = \frac{1}{n(n-1)} \left(\delta^d_b\, g_{ac} - \delta^d_c\, g_{ab} \right) R$$
$$R_{ab} = \frac{1}{n}\, g_{ab}\, R$$

In the above, we note especially that

$$R^d{}_{abc} = B^d{}_{abc}(\Delta)$$
$$R_{ab} = B_{ab}(\Delta)$$
$$R = B(\Delta)$$

Furthermore, after some lengthy algebra, we obtain, in general, the following *generalized* Bianchi identities:

$$
\begin{aligned}
& R^{a}_{\ bcd} + R^{a}_{\ cdb} + R^{a}_{\ dbc} = -2\left(\partial_d \Gamma^{a}_{[bc]} + \partial_b \Gamma^{a}_{[cd]} + \partial_c \Gamma^{a}_{[db]} + \Gamma^{a}_{eb}\,\Gamma^{e}_{[cd]} + \Gamma^{a}_{ec}\,\Gamma^{e}_{[db]} + \Gamma^{a}_{ed}\,\Gamma^{e}_{[bc]}\right) \\
& \nabla_e R^{a}_{\ bcd} + \nabla_c R^{a}_{\ bde} + \nabla_d R^{a}_{\ bec} = 2\left(\Gamma^{f}_{[cd]}\, R^{a}_{\ bfe} + \Gamma^{f}_{[de]}\, R^{a}_{\ bfc} + \Gamma^{f}_{[ec]}\, R^{a}_{\ bfd}\right) \\
& \nabla_a\left(R^{ab} - \frac{1}{2} g^{ab} R\right) = 2\, g^{ab}\, \Gamma^{c}_{[da]}\, R^{d}_{\ c} + \Gamma^{a}_{[cd]}\, R^{cdb}{}_{a}
\end{aligned}
$$

for any metric-compatible manifold endowed with both curvature and torsion.

In the last of the above set of equations, we have introduced the generalized Einstein tensor, i.e.,

$$
G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab}\, R
$$

In particular, we also have the following specialized identities, i.e., the *regular* Bianchi identities:

$$
\begin{aligned}
& B^{a}_{\ bcd} + B^{a}_{\ cdb} + B^{a}_{\ dbc} = 0 \\
& \hat{\nabla}_e B^{a}_{\ bcd} + \hat{\nabla}_c B^{a}_{\ bde} + \hat{\nabla}_d B^{a}_{\ bec} = 0 \\
& \hat{\nabla}_a\left(B^{ab} - \frac{1}{2} g^{ab} B\right) = 0
\end{aligned}
$$

In general, these hold in the case of a symmetric, metric-compatible connection. Non-metric differential geometry is beyond the scope of our present consideration.

We now define the so-called Lie derivative which can be used to define a diffeomorphism invariant in C^{∞}. for a vector field U and a tensor field T, both arbitrary, the invariant derivative represented (in component notation) by

$$
\begin{aligned}
L_U\, T^{ab\ldots g}_{cd\ldots h} = {} & \partial_m T^{ab\ldots g}_{cd\ldots h}\, U^m + T^{ab\ldots g}_{md\ldots h}\, \partial_c U^m + T^{ab\ldots g}_{cm\ldots h}\, \partial_d U^m + \ldots + T^{ab\ldots g}_{cd\ldots m}\, \partial_h U^m \\
& - T^{mb\ldots g}_{cd\ldots h}\, \partial_m U^a - T^{am\ldots g}_{cd\ldots h}\, \partial_m U^b - \ldots - T^{ab\ldots m}_{cd\ldots h}\, \partial_m U^g
\end{aligned}
$$

defines the Lie derivative of T with respect to U. With the help of the torsion tensor and the relation

$$
\partial_b U^a = \nabla_b U^a - \Gamma^{a}_{cb}\, U^c = \nabla_b U^a - \left(\Gamma^{a}_{bc} - 2\,\Gamma^{a}_{[bc]}\right) U^c
$$

we can write

$$
\begin{aligned}
L_U\, T^{ab\ldots g}_{cd\ldots h} &= \nabla_m\, T^{ab\ldots g}_{cd\ldots h}\ U^m + T^{ab\ldots g}_{md\ldots h}\ \nabla_c\, U^m + T^{ab\ldots g}_{cm\ldots h}\ \nabla_d\, U^m + \ldots + T^{ab\ldots g}_{cd\ldots m}\ \nabla_h\, U^m \\
&\quad - T^{mb\ldots g}_{cd\ldots h}\ \nabla_m U^a - T^{am\ldots g}_{cd\ldots.h}\ \nabla_m U^b - \ldots - T^{ab\ldots m}_{cd\ldots h}\ \nabla_m U^g \\
&\quad + 2\,\Gamma^a_{[mp]}\, T^{mb\ldots g}_{cd\ldots h}\ U^p + 2\,\Gamma^b_{[mp]}\, T^{am\ldots g}_{cd\ldots h}\ U^p + \ldots + 2\,\Gamma^g_{[mp]}\, T^{ab\ldots m}_{cd\ldots h}\ U^p \\
&\quad - 2\,\Gamma^m_{[cp]}\, T^{ab\ldots g}_{md\ldots h}\ U^p + 2\,\Gamma^m_{[dp]}\, T^{ab\ldots g}_{cm\ldots h}\ U^p - \ldots - 2\,\Gamma^m_{[hp]}\, T^{ab\ldots g}_{cd\ldots m}\ U^p
\end{aligned}
$$

Hence, noting that the components of the torsion tensor, namely, $\Gamma^i_{[kl]}$, indeed transform as components of a tensor field, it is seen that the $L_U\, T^{ij\ldots s}_{kl\ldots r}$ do transform as components of a tensor field. Apparently, the beautiful property of the Lie derivative (applied to an arbitrary tensor field) is that it is connection-independent even in a curved manifold.

We will need a few identities derived in this section later on.

3. The Generalized Four-Dimensional Linear Constitutive Field Equations

We shall now present a four-dimensional linear continuum theory of the classical physical fields capable of describing microspin phenomena in addition to the gravitational and electromagnetic fields. By microspin phenomena, we mean those phenomena generated by rotation of points in the four-dimensional space-time manifold (continuum) S^4 with local coordinates x^μ in the manner described by the so-called Cosserat continuum theory.

We start with the following constitutive equation in four dimensions:

$$T^{\mu\nu} = C^{\mu\nu}{}_{\rho\sigma}\ D^{\rho\sigma} = \frac{1}{\kappa}\left(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu}\ R\right)$$

where now the Greek indices run from 0 to 3. In the above equation, $T^{\mu\nu}$ are the contravariant components of the generally asymmetric energy-momentum tensor, $C^{\mu\nu}{}_{\rho\sigma}$ are the mixed components of the generalized four-dimensional elasticity tensor, $D^{\rho\sigma}$ are the contravariant components of the four-dimensional displacement gradient tensor, $R^{\mu\nu}$ are the contravariant components of the generalized (asymmetric) four-dimensional Ricci curvature tensor, $\kappa = -8\pi$ is the Einstein coupling constant (in geometrized units), and $R = R^\mu_\mu$ is the generalized Ricci four-dimensional curvature scalar.

Furthermore, we can decompose our four-dimensional elasticity tensor into its holonomic and anholonomic parts as follows:

$$C^{\mu\nu}{}_{\rho\sigma} = A^{\mu\nu}{}_{\rho\sigma} + B^{\mu\nu}{}_{\rho\sigma}$$

where

$$A^{\mu\nu}{}_{\rho\sigma} = A^{(\mu\nu)}{}_{(\rho\sigma)} = A_{\rho\sigma}{}^{\mu\nu}$$
$$B^{\mu\nu}{}_{\rho\sigma} = B^{[\mu\nu]}{}_{[\rho\sigma]} = B_{\rho\sigma}{}^{\mu\nu}$$

such that

$$C^{\mu\nu}{}_{\rho\sigma} = C_{\rho\sigma}{}^{\mu\nu}$$

Therefore, we can express the fully covariant components of the generalized four-dimensional elasticity tensor in terms of the covariant components of the symmetric metric tensor $g_{\mu\nu}$ (satisfying, as before, $g_{\nu\sigma}\, g^{\mu\sigma} = \delta^{\mu}_{\nu}$) as

$$\begin{aligned} C_{\mu\nu\rho\sigma} &= \alpha\, g_{\mu\nu}\, g_{\rho\sigma} + \beta\, g_{\mu\rho}\, g_{\nu\sigma} + \gamma\, g_{\mu\sigma}\, g_{\nu\rho} \\ &= \alpha\, g_{\mu\nu}\, g_{\rho\sigma} + \lambda \left(g_{\mu\rho}\, g_{\nu\sigma} + g_{\mu\sigma}\, g_{\nu\rho}\right) + \omega \left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right) \end{aligned}$$

where $\alpha, \beta, \gamma, \lambda,$ and ω are constitutive invariants that are not necessarily constant. It is therefore seen that

$$A_{\mu\nu\rho\sigma} = \alpha\, g_{\mu\nu}\, g_{\rho\sigma} + \lambda \left(g_{\mu\rho}\, g_{\nu\sigma} + g_{\mu\sigma}\, g_{\nu\rho}\right)$$
$$B_{\mu\nu\rho\sigma} = \omega \left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right)$$

