A New Semi-Symmetric Unified Field Theory of the Classical Fields of Gravity and Electromagnetism

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Introduction

It is now well-known that there are various paths available, provided by geometry alone, to a unified description of physical phenomena. The different possibilities for the interpretation of the underlying nature and fabric of the Universe in a purely geometric fashion imply that there is a deep underlying structural reason for singular harmony that lies in the depths of Nature's unity. Even in the mental sense, it appears that the Universe is a self-descriptive continuum which connects what seem to be purely intrinsic mathematical objects to physical observables. It is the belief that analytical geometry alone is able to provide the profoundest description of the complexity and harmony of our structured Universe that has led generations of mathematicians and physicists to undertake the task of geometrizing the apparently systematic laws of Nature. Indeed this is, as Einstein once described, the effect of the sense of universal causation on the inquisitive mind.

The above-mentioned wealth of the inherent mathematical possibility for the geometrization of physics has resulted in the myriad forms of unified field theory which have been proposed from time to time, roughly since 1918 when H. Weyl's applied his so-called purely infinitesimal geometry which was a relaxation of the geometry of Riemann spaces to the task of geometrizing the electromagnetic field in the hope to unify it with the already geometrized gravitational field of general relativity [6]. However, often for want of simplicity, this fact which basically gives us a vision of a solid, reified reality may also lead us to think that the Universe of phenomena must be ultimately describable in the somewhat simplest and yet perhaps most elegant mathematical (i.e., geometric) formalism. Furthermore, when one is exposed to the different forms of unified field theory, especially for the first time, I believe it is better for one to see a less complicated version, otherwise one might get overloaded mentally and it follows that there is a chance that such a thing will just prevent one from absorbing the essence of our desired simplicity which is intuitively expected to be present in any objective task of unification.

Given the freedom of choice, we do not attempt, in this work, to speak about which version of unified field theory out of many is true, rather we shall present what I believe should qualify among the logically simplest geometric descriptions of the classical fields of gravity and electromagnetism. Indeed, for the reason that we may not still be fully aware of the many hidden aspects of the Universe on the microscopic (quantum) scales, at present we shall restrict our attention to the unification and geometrization of the classical fields alone.

As we know, there are many types of differential geometry, from affine geometry to non-affine geometry, from metric (i.e., metric-compatible) geometry to non-metric geometry. However, the different systems of differential geometry that have been developed over hundreds of years can be most elegantly cast in the language of Cartan geometry. The geometric system I will use throughout this physical part of our work is a metric-compatible geometry endowed with a semi-symmetric Cartan connection. It therefore is a variant of the so-called Riemann-Cartan geometry presented in Sections 1.1-1.6. As we know, the standard form of general relativity adopts the symmetric, torsion-free, metric-compatible Christoffel connection. We are also aware that the various extensions of standard general relativity [7] tend to employ more general connections that are often asymmetric (e.g., the Sciama-Kibble theory [8, 9]) and even non-metric in general (e.g., the Weyl theory [6]). However, in the present work, we shall insist on logical simplicity and on having meaningful physical consequences. Once again, we are in no way interested in pointing out which geometric system is most relevant to physics, rather we are simply concerned with describing in detail what appears to be among the most consistent and accurate views of the physical world. We only wish to construct a unified field theory on the common foundation of beauty, simplicity, and observational accuracy without having to deal with unnecessarily complex physical implications that might dull our perspective on the workings of Nature. I myself have always been fond of employing the most general type of connection for the purpose of unification. However, after years of poring over the almost universally held and (supposedly) objectively existing physical evidence, I have come to the conclusion that there is more reason to impose a simpler geometric formulation than a more general type of geometry such as non-metric geometry. In this work, it is my hope to dovetail the classical fields of gravity and electromagnetism with the conventional Riemann-Cartan geometry in general and with a newly constructed semi-symmetric Cartan connection in particular. Our resulting field equations are then just the distillation of this motive, which will eventually give us a penetrating and unified perspective on the nature of the classical fields of gravity and electromagnetism as *intrinsic* geometric fields, as well as on the possible interaction between the translational and rotational symmetries of the space-time manifold.