An infinitesimal displacement (diffeomorphism) in the space-time manifold S^4 from an initial point P to a neighboring point Q is given as usual by

$$x^{\mu}(Q) = x^{\mu}(P) + \xi^{\mu}$$

where ξ^{μ} are the components of the four-dimensional infinitesimal displacement field vector. The generally asymmetric four-dimensional displacement gradient tensor is then given by

$$D_{\mu\nu} = \nabla_{\nu}\, \xi_{\mu}$$

The decomposition $D_{\mu\nu} = D_{(\mu\nu)} + D_{[\mu\nu]}$ and the supplementary infinitesimal point-rotation condition $\Gamma^{\alpha}_{[\mu\nu]}\, \xi^{\mu} = 0$ allow us to define the symmetric four-dimensional displacement ("dilation") tensor by

$$\Phi_{\mu\nu} = D_{(\mu\nu)} = \frac{1}{2}\left(\nabla_{\mu}\, \xi_{\nu} + \nabla_{\nu}\, \xi_{\mu}\right) = \frac{1}{2}\, L_{\xi}\, g_{\mu\nu}$$

from which the "dilation" scalar is given by

$$\Phi = \Phi^{\mu}_{\mu} = D^{\mu}_{\mu} = \frac{1}{2} g^{\mu\nu} L_{\xi} g_{\mu\nu} = \nabla_{\mu} \xi^{\mu}$$

as well as the anti-symmetric four-dimensional intrinsic spin (vorticity) tensor by

$$\omega_{\mu\nu} = D_{[\mu\nu]} = \frac{1}{2} \left(\nabla_{\nu} \xi_{\mu} - \nabla_{\mu} \xi_{\nu} \right)$$

Let us now decompose the four-dimensional infinitesimal displacement field vector as follows:

$$\xi^{\mu} = \partial^{\mu} F + \psi^{\mu}$$

Here the continuous scalar function F represents the integrable part of the four-dimensional macroscopic displacement field vector while the remaining parts are given by ψ^{μ} via

$$\psi^{\mu} = \sigma^{\mu} + \phi^{\mu} + 2\bar{e}\,\varphi^{\mu}$$

where σ^{μ} are the components of the non-integrable four-dimensional macroscopic displacement field vector, ϕ^{μ} are the components of the four-dimensional microscopic (micropolar) intrinsic spin vector, $\bar{e}$ is a constant proportional to the electric charge, and φ^{μ} are the components of the electromagnetic four-potential vector. We assume that in general σ^{μ}, ϕ^{μ}, and φ^{μ} are linearly independent of each other.

The intrinsic four-dimensional macroscopic spin ("angular momentum") tensor is then given by

$$\Omega_{\mu\nu} = \frac{1}{2} \left(\nabla_{\nu} \sigma_{\mu} - \nabla_{\mu} \sigma_{\nu} \right)$$

Likewise, the intrinsic four-dimensional microscopic (micropolar) spin tensor is given by

$$S_{\mu\nu} = \frac{1}{2} \left(\nabla_{\nu} \phi_{\mu} - \nabla_{\mu} \phi_{\nu} \right)$$

Note that this tensor vanishes when the points are not allowed to rotate such as in conventional (standard) cases.

Meanwhile, the electromagnetic field tensor is given by

$$F_{\mu\nu} = \nabla_{\nu} \varphi_{\mu} - \nabla_{\mu} \varphi_{\nu}$$

In this case, we especially note that, by means of the condition $\Gamma^{\alpha}_{[\mu\nu]}\,\xi^{\mu} = 0$, the above expression reduces to the usual Maxwellian relation

$$F_{\mu\nu} = \partial_{\nu}\,\varphi_{\mu} - \partial_{\mu}\,\varphi_{\nu}$$

We can now write the intrinsic spin tensor as

$$\omega_{\mu\nu} = \Omega_{\mu\nu} + S_{\mu\nu} + \bar{e}\,F_{\mu\nu}$$

Hence the full electromagnetic content of the theory becomes visible. We also see that our space-time continuum can be considered as a dynamically polarizable medium possessing chirality. As such, the gravitational and electromagnetic fields, i.e., the familiar classical fields, are intrinsic geometric objects in the theory.

Furthermore, from the contorsion tensor, let us define a geometric spin vector via

$$A_{\mu} \equiv K^{\sigma}_{\mu\sigma} = 2\,\Gamma^{\sigma}_{[\mu\sigma]}$$

Now, in a somewhat restrictive case, in connection with the spin fields represented by σ^{μ}, ϕ^{μ}, and φ^{μ}, the selection

$$A_{\mu} = c_1\,\sigma_{\mu} + c_2\,\phi_{\mu} + 2\bar{e}\,c_3\,\varphi_{\mu} = \in\psi_{\mu}$$

i.e.,

$$\in = \frac{c_1\,\sigma_{\mu} + c_2\,\phi_{\mu} + 2\bar{e}\,c_3\,\varphi_{\mu}}{\sigma_{\mu} + \phi_{\mu} + 2\bar{e}\,\varphi_{\mu}}$$

will directly attribute the contorsion tensor to the intrinsic spin fields of the theory. However, we would in general expect the intrinsic spin fields to remain in the case of a semi-symmetric connection, for which $A_{\mu} = 0$, and so we cannot carry this proposition any further.

At this point, we see that the holonomic part of the generalized four-dimensional elasticity tensor given by $A_{\mu\nu\rho\sigma}$ is responsible for (centrally symmetric) gravitational phenomena while the anholonomic part given by $B_{\mu\nu\rho\sigma}$ owes its existence to the (con)torsion tensor which is responsible for the existence of the intrinsic spin fields in our consideration.

Furthermore, we see that the components of the energy-momentum tensor can now be expressed as

$$T_{\mu\nu} = \alpha \, g_{\mu\nu} \, \Phi + \beta \, D_{\mu\nu} + \gamma \, D_{\nu\mu}$$

In other words,

$$T_{(\mu\nu)} = \alpha \, g_{\mu\nu} \, \Phi + (\beta + \gamma) \, \Phi_{\mu\nu}$$
$$T_{[\mu\nu]} = (\beta - \gamma) \, \omega_{\mu\nu}$$

Alternatively,

$$T_{(\mu\nu)} = \frac{1}{2} \alpha \, g_{\mu\nu} \, g^{\alpha\beta} \, L_\xi \, g_{\alpha\beta} + \frac{1}{2} (\beta + \gamma) \, L_\xi \, g_{\mu\nu}$$
$$T_{[\mu\nu]} = (\beta - \gamma) \left(\Omega_{\mu\nu} + S_{\mu\nu} + \bar{e} \, F_{\mu\nu} \right)$$

We may note that, in a sense analogous to that of the ordinary mechanics of continuous media, the generally asymmetric character of the energy-momentum tensor means that a material object in motion is generally subject to distributed body couples.

We also have

$$T = T^{\mu}_{\mu} = (4\alpha + \beta + \gamma) \, \Phi = - \kappa^{-1} \, R$$

Let us briefly relate our description to the standard material description given by general relativity. For this purpose, let us assume that the intrinsic spin fields other than the electromagnetic field are negligible. If we denote the material density and the pressure by ρ and p, respectively, then it can be directly verified that

$$\Phi = \frac{\rho - 4p}{4\alpha + \beta + \gamma}$$

is a solution to the ordinary expression

$$T_{(\mu\nu)} = \rho \, u_\mu \, u_\nu - p \, g_{\mu\nu} - \frac{1}{4\pi} \left(F_{\mu\sigma} \, F_\nu^{\ \sigma} - \frac{1}{4} \, g_{\mu\nu} \, F_{\alpha\beta} \, F^{\alpha\beta} \right)$$

where u_μ are the covariant components of the unit velocity vector. This is true whether the electromagnetic field is present or not since the (symmetric) energy-momentum tensor of the electromagnetic field given by

$$J_{\mu\nu} = - \frac{1}{4\pi} \left(F_{\mu\sigma} \, F_\nu^{\ \sigma} - \frac{1}{4} \, g_{\mu\nu} \, F_{\alpha\beta} \, F^{\alpha\beta} \right)$$

is traceless.

At this point, however, we may note that the covariant divergence

$$\nabla_\mu T^{\mu\nu} = g^{\mu\nu}\, \nabla_\mu \left(\alpha\,\Phi\right) + \beta\, \nabla_\mu D^{\mu\nu} + \gamma\, \nabla_\mu D^{\nu\mu} + D^{\mu\nu}\, \nabla_\mu \beta + D^{\nu\mu}\, \nabla_\mu \gamma$$

need not vanish in general since

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\kappa}\, \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu}\, R \right) = \frac{1}{\kappa} \left(2\, g^{\mu\nu}\, \Gamma^{\rho}_{[\sigma\mu]}\, R^{\sigma}_{\rho} + \Gamma^{\lambda}_{[\rho\sigma]}\, R^{\rho\sigma\nu}{}_{\lambda}\right)$$

In an isotropic, homogeneous Universe, for which the constitutive invariants $\alpha, \beta, \gamma, \lambda,$ and ω are constant, the above expression reduces to

$$\nabla_\mu T^{\mu\nu} = \alpha\, g^{\mu\nu}\, \nabla_\mu \Phi + \beta\, \nabla_\mu D^{\mu\nu} + \gamma\, \nabla_\mu D^{\nu\mu}$$

If we require the above divergence to vanish, however, we see that the motion described by this condition is still more general than the pure geodesic motion for point-particles.

Still in the case of an isotropic, homogeneous Universe, possibly on large cosmological scales, then our expression for the energy-momentum tensor relates the generalized Ricci curvature scalar directly to the "dilation" scalar. In general, we have

$$R = -\,\kappa \left(4\alpha + \beta + \gamma\right) \Phi = -\,\kappa\, \Lambda\, \Phi = -\,\frac{1}{2}\, \kappa\, \Lambda\, g^{\mu\nu}\, L_\xi\, g_{\mu\nu}$$

Now, for the generalized Ricci curvature tensor, we obtain the following asymmetric constitutive field equation:

$$\begin{aligned} R_{\mu\nu} &= \kappa \left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T \right) \\ &= \kappa \left(\theta\, g_{\mu\nu} + \beta\, D_{\mu\nu} + \gamma\, D_{\nu\mu}\right) \end{aligned}$$

where

$$\theta = -\,\frac{1}{2} \left(2\alpha + \beta + \gamma\right) \Phi$$

In other words,

$$\begin{aligned} R_{(\mu\nu)} &= \kappa \left(\theta\, g_{\mu\nu} + \left(\beta + \gamma\right) \Phi_{\mu\nu}\right) \\ R_{[\mu\nu]} &= \kappa \left(\beta - \gamma\right) \omega_{\mu\nu} \end{aligned}$$

Inserting the value of κ, we can alternatively write

$$R_{(\mu\nu)} = -8\pi \left(\theta\, g_{\mu\nu} + \frac{1}{2}(\beta + \gamma)\, L_{\xi}\, g_{\mu\nu} \right)$$
$$R_{[\mu\nu]} = -8\pi\,(\beta - \gamma)\left(\Omega_{\mu\nu} + S_{\mu\nu} + \bar{e}\, F_{\mu\nu}\right)$$

Hence, the correspondence between the generalized Ricci curvature tensor and the physical fields in our theory becomes complete. The present theory shows that in a curved space-time with a particular spherical symmetry and in a flat Minkowski space-time (both space-times are solutions to the equation $\Phi_{\mu\nu} = 0$, i.e., $L_{\xi}\, g_{\mu\nu} = 0$) it is in general still possible for the spin fields to exist. One possible geometry that complies with such a space-time symmetry is the geometry of distant parallelism with vanishing space-time curvature (but non-vanishing Riemann-Christoffel curvature) and non-vanishing torsion.

Now let us recall that in four dimensions, with the help of the Weyl tensor W, we have the decomposition

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + \frac{1}{2}\left(g_{\mu\rho}\, R_{\nu\sigma} + g_{\nu\sigma}\, R_{\mu\rho} - g_{\mu\sigma}\, R_{\nu\rho} - g_{\nu\rho}\, R_{\mu\sigma}\right) + \frac{1}{6}\left(g_{\mu\sigma}\, g_{\nu\rho} - g_{\mu\rho}\, g_{\nu\sigma}\right) R$$

We obtain, upon setting $\bar{\alpha} = \frac{1}{2}\kappa\,\theta,\ \bar{\beta} = \frac{1}{2}\kappa\,\beta,\ \bar{\gamma} = \frac{1}{2}\kappa\,\gamma,$ and $\bar{\lambda} = \frac{1}{6}\kappa\,\Lambda$

$$\begin{aligned} R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} &+ 2\bar{\alpha}\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right) + \bar{\beta}\left(g_{\mu\rho}\, D_{\nu\sigma} + g_{\nu\sigma}\, D_{\mu\rho} - g_{\mu\sigma}\, D_{\nu\rho} - g_{\nu\rho}\, D_{\mu\sigma}\right) \\ &+ \bar{\gamma}\left(g_{\mu\rho}\, D_{\sigma\nu} + g_{\nu\sigma}\, D_{\rho\mu} - g_{\mu\sigma}\, D_{\rho\nu} - g_{\nu\rho}\, D_{\sigma\mu}\right) + \bar{\lambda}\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right)\Phi \end{aligned}$$

Therefore, in terms of the anholonomic part of the generalized elasticity tensor, we have

$$\begin{aligned} R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} &+ 2\frac{\bar{\alpha}}{\omega}\, B_{\mu\nu\rho\sigma} + \bar{\beta}\left(g_{\mu\rho}\, D_{\nu\sigma} + g_{\nu\sigma}\, D_{\mu\rho} - g_{\mu\sigma}\, D_{\nu\rho} - g_{\nu\rho}\, D_{\mu\sigma}\right) \\ &+ \bar{\gamma}\left(g_{\mu\rho}\, D_{\sigma\nu} + g_{\nu\sigma}\, D_{\rho\mu} - g_{\mu\sigma}\, D_{\rho\nu} - g_{\nu\rho}\, D_{\sigma\mu}\right) + \bar{\lambda}\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right)\Phi \end{aligned}$$

In the special case of a pure gravitational field, the torsion of the space-time continuum vanishes. In this situation our intrinsic spin fields vanish and consequently, we are left simply with

$$\begin{aligned} R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} &+ \frac{1}{2}\left(\bar{\beta} + \bar{\gamma}\right)\left(g_{\mu\rho}\, D_{\nu\sigma} + g_{\nu\sigma}\, D_{\mu\rho} - g_{\mu\sigma}\, D_{\nu\rho} - g_{\nu\rho}\, D_{\mu\sigma}\right) \\ &+ \bar{\lambda}\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right)\Phi \end{aligned}$$

In standard general relativity, this gives the explicit form of the Riemann-Christoffel curvature tensor in terms of the Lie derivative $L_\xi\, g_{\mu\nu} = 2\, \Phi_{\mu\nu}$. For a space-time satisfying the symmetry $L_\xi\, g_{\mu\nu} = 0$, we simply have $R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma}$, i.e., the space-time is devoid of material sources or "empty". This condition is relatively weaker than the case of a space-time with constant sectional curvature, $R = const.$ for which the Weyl tensor vanishes.

4. The Generalized Four-Dimensional Non-Linear Constitutive Field Equations

In reference to the preceding section, let us now present, in a somewhat concise manner, a non-linear extension of the formulation presented in the preceding section. The resulting non-linear constitutive field equations will therefore not be limited to weak fields only. In general, it can be shown that the full curvature tensor contains terms quadratic in the displacement gradient tensor and this gives us the reason to express the energy-momentum tensor which is quadratic in the displacement gradient tensor.

We start with the non-linear constitutive field equation

$$T^{\mu\nu} = C^{\mu\nu}{}_{\rho\sigma}\, D^{\rho\sigma} + K^{\mu\nu}{}_{\rho\sigma\lambda\eta}\, D^{\rho\sigma}\, D^{\lambda\eta} = \frac{1}{\kappa}\left(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu}\, R \right)$$

where

$$\begin{aligned} K_{\mu\nu\rho\sigma\lambda\eta} = {} & a_1\, g_{\mu\nu}\, g_{\rho\sigma}\, g_{\lambda\eta} + a_2\, g_{\mu\nu}\, g_{\rho\lambda}\, g_{\sigma\eta} + a_3\, g_{\mu\nu}\, g_{\rho\eta}\, g_{\sigma\lambda} + a_4\, g_{\rho\sigma}\, g_{\mu\lambda}\, g_{\nu\eta} \\ & + a_5\, g_{\rho\sigma}\, g_{\mu\eta}\, g_{\nu\lambda} + a_6\, g_{\lambda\eta}\, g_{\mu\rho}\, g_{\nu\sigma} + a_7\, g_{\lambda\eta}\, g_{\mu\sigma}\, g_{\nu\rho} + a_8\, g_{\mu\lambda}\, g_{\nu\rho}\, g_{\sigma\eta} \\ & + a_9\, g_{\mu\lambda}\, g_{\nu\sigma}\, g_{\rho\eta} + a_{10}\, g_{\mu\eta}\, g_{\nu\rho}\, g_{\sigma\lambda} + a_{11}\, g_{\mu\eta}\, g_{\nu\sigma}\, g_{\rho\lambda} + a_{12}\, g_{\nu\lambda}\, g_{\mu\rho}\, g_{\sigma\eta} \\ & + a_{13}\, g_{\nu\lambda}\, g_{\mu\sigma}\, g_{\rho\eta} + a_{14}\, g_{\nu\eta}\, g_{\mu\rho}\, g_{\sigma\lambda} + a_{15}\, g_{\nu\eta}\, g_{\mu\sigma}\, g_{\rho\lambda} \end{aligned}$$

where the fifteen constitutive invariants $a_1, a_2, ..., a_{15}$ are not necessarily constant.