I believe that the semi-symmetric nature of the present theory (which keeps us as close as possible to the profound, observable physical implications of standard general relativity) is of great generality such that it can be applied to a large class of problems, especially problems related to the more general laws of motion for objects with structure.

1. A Comprehensive Evaluation of the Differential Geometry of Cartan Connections with Metric Structure

The splendid, profound, and highly intuitive interpretation of differential geometry by E. Cartan, which was first applied to Riemann spaces, has resulted in a highly systematic description of a vast range of geometric and topological properties of differentiable manifolds. Although it possesses a somewhat abstract analytical foundation, to my knowledge there is no instance where Riemann-Cartan geometry, cast in the language of differential forms (i.e., exterior calculus), gives a description that is in conflict with the classical tensor analysis as formalized, e.g., by T. Levi-Civita. Given all its successes, one might expect that any physical theory, which relies on the concept of a field, can be elegantly built on its rigorous foundation. Therefore, as long as the reality of metric structure (i.e., metric compatibility) is assumed, it appears that a substantial modified geometry is not needed to supersede Riemann-Cartan geometry.

A common overriding theme in both mathematics and theoretical physics is that of unification. And as long as physics can be thought of as geometry, the geometric objects within Riemann-Cartan geometry (such as curvature for gravity and torsion for intrinsic spin) certainly help us visualize and conceptualize the essence of unity in physics. Because of its intrinsic unity and its breadth of numerous successful applications, it might be possible for nearly all the laws governing physical phenomena to be combined and written down in compact form via the structural equations. By the intrinsic unity of Riemann-Cartan geometry, I simply refer to its tight interlock between algebra, analysis, group representation theory, and geometry. At least in mathematics alone, this is just as close as one can get to a "final" unified description of things. I believe that the unifying power of this beautiful piece of mathematics extends further still.

I'm afraid the title I have given to this first part of our work (which deals with the essential mathematics) has a somewhat narrow meaning, unlike the way it sounds. In writing this article, my primary goal has been to present Riemann-Cartan geometry in a somewhat simpler, more concise, and therefore more efficient form than others dealing with the same subject have done before [1-4]. I have therefore had to drop whatever mathematical elements or representations that might seem somewhat highly counterintuitive at first. After all, not everyone, unless perhaps he or she is a mathematician, is familiar with abstract concepts from algebra, analysis, and topology, just to name a few. Nor is he or she expected to understand these things. But one thing remains essential, namely, one's ability to catch at least a glimpse of the beauty of the presented subject via deep, often simple, real understanding of its basics. As a non-mathematician (or simply a "dabbler" in pure mathematics), I do think that pure mathematics as a whole has grown extraordinarily "strange", if not complex (the weight of any complexity is really relative of course), with a myriad of seemingly separate branches, each of which might only be understood at a certain level of depth by the pure mathematicians specializing in that particular branch themselves. As such, a comparable complexity may also have occurred in the case of theoretical physics itself as it necessarily feeds on the latest formalism of the relevant mathematics each time. Whatever may be the case, the real catch is in the *essential understanding* of the basics. I believe simplicity alone will reveal it without necessarily having to diminish one's perspectives at the same time.