We shall set

$$K_{\mu\nu\rho\sigma\lambda\eta} = K_{\rho\sigma\mu\nu\lambda\eta} = K_{\lambda\eta\mu\nu\rho\sigma} = K_{\mu\nu\lambda\eta\rho\sigma}$$

Letting

$$\begin{aligned} K_{\mu\nu\rho\sigma\lambda\eta} &= P_{\mu\nu\rho\sigma\lambda\eta} + Q_{\mu\nu\rho\sigma\lambda\eta} \\ P_{\mu\nu\rho\sigma\lambda\eta} &= P_{(\mu\nu)(\rho\sigma)(\lambda\eta)} \\ Q_{\mu\nu\rho\sigma\lambda\eta} &= Q_{[\mu\nu][\rho\sigma][\lambda\eta]} \end{aligned}$$

we have

$$P_{\mu\nu\rho\sigma\lambda\eta} = P_{\rho\sigma\mu\nu\lambda\eta} = P_{\lambda\eta\mu\nu\rho\sigma} = P_{\mu\nu\lambda\eta\rho\sigma}$$
$$Q_{\mu\nu\rho\sigma\lambda\eta} = Q_{\rho\sigma\mu\nu\lambda\eta} = Q_{\lambda\eta\mu\nu\rho\sigma} = Q_{\mu\nu\lambda\eta\rho\sigma}$$

Introducing the eleven constitutive invariants $b_1, b_2, ..., b_{11}$, we can write

$$\begin{aligned} K_{\mu\nu\rho\sigma\lambda\eta} = {} & b_1\, g_{\mu\nu}\, g_{\rho\sigma}\, g_{\lambda\eta} + b_2\, g_{\mu\nu}\left(g_{\rho\lambda}\, g_{\sigma\eta} + g_{\rho\eta}\, g_{\sigma\lambda}\right) + b_3\, g_{\mu\nu}\left(g_{\rho\lambda}\, g_{\sigma\eta} - g_{\rho\eta}\, g_{\sigma\lambda}\right) \\ & + b_4\, g_{\rho\sigma}\left(g_{\mu\lambda}\, g_{\nu\eta} + g_{\mu\eta}\, g_{\nu\lambda}\right) + b_5\, g_{\rho\sigma}\left(g_{\mu\lambda}\, g_{\nu\eta} - g_{\mu\eta}\, g_{\nu\lambda}\right) \\ & + b_6\, g_{\lambda\eta}\left(g_{\mu\rho}\, g_{\nu\sigma} + g_{\mu\sigma}\, g_{\nu\rho}\right) + b_7\, g_{\lambda\eta}\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right) \\ & + b_8\, g_{\mu\lambda}\left(g_{\upsilon\rho}\, g_{\sigma\eta} + g_{\nu\sigma}\, g_{\rho\eta}\right) + b_9\, g_{\mu\lambda}\left(g_{\nu\rho}\, g_{\sigma\eta} - g_{\nu\sigma}\, g_{\rho\eta}\right) \\ & + b_{10}\, g_{\nu\lambda}\left(g_{\mu\rho}\, g_{\sigma\eta} + g_{\mu\sigma}\, g_{\rho\eta}\right) + b_{11}\, g_{\nu\lambda}\left(g_{\mu\rho}\, g_{\sigma\eta} - g_{\mu\sigma}\, g_{\rho\eta}\right) \end{aligned}$$

The energy-momentum tensor is therefore given by

$$\begin{aligned} T_{\mu\nu} = {} & \left(\alpha\, \Phi + b_1\, \Phi^2 + 2b_2\, \Phi_{\rho\sigma}\, \Phi^{\rho\sigma} + 2b_3\, \omega_{\rho\sigma}\, \omega^{\rho\sigma}\right) g_{\mu\nu} + \beta\, D_{\mu\nu} + \gamma\, D_{\nu\mu} \\ & + 2\left(b_4 + b_6\right) \Phi\, \Phi_{\mu\nu} + 2\left(b_5 + b_7\right) \Phi\, \omega_{\mu\nu} + 2b_8\, D_\mu^{\ \rho}\, \Phi_{\nu\rho} + 2b_9\, D_\mu^{\ \rho}\, \omega_{\nu\rho} \\ & + 2b_{10}\, D_\nu^{\ \rho}\, \Phi_{\mu\rho} + 2b_{11}\, D_\nu^{\ \rho}\, \omega_{\mu\rho} \end{aligned}$$

In other words,

$$\begin{aligned} T_{(\mu\nu)} = {} & \left(\alpha\, \Phi + b_1\, \Phi^2 + 2b_2\, \Phi_{\rho\sigma}\, \Phi^{\rho\sigma} + 2b_3\, \omega_{\rho\sigma}\, \omega^{\rho\sigma}\right) g_{\mu\nu} + \left(\beta + \gamma\right) \Phi_{\mu\nu} \\ & 2\left(b_4 + b_6\right) \Phi\, \Phi_{\mu\nu} + \left(b_8 + b_{10}\right)\left(D_\mu^{\ \rho}\, \Phi_{\nu\rho} + D_\nu^{\ \rho}\, \Phi_{\mu\rho}\right) + \left(b_9 + b_{11}\right)\left(D_\mu^{\ \rho}\, \omega_{\nu\rho} + D_\nu^{\ \rho}\, \omega_{\mu\rho}\right) \\ T_{[\mu\nu]} = {} & \left(\beta - \gamma\right) \omega_{\mu\nu} + 2\left(b_4 + b_6\right) \Phi\, \omega_{\mu\nu} + \left(b_8 + b_{10}\right)\left(D_\mu^{\ \rho}\, \Phi_{\nu\rho} - D_\nu^{\ \rho}\, \Phi_{\mu\rho}\right) \\ & + \left(b_9 + b_9\right)\left(D_\mu^{\ \rho}\, \omega_{\nu\rho} - D_\nu^{\ \rho}\, \omega_{\mu\rho}\right) \end{aligned}$$

We also have

$$T = \mu_1\, \Phi + \mu_2\, \Phi^2 + \mu_3\, \Phi_{\mu\nu}\, \Phi^{\mu\nu} + \mu_4\, \omega_{\mu\nu}\, \omega^{\mu\nu}$$

where we have set

$$\begin{aligned} \mu_1 &= 4\alpha + \beta + \gamma \\ \mu_2 &= 4b_1 + 2\left(b_4 + b_6\right) \\ \mu_3 &= 8b_2 + 2\left(b_8 + b_{10}\right) \\ \mu_4 &= 8b_3 + 2\left(b_9 - b_{11}\right) \end{aligned}$$

for the sake of simplicity.

For the generalized Ricci curvature tensor, we obtain

$$R_{\mu\nu} = \kappa \Big(\left(c_1\ \Phi + c_2\ \Phi^2 + c_3\ \Phi_{\rho\sigma}\ \Phi^{\rho\sigma} + c_4\ \omega_{\rho\sigma}\ \omega^{\rho\sigma} \right) g_{\mu\nu}$$
$$+ c_5\ D_{\mu\nu} + c_6\ D_{\nu\mu} + c_7\ \Phi\ \Phi_{\mu\nu} + c_8\ \Phi\ \omega_{\mu\nu}$$
$$+ c_9\ D_\mu^{\ \rho}\ \Phi_{\nu\rho} + c_{10}\ D_\mu^{\ \rho}\ \omega_{\nu\rho} + c_{11}\ D_\nu^{\ \rho}\ \Phi_{\mu\rho} + c_{12}\ D_\nu^{\ \rho}\ \omega_{\mu\rho} \Big)$$

where

$$c_1 = -\frac{1}{2}\left(2\alpha + \beta + \gamma\right)$$
$$c_2 = -\left(b_1 + b_4 + b_6\right)$$
$$c_3 = -\left(2b_2 + b_8 + b_{10}\right)$$
$$c_4 = -\left(2b_3 + b_9 - b_{11}\right)$$
$$c_5 = \beta$$
$$c_6 = \gamma$$
$$c_7 = 2\left(b_4 + b_6\right)$$
$$c_8 = 2\left(b_5 + b_7\right)$$
$$c_9 = 2b_8$$
$$c_{10} = 2b_9$$
$$c_{11} = 2b_{10}$$
$$c_{12} = 2b_{11}$$

i.e.,

$$R_{(\mu\nu)} = \kappa \Big(\left(c_1\ \Phi + c_2\ \Phi^2 + c_3\ \Phi_{\rho\sigma}\ \Phi^{\rho\sigma} + c_4\ \omega_{\rho\sigma}\ \omega^{\rho\sigma} \right) g_{\mu\nu}$$
$$+ \left(c_5 + c_6\right) \Phi_{\mu\nu} + c_7\ \Phi\ \Phi_{\mu\nu} + \frac{1}{2}\left(c_9 + c_{11}\right)\left(D_\mu^{\ \rho}\ \Phi_{\nu\rho} + D_\nu^{\ \rho}\ \Phi_{\mu\rho}\right)$$
$$+ \frac{1}{2}\left(c_{10} + c_{12}\right)\left(D_\mu^{\ \rho}\ \omega_{\nu\rho} + D_\nu^{\ \rho}\ \omega_{\mu\rho}\right)\Big)$$

$$R_{[\mu\nu]} = \kappa \Big(\left(c_5 - c_6\right) \omega_{\mu\nu} + c_8\ \Phi\ \omega_{\mu\nu} + \frac{1}{2}\left(c_9 + c_{11}\right)\left(D_\mu^{\ \rho}\ \Phi_{\nu\rho} - D_\nu^{\ \rho}\ \Phi_{\mu\rho}\right)$$
$$+ \frac{1}{2}\left(c_{10} + c_{11}\right)\left(D_\mu^{\ \rho}\ \omega_{\nu\rho} - D_\nu^{\ \rho}\ \omega_{\mu\rho}\right)\Big)$$

The generalized Ricci curvature scalar is then

$$R = \kappa\left(h_1\ \Phi + h_2\ \Phi^2 + h_3\ \Phi_{\mu\nu}\ \Phi^{\mu\nu} + h_4\ \omega_{\mu\nu}\ \omega^{\mu\nu}\right)$$

where

$$
\begin{aligned}
h_1 &= 4c_1 + c_5 + c_6 \\
h_2 &= 4c_2 + c_5 \\
h_3 &= 4c_3 + c_9 + c_{11} \\
h_4 &= 4c_4 + c_{10} + c_{12}
\end{aligned}
$$

Finally, we obtain, for the curvature tensor, the following expression:

$$
\begin{aligned}
R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} &+ \left(f_1\, \Phi + f_2\, \Phi^2 + f_3\, \Phi_{\lambda\eta}\, \Phi^{\lambda\eta} + f_4\, \omega_{\lambda\eta}\, \omega^{\lambda\eta}\right)\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right) \\
&+ \left(\overline{\beta} + f_5\, \Phi\right)\left(g_{\mu\rho}\, \Phi_{\nu\sigma} + g_{\nu\sigma}\, \Phi_{\mu\rho} - g_{\mu\sigma}\, \Phi_{\nu\rho} - g_{\nu\rho}\, \Phi_{\mu\sigma}\right) \\
&+ \left(\overline{\beta} + f_6\, \Phi\right)\left(g_{\mu\rho}\, \omega_{\nu\sigma} + g_{\nu\sigma}\, \omega_{\mu\rho} - g_{\mu\sigma}\, \omega_{\nu\rho} - g_{\nu\rho}\, \omega_{\mu\sigma}\right) \\
&+ \overline{\gamma}\left(g_{\mu\rho}\, D_{\sigma\nu} + g_{\nu\sigma}\, D_{\rho\mu} - g_{\mu\sigma}\, D_{\rho\nu} - g_{\nu\rho}\, D_{\sigma\mu}\right) \\
&+ f_7\left(D_\nu^{\ \lambda}\, \Phi_{\sigma\lambda}\, g_{\mu\rho} + D_\mu^{\ \lambda}\, \Phi_{\rho\lambda}\, g_{\nu\sigma} - D_\nu^{\ \lambda}\, \Phi_{\rho\lambda}\, g_{\mu\sigma} - D_\mu^{\ \lambda}\, \Phi_{\sigma\lambda}\, g_{\nu\rho}\right) \\
&+ f_8\left(D_\nu^{\ \lambda}\, \omega_{\sigma\lambda}\, g_{\mu\rho} + D_\mu^{\ \lambda}\, \omega_{\rho\lambda}\, g_{\nu\sigma} - D_\nu^{\ \lambda}\, \omega_{\rho\lambda}\, g_{\mu\sigma} - D_\mu^{\ \lambda}\, \omega_{\sigma\lambda}\, g_{\nu\rho}\right) \\
&+ f_9\left(D_\sigma^{\ \lambda}\, \Phi_{\nu\lambda}\, g_{\mu\rho} + D_\rho^{\ \lambda}\, \Phi_{\mu\lambda}\, g_{\nu\sigma} - D_\rho^{\ \lambda}\, \Phi_{\nu\lambda}\, g_{\mu\sigma} - D_\sigma^{\ \lambda}\, \Phi_{\mu\lambda}\, g_{\nu\rho}\right) \\
&+ f_{10}\left(D_\sigma^{\ \lambda}\, \omega_{\nu\lambda}\, g_{\mu\rho} + D_\rho^{\ \lambda}\, \omega_{\mu\lambda}\, g_{\nu\sigma} - D_\rho^{\ \lambda}\, \omega_{\nu\lambda}\, g_{\mu\sigma} - D_\sigma^{\ \lambda}\, \omega_{\mu\lambda}\, g_{\nu\rho}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
f_1 &= c_1 = 2\overline{\alpha} + \overline{\lambda} \\
f_2 &= \left(1 - \frac{2}{3}\kappa\right) c_2 + \frac{1}{6}\kappa\, c_7 \\
f_3 &= \left(1 - \frac{2}{3}\kappa\right) c_3 + \frac{1}{6}\kappa\left(c_9 + c_{11}\right) \\
f_4 &= \left(1 - \frac{2}{3}\kappa\right) c_4 + \frac{1}{6}\kappa\left(c_{10} - c_{12}\right) \\
f_5 &= c_7 \\
f_6 &= c_8 \\
f_7 &= c_9 \\
f_8 &= c_{10} \\
f_9 &= c_{11} \\
f_{10} &= c_{12}
\end{aligned}
$$

At this point, the apparent main difficulty lies in the fact that there are too many constitutive invariants that need to be exactly determined. As such, the linear theory is comparatively preferable since it only contains three constitutive invariants. However, by presenting the most general structure of the non-linear continuum theory in this section, we have acquired a quite general picture of the most general behavior of the space-time continuum in the presence of the classical fields.

5. The Equations of Motion

Let us now investigate the local translational-rotational motion of points in the space-time continuum S^4. Consider an infinitesimal displacement in the manner described in the preceding section. Keeping the initial position fixed, the unit velocity vector is given by

$$u^\mu = \frac{d\xi^\mu}{ds} = \frac{dx^\mu}{ds}$$
$$1 = g_{\mu\nu}\, u^\mu\, u^\nu$$

such that, at any proper time given by the world-line s, the parametric representation

$$d\xi^\mu = u^\mu\left(x^\alpha, s\right) ds$$

describes space-time curves whose tangents are everywhere directed along the direction of a particle's motion. As usual, the world-line can be parametrized by a scalar ς via $s = a\,\varsigma + b$, where a and b are constants of motion.

The local equations of motion along arbitrary curves in the space-time continuum S^4 can be described by the quadruplet of unit space-time vectors (u, v, w, z) orthogonal to each other where the first three unit vectors, or the triplet (u, v, w), may be defined as (a set of) local tangent vectors in the (three-dimensional) hypersurface $\Sigma(t)$ such that the unit vector z is normal to it. More explicitly, the hypersurface $\Sigma(t)$ is given as the time section $t = x^0 = const.$ of S^4. This way, the equations of motion will be derived by generalizing the ordinary Frenet equations of orientable points along an arbitrary curve in three-dimensional Euclidean space, i.e., by recasting them in a four-dimensional manner. Of course, we will also include effects of microspin generated by the torsion of space-time.

With respect to the anholonomic space-time basis $\omega_\mu = \omega_\mu\left(x^\alpha\left(X^k\right)\right) = e^i_\mu \dfrac{\partial}{\partial X^i}$, we can write

$$u = u^\mu\, \omega_\mu$$
$$v = v^\mu\, \omega_\mu$$
$$w = w^\mu\, \omega_\mu$$
$$z = z^\mu\, \omega_\mu$$

we obtain, in general, the following set of equations of motion of points, i.e., point-like particles, along an arbitrary curve ℓ in the space-time continuum S^4:

$$\frac{Du^\mu}{Ds} = \phi\, v^\mu$$
$$\frac{Dv^\mu}{Ds} = \tau\, w^\mu - \phi\, u^\mu$$
$$\frac{Dw^\mu}{Ds} = \tau\, v^\mu + \varphi\, z^\mu$$
$$\frac{Dz^\mu}{Ds} = \varphi\, w^\mu$$

where the operator $\frac{D}{Ds} = u^\mu\, \nabla_\mu$ represents the absolute covariant derivative. In the above equations we have introduced the following invariants:

$$\phi = \left(g_{\mu\nu} \frac{Du^\mu}{Ds} \frac{Du^\nu}{Ds} \right)^{1/2}$$
$$\tau = \in_{\mu\nu\rho\sigma} u^\mu\, v^\nu\, \frac{Dv^\rho}{Ds}\, z^\sigma$$
$$\varphi = \left(g_{\mu\nu} \frac{Dz^\mu}{Ds} \frac{Dz^\nu}{Ds} \right)^{1/2}$$

In particular, we note that, the torsion scalar τ measures the twist of the curve ℓ in S^4 due to microspin.