1.1 A brief elementary introduction to the Cartan(-Hausdorff) manifold C^∞

Let $\omega_a = \dfrac{\partial X^i}{\partial x^a} E_i = \partial_a X^i E_i$ (summation convention employed throughout this article) be the covariant (*frame*) basis spanning the $n-$dimensional base manifold C^∞ with local coordinates $x^a = x^a\left(X^k\right)$. The contravariant (*coframe*) basis θ^b is then given via the orthogonal projection $\left\langle \theta^b, \omega_a \right\rangle = \delta^b_a$, where δ^b_a are the components of the Kronecker delta (whose value is unity if the indices coincide or null otherwise). Now the set of linearly independent local directional derivatives $E_i = \dfrac{\partial}{\partial X^i} = \partial_i$ gives the coordinate basis of the locally flat tangent space $T_x(M)$ at a point $x \in C^\infty$. Here M denotes the topological space of the so-called $n-$tuples $h(x) = h\left(x^1, ..., x^n\right)$ such that relative to a given chart $(U, h(x))$ on a neighborhood U of a local coordinate point x, our $C^\infty-$differentiable manifold itself is a topological space. The dual basis to E_i spanning the locally flat cotangent space $T^*_x(M)$ will then be given by the differential elements dX^k via the relation $\left\langle dX^k, \partial_i \right\rangle = \delta^k_i$. In fact and in general, the *one-forms* dX^k indeed act as a linear map $T_x(M) \to IR$ when applied to an arbitrary vector field $F \in T_x(M)$ of the explicit form $F = F^i \dfrac{\partial}{\partial X^i} = f^a \dfrac{\partial}{\partial x^a}$. Then it is easy to see that $F^i = F\, X^i$ and $f^a = F\, x^a$, from which we obtain the usual transformation laws for the contravariant components of a vector field, i.e., $F^i = \partial_a X^i\, f^a$ and $f^i = \partial_i x^a\, F^i$, relating the localized components of F to the general ones and vice versa. In addition, we also see that $\left\langle dX^k, F \right\rangle = F\, X^k = F^k$.

The components of the metric tensor $g = g_{ab}\, \theta^a \otimes \theta^b$ of the base manifold C^∞ are readily given by

$$g_{ab} = \left\langle \omega_a, \omega_b \right\rangle$$

The components of the metric tensor $g\left(x_N\right) = \eta_{ik}\, dX^i \otimes dX^k$ describing the locally flat tangent space $T_x(M)$ of rigid frames at a point $x_N = x_N\left(x^a\right)$ are given by

$$\eta_{ik} = \left\langle E_i, E_k \right\rangle = diag\left(\pm 1, \pm 1, ..., \pm 1\right)$$

In four dimensions, the above may be taken to be the components of the Minkowski metric tensor, i.e., $\eta_{ik} = \left\langle E_i, E_k \right\rangle = diag\left(1, -1, -1, -1\right)$.

Then we have the expression

$$g_{ab} = \eta_{ik}\, \partial_a X^i\, \partial_b X^k$$

satisfying

$$g_{ac}\, g^{bc} = \delta^b_a$$

where $g^{ab} = \left\langle \theta^a , \theta^b \right\rangle$.

The manifold C^∞ is a metric space whose line-element in this formalism of a differentiable manifold is directly given by the metric tensor itself, i.e.,

$$ds^2 = g = g_{ab} \left(\partial_i x^a \; \partial_k x^b \right) dX^i \otimes dX^k$$

where the coframe basis is given by the one-forms $\theta^a = \partial_i x^a \; dX^i$.

1.2 Exterior calculus in n dimensions

As we know, an arbitrary tensor field $T \in C^\infty$ of rank (p, q) is the object

$$T = T^{i_1 i_2 \ldots i_q}_{j_1 j_2 \ldots j_p} \;\; \omega_{i_1} \otimes \omega_{i_2} \otimes \ldots \otimes \omega_{i_q} \otimes \theta^{j_1} \otimes \theta^{j_2} \otimes \ldots \otimes \theta^{j_p}$$

Given the existence of a local coordinate transformation via $x^i = x^i \left(\bar{x}^\alpha \right)$ in C^∞, the components of $T \in C^\infty$ transform according to

$$T^{ij\ldots s}_{kl\ldots r} = T^{\alpha\beta\ldots\lambda}_{\mu\nu\ldots\eta} \; \partial_\alpha x^i \; \partial_\beta x^j \; \ldots \; \partial_\lambda x^s \; \partial_k \bar{x}^\mu \; \partial_l \bar{x}^\nu \; \ldots \; \partial_r \bar{x}^\eta$$

Taking a local coordinate basis $\theta^i = dx^i$, a Pfaffian $p-$form ω is the completely anti-symmetric tensor field

$$\omega = \omega_{i_1 i_2 \ldots i_p} \;\; dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}$$

where

$$dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \equiv \frac{1}{p!} \, \delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p} \;\; dx^{j_1} \otimes dx^{j_2} \otimes \ldots \otimes dx^{j_p}$$

In the above, the $\delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p}$ are the components of the generalized Kronecker delta. They are given by

$$\delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p} = \in_{j_1 j_2 \ldots j_p} \in^{i_1 \ldots i_p} = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \delta^{i_2}_{j_1} & \ldots & \delta^{i_p}_{j_1} \\ \delta^{i_1}_{j_2} & \delta^{i_2}_{j_2} & \ldots & \delta^{i_p}_{j_2} \\ \ldots & \ldots & \ldots & \ldots \\ \delta^{i_1}_{j_p} & \delta^{i_2}_{j_p} & \ldots & \delta^{i_p}_{j_p} \end{pmatrix}$$

where $\in_{j_1 j_2 \ldots j_p} = \sqrt{\det(g)}\, \varepsilon_{j_1 j_2 \ldots j_p}$ and $\in^{i_1 i_2 \ldots i_p} = \frac{1}{\sqrt{\det(g)}}\, \varepsilon^{i_1 i_2 \ldots i_p}$ are the covariant and contravariant components of the completely anti-symmetric Levi-Civita permutation tensor, respectively, with the ordinary permutation symbols being given as usual by $\varepsilon_{j_1 j_2 \ldots j_q}$ and $\varepsilon^{i_1 i_2 \ldots i_p}$.

We can now write

$$\omega = \frac{1}{p!}\, \delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p}\, \omega_{i_1 i_2 \ldots i_p}\, dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_p}$$

such that for a null $p-$form $\omega = 0$ its components satisfy the relation $\delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p}\, \omega_{i_1 i_2 \ldots i_p} = 0$.

By meticulously moving the dx^i from one position to another, we see that

$$\begin{aligned} &dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{p-1}} \wedge dx^{i_p} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_q} \\ &\qquad = (-1)^p\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{p-1}} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_q} \wedge dx^{i_p} \end{aligned}$$

and

$$\begin{aligned} &dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_q} \\ &\qquad = (-1)^{pq}\, dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_q} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \end{aligned}$$

Let ω and π be a $p-$form and a $q-$form, respectively. Then, in general, we have the following relations:

$$\begin{aligned} &\omega \wedge \pi = (-1)^{pq}\, \pi \wedge \omega = \omega_{i_1 i_2 \ldots i_p}\, \pi_{j_1 j_2 \ldots j_q}\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^p \wedge dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_q} \\ &d(\omega + \pi) = d\omega + d\pi \\ &d(\omega \wedge \pi) = d\omega \wedge \pi + (-1)^p\, \omega \wedge d\pi \end{aligned}$$

Note that the mapping $d: \omega = d\omega$ is a $(p+1)-$form. Explicitly, we have

$$d\omega = \frac{(-1)^p}{(p+1)!}\,\delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p}\,\frac{\partial\,\omega_{i_1 i_2 \ldots i_p}}{\partial\, x^{i_{p+1}}}\; dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_p} \wedge dx^{i_{p+1}}$$

For instance, given a (continuous) function f, the one-form $df = \partial_i\, f\, dx^i$ satisfies $d^2 f \equiv ddf = \partial_k\, \partial_i\, f\, dx^k \wedge dx^i = 0$. Likewise, for the one-form $A = A_i\; dx^i$, we have $dA = \partial_k\, A_i\; dx^k \wedge dx^i$ and therefore $d^2 A = \partial_l\, \partial_k\, A_i\; dx^l \wedge dx^k \wedge dx^i = 0$, i.e., $\delta^{ikl}_{rst}\; \partial_l\, \partial_k\, A_i = 0$ or $\partial_l\, \partial_k\, A_i + \partial_k\, \partial_i\, A_l + \partial_i\, \partial_l\, A_k = 0$. Obviously, the last result holds for arbitrary $p-$forms $\Pi^{ij\ldots s}_{kl\ldots r}$, i.e.,

$$d^2 \Pi^{ij\ldots s}_{kl\ldots r} = 0$$

Let us now consider a simple two-dimensional case. From the transformation law $dx^i = \partial_\alpha\, x^i\, d\bar{x}^\alpha$, we have, upon employing a positive definite Jacobian, i.e., $\dfrac{\partial\left(x^i, x^j\right)}{\partial\left(\bar{x}^\alpha, \bar{x}^\beta\right)} > 0$, the following:

$$dx^i \wedge dx^j = \partial_\alpha\, x^i\, \partial_\beta\, x^j\;\; d\bar{x}^\alpha \wedge d\bar{x}^\beta = \frac{1}{2}\,\frac{\partial\left(x^i, x^j\right)}{\partial\left(\bar{x}^\alpha, \bar{x}^\beta\right)}\, d\bar{x}^\alpha \wedge d\bar{x}^\beta$$