At this point, we see that our equations of motion describe a "minimal" geodesic motion (with intrinsic spin) when $\phi = 0$. In other words, if

$$\frac{Du^\mu}{Ds} = 0$$
$$\frac{Dv^\mu}{Ds} = \tau\, w^\mu$$
$$\frac{Dw^\mu}{Ds} = \tau\, v^\mu + \varphi\, z^\mu$$
$$\frac{Dz^\mu}{Ds} = \varphi\, w^\mu$$

However, in general, any material motion in S^4 will not follow the condition $\phi = 0$. This is true especially for the motion of a physical object with structure. In general, any physical object can be regarded as a collection of points (with different orientations) obeying our general equations of motion. It is therefore clear that $\phi \neq 0$ for a moving finite physical object (with structure) whose material points cannot be homogeneously oriented.

Furthermore, it can be shown that the gradient of the unit velocity vector can be decomposed according to

$$\nabla_\nu u_\mu = \alpha_{\mu\nu} + \beta_{\mu\nu} + \frac{1}{6} h_{\mu\nu} \bar{\theta} + u_\nu a_\mu$$

where

$$\begin{aligned}
h_{\mu\nu} &= g_{\mu\nu} - u_\mu u_\nu \\
\alpha_{\mu\nu} &= \frac{1}{4} h_\mu^\alpha h_\nu^\beta \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right) = \frac{1}{4} h_\mu^\alpha h_\nu^\beta \left(\hat{\nabla}_\alpha u_\beta + \hat{\nabla}_\beta u_\alpha\right) - \frac{1}{2} h_\mu^\alpha h_\nu^\beta K_{(\alpha\beta)}^\sigma u_\sigma \\
\beta_{\mu\nu} &= \frac{1}{4} h_\mu^\alpha h_\nu^\beta \left(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha\right) = \frac{1}{4} h_\mu^\alpha h_\nu^\beta \left(\hat{\nabla}_\alpha u_\beta - \hat{\nabla}_\beta u_\alpha\right) - \frac{1}{2} h_\mu^\alpha h_\nu^\beta K_{[\alpha\beta]}^\sigma u_\sigma \\
\bar{\theta} &= \nabla_\mu u^\mu \\
a_\mu &= \frac{D u_\mu}{Ds}
\end{aligned}$$

Note that

$$\begin{aligned}
& h_{\mu\nu} u^\nu = \alpha_{\mu\nu} u^\nu = \beta_{\mu\nu} u^\nu = 0 \\
& K_{(\alpha\beta)}^\sigma = - g^{\sigma\lambda} \left(g_{\alpha\eta} \Gamma_{[\lambda\beta]}^\eta + g_{\beta\eta} \Gamma_{[\lambda\alpha]}^\eta\right) \\
& K_{[\alpha\beta]}^\sigma = \Gamma_{[\alpha\beta]}^\sigma
\end{aligned}$$

Meanwhile, with the help of the identities

$$\begin{aligned}
& u^\lambda \nabla_\nu \nabla_\lambda u_\mu = \nabla_\nu \left(u^\lambda \nabla_\lambda u_\mu\right) - \left(\nabla_\nu u_\lambda\right)\left(\nabla^\lambda u_\mu\right) = \nabla_\nu a_\mu - \left(\nabla_\nu u_\lambda\right)\left(\nabla^\lambda u_\mu\right) \\
& u^\lambda \left(\nabla_\nu \nabla_\lambda - \nabla_\lambda \nabla_\nu\right) u_\mu = R^\sigma_{\mu\lambda\nu} u_\sigma u^\lambda - 2 \Gamma_{[\lambda\nu]}^\sigma u^\lambda \nabla_\sigma u_\mu
\end{aligned}$$

we obtain

$$\frac{D\bar{\theta}}{Ds} = \nabla_\mu a^\mu - \left(\nabla_\mu u^\nu\right)\left(\nabla_\nu u^\mu\right) - R_{\mu\nu} u^\mu u^\nu + 2 \Gamma_{[\mu\nu]}^\sigma u^\mu \nabla_\sigma u^\nu$$

for the "rate of shear" of a moving material object with respect to the world-line.

6. The Variational Principle for the Theory

Let us now derive the field equations of the present theory by means of the variational principle. Considering thermodynamic effects, in general, our theory can best be described by the following Lagrangian density:

$$\bar{L} = \bar{L}_1 + \bar{L}_2 + \bar{L}_3$$

where

$$\bar{L}_1 = \frac{1}{\kappa}\sqrt{\det(g)}\left(R^{\mu\nu}\left(\nabla_\nu \xi_\mu - D_{\mu\nu}\right) - \frac{1}{2}\left(\Phi - D^\mu_{\ \mu}\right)R\right)$$
$$\bar{L}_2 = \sqrt{\det(g)}\left(\frac{1}{2}C^{\mu\nu}{}_{\rho\sigma}\, D_{\mu\nu}\, D^{\rho\sigma} + \frac{1}{3}K^{\mu\nu}{}_{\rho\sigma\lambda\eta}\, D_{\mu\nu}\, D^{\rho\sigma}\, D^{\lambda\eta} - \Theta\, D^\mu_{\ \mu}\,\Delta T\right)$$
$$\bar{L}_3 = \sqrt{\det(g)}\, u^\mu \left(\nabla_\mu \xi_\nu\right)\left(f\, \xi^\nu - \rho\, u^\nu\right)$$

where Θ is a thermal coefficient, ΔT is (the change in) the temperature, and f is a generally varying scalar entity. Note that here we have only explicitly assumed that $\Phi = \nabla_\mu \xi^\mu$.

Alternatively, we can express $\bar{L}_1$ as follows:

$$\bar{L}_1 = \frac{1}{\kappa}\sqrt{\det(g)}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\left(\nabla_\nu \xi_\mu - D_{\mu\nu}\right)$$

Hence we have

$$\bar{L} = \sqrt{\det(g)}\Big(T^{\mu\nu}\left(\nabla_\nu \xi_\mu - D_{\mu\nu}\right) + \frac{1}{2}C^{\mu\nu}{}_{\rho\sigma}\, D_{\mu\nu}\, D^{\rho\sigma} + \frac{1}{3}K^{\mu\nu}{}_{\rho\sigma\lambda\eta}\, D_{\mu\nu}\, D^{\rho\sigma}\, D^{\lambda\eta}$$
$$- \Theta\, D^\mu_{\ \mu}\,\Delta T + u^\mu\left(\nabla_\mu \xi_\nu\right)\left(f\,\xi^\nu - \rho\, u^\nu\right)\Big)$$

We then arrive at the following invariant integral:

$$I = \int_{S^4}\Big(T^{\mu\nu}\left(\nabla_{(\nu}\xi_{\mu)} - \Phi_{\mu\nu}\right) + T^{\mu\nu}\left(\nabla_{[\nu}\xi_{\mu]} - \omega_{\mu\nu}\right) + \frac{1}{2}A^{\mu\nu}{}_{\rho\sigma}\,\Phi_{\mu\nu}\,\Phi^{\rho\sigma} + \frac{1}{2}B^{\mu\nu}{}_{\rho\sigma}\,\omega_{\mu\nu}\,\omega^{\rho\sigma}$$
$$+ \frac{1}{3}P^{\mu\nu}{}_{\rho\sigma\lambda\eta}\,\Phi_{\mu\nu}\,\Phi^{\rho\sigma}\,\Phi^{\lambda\eta} + \frac{1}{3}Q^{\mu\nu}{}_{\rho\sigma\lambda\eta}\,\omega_{\mu\nu}\,\omega^{\rho\sigma}\,\omega^{\lambda\eta} - \Theta\, D^\mu_{\ \mu}\,\Delta T$$
$$+ u^\mu\left(\nabla_\mu \xi_\nu\right)\left(f\,\xi^\nu - \rho\, u^\nu\right)\Big)\, d\Sigma$$

where $d\Sigma = \sqrt{\det(g)}\ dx^0\, dx^1\, dx^2\, dx^3$ is the proper four-dimensional differential volume.

Writing $\overline{L} = \sqrt{\det(g)}\ L$ and employing the variational principle, we then have

$$\delta I = \int_{S^4} \left(\frac{\partial L}{\partial T^{\mu\nu}}\ \delta T^{\mu\nu} + \frac{\partial L}{\partial \Phi^{\mu\nu}}\ \delta \Phi^{\mu\nu} + \frac{\partial L}{\partial \omega^{\mu\nu}}\ \delta \omega^{\mu\nu} + \frac{\partial L}{\partial \left(\nabla_\mu \xi_\nu\right)}\ \delta\left(\nabla_\mu \xi_\nu\right) \right) d\Sigma = 0$$

Now

$$\int_{S^4} \frac{\partial L}{\partial\left(\nabla_\mu \xi_\nu\right)}\ \delta\left(\nabla_\mu \xi_\nu\right) d\Sigma = \int_{S^4} \nabla_\mu \left(\frac{\partial L}{\partial\left(\nabla_\mu \xi_\nu\right)}\ \delta \xi_\nu \right) d\Sigma - \int_{S^4} \nabla_\mu \left(\frac{\partial L}{\partial\left(\nabla_\mu \xi_\nu\right)} \right) \delta \xi_\nu\ d\Sigma$$
$$= - \int_{S^4} \nabla_\mu \left(\frac{\partial L}{\partial\left(\nabla_\mu \xi_\nu\right)} \right) \delta \xi_\nu\ d\Sigma$$

since the first term on the right-hand-side of the first line is an absolute differential that can be transformed away on the boundary of integration by means of the divergence theorem. Hence we have

$$\delta I = \int_{S^4} \left(\frac{\partial L}{\partial T^{\mu\nu}}\ \delta T^{\mu\nu} + \frac{\partial L}{\partial \Phi^{\mu\nu}}\ \delta \Phi^{\mu\nu} + \frac{\partial L}{\partial \omega^{\mu\nu}}\ \delta \omega^{\mu\nu} - \nabla_\mu \left(\frac{\partial L}{\partial\left(\nabla_\mu \xi_\nu\right)} \right) \delta \xi_\nu \right) d\Sigma = 0$$

where each term in the integrand is independent of the others. We may also note that the variations $\delta T^{\mu\nu}, \delta \Phi^{\mu\nu}, \delta \omega^{\mu\nu}$, and $\delta \xi_\nu$ are arbitrary.

From $\dfrac{\partial L}{\partial T^{\mu\nu}} = 0$, we obtain

$$\Phi_{\mu\nu} = \nabla_{(\nu}\, \xi_{\mu)}$$
$$\omega_{\mu\nu} = \nabla_{[\nu}\, \xi_{\mu]}$$

i.e., the covariant components of the "dilation" and intrinsic spin tensors, respectively.

From $\dfrac{\partial L}{\partial \Phi^{\mu\nu}} = 0$, we obtain

$$T^{(\mu\nu)} = \frac{1}{\kappa}\left(R^{(\mu\nu)} - \frac{1}{2}\, g^{\mu\nu}\ R \right) = A^{\mu\nu}{}_{\rho\sigma}\ \Phi^{\rho\sigma} + P^{\mu\nu}{}_{\rho\sigma\lambda\eta}\ \Phi^{\rho\sigma}\ \Phi^{\lambda\eta} - \Theta\ g^{\mu\nu}\ \Delta T$$

i.e., the symmetric contravariant components of the energy-momentum tensor.

From $\dfrac{\partial L}{\partial \omega^{\mu\nu}} = 0$, we obtain

$$T^{[\mu\nu]} = \frac{1}{\kappa} R^{[\mu\nu]} = B^{\mu\nu}{}_{\rho\sigma}\ \omega^{\rho\sigma} + Q^{\mu\nu}{}_{\rho\sigma\lambda\eta}\ \omega^{\rho\sigma}\ \omega^{\lambda\eta}$$

i.e., the anti-symmetric contravariant components of the energy-momentum tensor.

In other words,

$$T^{\mu\nu} = \frac{1}{\kappa}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}\ R\right) = C^{\mu\nu}{}_{\rho\sigma}\ D^{\rho\sigma} + K^{\mu\nu}{}_{\rho\sigma\lambda\eta}\ D^{\rho\sigma}\ D^{\lambda\eta} - \Theta\ g^{\mu\nu}\ \Delta T$$

Finally, we now show in detail that the fourth variation yields an important equation of motion. We first see that

$$\frac{\partial L}{\partial\left(\nabla_\mu \xi_\nu\right)} = T^{\mu\nu} + u^\mu \left(f\ \xi^\nu - \rho\ u^\nu\right)$$

Hence

$$\nabla_\mu\left(\frac{\partial L}{\partial\left(\nabla_\mu \xi_\nu\right)}\right) = \nabla_\mu T^{\mu\nu} + \nabla_\mu\left(f\ u^\mu\right)\xi^\nu + f\ u^\mu\ \nabla_\mu \xi^\nu - \nabla_\mu\left(\rho\ u^\mu\right)u^\nu - \rho\ u^\mu\ \nabla_\mu u^\nu$$

Let us define the "extended" shear scalar and the mass current density vector, respectively, via

$$l = \nabla_\mu\left(f\ u^\mu\right)$$
$$J^\mu = \rho\ u^\mu$$

We can now readily identify the acceleration vector and the body force per unit mass, respectively, by

$$a^\mu = u^\nu\ \nabla_\nu\ a^\mu = \frac{Du^\mu}{Ds}$$
$$b^\mu = \frac{1}{\rho}\left(l\ \xi^\mu + f\ \left(1 - \nabla_\nu\ J^\nu\right)u^\mu\right)$$

In the conservative case, the condition $\nabla_\mu J^\mu = 0$ gives

$$\frac{D\rho}{Ds} = -\rho \nabla_\mu u^\mu$$

In the weak-field limit for which $u^\mu = \left(1, u^A\right)$ where $A = 1,2,3$, we obtain the ordinary continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla_A \left(\rho u^A\right) = 0$$

Finally, we have

$$\int_{S^4} \left(\nabla_\mu T^{\mu\nu} + \rho b^\nu - \rho a^\nu\right) \delta \xi_\nu \, d\Sigma = 0$$

i.e., the equation of motion

$$\nabla_\mu T^{\mu\nu} = \rho \left(a^\nu - b^\nu\right)$$

or

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = \kappa \rho \left(a^\nu - b^\nu\right)$$

If we restrict our attention to point-like particles, the body force vanishes since it cannot act on a structureless (zero-dimensional) object. And since the motion is geodesic, i.e., $a^\mu = 0$, we have the conservation law

$$\nabla_\mu T^{\mu\nu} = 0$$

In this case, this conservation law is true regardless of whether the energy-momentum tensor is symmetric or not.

Let us now discuss the so-called couple stress, i.e., the couple per unit area which is also known as the distributed moment. We denote the couple stress tensor by the second-rank tensor field M. In analogy to the linear constitutive relations relating the energy-momentum tensor to the displacement gradient tensor, we write

$$M^{\mu\nu} = J^{\mu\nu}{}_{\rho\sigma} L^{\rho\sigma} + H^{\mu\nu}{}_{\rho\sigma\lambda\eta} L^{\rho\sigma} L^{\lambda\eta}$$

where

$$J_{\mu\nu\rho\sigma} = E_{\mu\nu\rho\sigma} + F_{\mu\nu\rho\sigma}$$
$$H_{\mu\nu\rho\sigma\lambda\eta} = U_{\mu\nu\rho\sigma\lambda\eta} + V_{\mu\nu\rho\sigma\lambda\eta}$$

These are assumed to possess the same symmetry properties as $C_{\mu\nu\rho\sigma}$ and $K_{\mu\nu\rho\sigma\lambda\eta}$, respectively, i.e., $E_{\mu\nu\rho\sigma}$ have the same symmetry properties as $A_{\mu\nu\rho\sigma}$, $F_{\mu\nu\rho\sigma}$ have the same symmetry properties as $B_{\mu\nu\rho\sigma}$, $U_{\mu\nu\rho\sigma\lambda\eta}$ have the same symmetry properties as $P_{\mu\nu\rho\sigma\lambda\eta}$, and $V_{\mu\nu\rho\sigma\lambda\eta}$ have the same symmetry properties as $Q_{\mu\nu\rho\sigma\lambda\eta}$.

Likewise, the asymmetric tensor given by

$$L_{\mu\nu} = L_{(\mu\nu)} + L_{[\mu\nu]}$$

is comparable to the displacement gradient tensor.

Introducing a new infinitesimal spin potential via ϕ_μ, let the covariant dual form of the intrinsic spin tensor be given by

$$\varpi_{\mu\nu} = \frac{1}{2} \in_{\mu\nu\rho\sigma} \omega^{\rho\sigma} = \frac{1}{2}\left(\nabla_\nu \phi_\mu - \nabla_\mu \phi_\nu\right)$$

Let us now introduce a completely anti-symmetric third-rank spin tensor via

$$S^{\mu\nu\rho} = -\frac{1}{2}\left(\beta - \gamma\right) \in^{\mu\nu\rho\sigma} \phi_\sigma$$

As a direct consequence, we see that

$$\nabla_\rho S^{\mu\nu\rho} = \left(\beta - \gamma\right) \omega^{\mu\nu}$$

In other words,

$$\nabla_\rho S^{\mu\nu\rho} = T^{[\mu\nu]} - N^{\mu\nu} = \frac{1}{\kappa}\left(R^{[\mu\nu]} - \Lambda^{\mu\nu}\right)$$

where

$$N^{\mu\nu} = 2\left(b_4 + b_6\right) \Phi\, \omega^{\mu\nu} + \left(b_8 + b_{10}\right)\left(D^{\mu\rho}\, \Phi^{\nu}{}_{\rho} - D^{\nu\rho}\, \Phi^{\mu}{}_{\rho}\right) + \left(b_9 + b_9\right)\left(D^{\mu\rho}\, \omega^{\nu}{}_{\rho} - D^{\nu\rho}\, \omega^{\mu}{}_{\rho}\right)$$
$$\Lambda^{\mu\nu} = c_8\, \Phi\, \omega^{\mu\nu} + \frac{1}{2}\left(c_9 + c_{11}\right)\left(D^{\mu\rho}\, \Phi^{\nu}{}_{\rho} - D^{\nu\rho}\, \Phi^{\mu}{}_{\rho}\right) + \frac{1}{2}\left(c_{10} + c_{11}\right)\left(D^{\mu\rho}\, \omega^{\nu}{}_{\rho} - D^{\nu\rho}\, \omega^{\mu}{}_{\rho}\right)$$

We can now form the second Lagrangian density of our theory as

$$\overline{H} = \sqrt{\det(g)}\,\Big(M^{\mu\nu}\left(\nabla_\nu \phi_\mu - L_{\mu\nu}\right) + \frac{1}{2} J^{\mu\nu}{}_{\rho\sigma}\, L_{\mu\nu}\, L^{\rho\sigma} + \frac{1}{3} H^{\mu\nu}{}_{\rho\sigma\lambda\eta}\, L_{\mu\nu}\, L^{\rho\sigma}\, L^{\lambda\eta}$$
$$- \in^{\mu}{}_{\rho\sigma\lambda}\left(\nabla_\nu \phi_\mu\right) S^{\rho\sigma\nu}\, u^\lambda + u^\mu \left(\nabla_\mu \phi_\nu\right)\left(h\,\phi^\nu - I\,\rho\, s^\nu\right)\Big)$$

where h is a scalar function, I is the moment of inertia, and s^ν are the components of the angular velocity vector.

Letting $L_{(\mu\nu)} = X_{\mu\nu}$ and $L_{[\mu\nu]} = Z_{\mu\nu}$, the corresponding action integral is

$$J = \int_{S^4} \Big(M^{\mu\nu}\left(\nabla_{(\nu} \phi_{\mu)} - X_{\mu\nu}\right) + M^{\mu\nu}\left(\nabla_{[\nu} \phi_{\mu]} - Z_{\mu\nu}\right) + \frac{1}{2} E^{\mu\nu}{}_{\rho\sigma}\, X_{\mu\nu}\, X^{\rho\sigma} + \frac{1}{2} F^{\mu\nu}{}_{\rho\sigma}\, Z_{\mu\nu}\, Z^{\rho\sigma}$$
$$+ \frac{1}{3} U^{\mu\nu}{}_{\rho\sigma\lambda\eta}\, X_{\mu\nu}\, X^{\rho\sigma}\, X^{\lambda\eta} + \frac{1}{3} V^{\mu\nu}{}_{\rho\sigma\lambda\eta}\, Z_{\mu\nu}\, Z^{\rho\sigma}\, Z^{\lambda\eta} - \in^{\mu}{}_{\rho\sigma\lambda}\left(\nabla_\nu \phi_\mu\right) S^{\rho\sigma\nu}\, u^\lambda$$
$$+ u^\mu \left(\nabla_\mu \phi_\nu\right)\left(h\,\phi^\nu - I\,\rho\, s^\nu\right)\Big)\, d\Sigma$$

As before, writing $\overline{H} = \sqrt{\det(g)}\, H$ and performing the variation $\delta J = 0$, we have

$$\delta J = \int_{S^4} \left(\frac{\partial H}{\partial M^{\mu\nu}}\, \delta M^{\mu\nu} + \frac{\partial H}{\partial X^{\mu\nu}}\, \delta X^{\mu\nu} + \frac{\partial H}{\partial Z^{\mu\nu}}\, \delta Z^{\mu\nu} - \nabla_\mu \left(\frac{\partial H}{\partial\left(\nabla_\mu \phi_\nu\right)} \right) \delta \phi_\nu \right) d\Sigma = 0$$

with arbitrary variations $\delta M^{\mu\nu}, \delta X^{\mu\nu}, \delta Z^{\mu\nu}$, and $\delta \phi_\nu$.

From $\dfrac{\partial H}{\partial M^{\mu\nu}} = 0$, we obtain

$$X_{\mu\nu} = \nabla_{(\nu} \phi_{\mu)}$$
$$Z_{\mu\nu} = \nabla_{[\nu} \phi_{\mu]}$$

From $\dfrac{\partial H}{\partial X^{\mu\nu}} = 0$, we obtain

$$M^{(\mu\nu)} = E^{\mu\nu}{}_{\rho\sigma}\, X^{\rho\sigma} + U^{\mu\nu}{}_{\rho\sigma\lambda\eta}\, X^{\rho\sigma}\, X^{\lambda\eta}$$

From $\dfrac{\partial H}{\partial Z^{\mu\nu}} = 0$, we obtain

$$M^{[\mu\nu]} = F^{\mu\nu}{}_{\rho\sigma}\, Z^{\rho\sigma} + V^{\mu\nu}{}_{\rho\sigma\lambda\eta}\, Z^{\rho\sigma}\, Z^{\lambda\eta}$$

We therefore have the constitutive relation

$$M^{\mu\nu} = J^{\mu\nu}{}_{\rho\sigma}\ L^{\rho\sigma} + H^{\mu\nu}{}_{\rho\sigma\lambda\eta}\ L^{\rho\sigma}\ L^{\lambda\eta}$$

Let us investigate the last variation

$$-\int_{S^4} \nabla_\mu \left(\frac{\partial H}{\partial\left(\nabla_\mu \phi_\nu\right)} \right) \delta\phi_\nu\ d\Sigma = 0$$

in detail.

Firstly,

$$\frac{\partial H}{\partial\left(\nabla_\mu \phi_\nu\right)} = M^{\mu\nu} - \in^\nu{}_{\lambda\rho\sigma}\ S^{\lambda\rho\mu}\ u^\sigma + u^\mu \left(h\ \phi^\nu - I\ \rho\ s^\nu\right)$$

Then we see that

$$\nabla_\mu \left(\frac{\partial H}{\partial\left(\nabla_\mu \phi_\nu\right)} \right) = \nabla_\mu M^{\mu\nu} - \in^\nu{}_{\mu\rho\sigma}\ T^{[\mu\rho]}\ u^\sigma - \in^\nu{}_{\lambda\rho\sigma}\ S^{\lambda\rho\mu}\ \nabla_\mu u^\sigma + \nabla_\mu \left(h\ u^\mu\right)\phi^\nu + h\ u^\mu\ \nabla_\mu \phi^\nu$$
$$- I\ \nabla_\mu \left(\rho\ u^\mu\right) s^\nu - I\ \rho\ u^\mu\ \nabla_\mu s^\nu$$

We now define the angular acceleration by

$$\alpha^\mu = u^\nu\ \nabla_\nu s^\mu = \frac{D s^\mu}{Ds}$$

and the angular body force per unit mass by

$$\beta^\mu = \frac{1}{\rho}\left(\bar{l}\ \phi^\mu + h\ \frac{D\phi^\mu}{Ds} - I\ \left(\nabla_\nu J^\nu\right) s^\mu \right)$$

where $\bar{l} = \nabla_\mu \left(h\ u^\mu\right)$.

We have

$$\int_{S^4} \left(\nabla_\mu M^{\mu\nu} - \in^\nu{}_{\mu\rho\sigma} \left(T^{[\mu\rho]}\ u^\sigma + S^{\mu\rho\lambda}\ \nabla_\lambda u^\sigma \right) + \rho\ \beta^\nu - I\ \rho\ \alpha^\nu \right) \delta\phi_\nu\ d\Sigma = 0$$

Hence we obtain the equation of motion

$$\nabla_{\mu} M^{\mu\nu} = \in^{\nu}{}_{\mu\rho\sigma} \left(\left(T^{[\mu\rho]} - N^{\mu\rho} \right) u^{\sigma} + S^{\mu\rho\lambda} \nabla_{\lambda} u^{\sigma} \right) + \rho \left(I \alpha^{\nu} - \beta^{\nu} \right)$$

i.e.,

$$\nabla_{\mu} M^{\mu\nu} = \in^{\nu}{}_{\mu\rho\sigma} \left(\frac{1}{\kappa} \left(R^{[\mu\rho]} - \Lambda^{\mu\rho} \right) u^{\sigma} + S^{\mu\rho\lambda} \nabla_{\lambda} u^{\sigma} \right) + \rho \left(I \alpha^{\nu} - \beta^{\nu} \right)$$

7. Final Remarks

We have seen that the classical fields of physics can be unified in a simple manner by treating space-time itself as a four-dimensional finite (but unbounded) elastic medium capable of undergoing extensions (dilations) and internal point-rotations in the presence of material-energy fields. In addition, we must note that this apparent simplicity still leaves the constitutive invariants undetermined. At the moment, we leave this aspect of the theory to more specialized attempts. However, it can be said, in general, that we expect the constitutive invariants of the theory to be functions of the known physical properties of matter such as material density, energy density, compressibility, material symmetry, etc. This way, we have successfully built a significant theoretical framework that holds in all classical physical situations.