It is easy to see that

$$dx^1 \wedge dx^2 = \frac{\partial\left(x^1, x^2\right)}{\partial\left(\bar{x}^1, \bar{x}^2\right)}\, d\bar{x}^1 \wedge d\bar{x}^2$$

which gives the correct transformation law of a surface element.

We can now elaborate on the so-called *Stokes theorem*. Given an arbitrary function f, the integration in a domain D in the manifold C^∞ is such that

$$\iint\limits_D f\left(x^i\right) dx^1 \wedge dx^2 = \iint\limits_D f\left(x^i\left(\bar{x}^\alpha\right)\right) \frac{\partial\left(x^1, x^2\right)}{\partial\left(\bar{x}^1, \bar{x}^2\right)}\, d\bar{x}^1\, d\bar{x}^2$$

Generalizing to n dimensions, for any $\psi^i = \psi^i\left(x^k\right)$ we have

$$d\psi^1 \wedge d\psi^2 \wedge \ldots \wedge d\psi^n = \frac{\partial\left(\psi^1, \psi^2, \ldots, \psi^n\right)}{\partial\left(x^1, x^2, \ldots, \bar{x}^n\right)}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

Therefore (in a particular domain)

$$\iint\ldots\int f\; d\psi^1 \wedge d\psi^2 \wedge\ldots\wedge d\psi^n = \iint\ldots\int f\left(x^i\right)\frac{\partial\left(\psi^1,\psi^2,\ldots,\psi^n\right)}{\partial\left(x^1,x^2,\ldots,x^n\right)}\, dx^1 \wedge dx^2 \wedge\ldots\wedge dx^n$$

Obviously, the value of this integral is independent of the choice of the coordinate system. Under the coordinate transformation given by $x^i = x^i\left(\bar{x}^\alpha\right)$, the Jacobian can be expressed as

$$\frac{\partial\left(\psi^1,\psi^2,\ldots,\psi^n\right)}{\partial\left(x^1,x^2,\ldots,x^n\right)} = \frac{\partial\left(\psi^1,\psi^2,\ldots,\psi^n\right)}{\partial\left(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^n\right)}\frac{\partial\left(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^n\right)}{\partial\left(x^1,x^2,\ldots,x^n\right)}$$

If we consider a $(n-m)-$dimensional subspace (hypersurface) $S \in C^\infty$ whose local coordinates u^A parametrize the coordinates x^i, we have

$$\begin{aligned}&\int\int\ldots\int f\; d\psi^1 \wedge d\psi^2 \wedge\ldots\wedge d\psi^n\\&= \int\int\ldots\int f\left(x^i\left(u^A\right)\right)\frac{\partial\left(\psi^1\left(x^i\left(u^A\right)\right),\psi^2\left(x^i\left(u^A\right)\right),\ldots,\psi^n\left(x^i\left(u^A\right)\right)\right)}{\partial\left(u^1,u^2,\ldots,u^{n-m}\right)}\, du^1\, du^2\ldots du^{n-m}\end{aligned}$$

1.3 Geometric properties of a curved manifold

Let us recall a few concepts from conventional tensor analysis for a while. Introducing a generally asymmetric connection Γ via the covariant derivative

$$\partial_b\, \omega_a = \Gamma^c_{ab}\; \omega_c$$

i.e.,

$$\Gamma^c_{ab} = \left\langle \theta^c, \partial_b\, \omega_a \right\rangle = \Gamma^c_{(ab)} + \Gamma^c_{[ab]}$$

where the round index brackets indicate symmetrization and the square ones indicate anti-symmetrization, we have, by means of the local coordinate transformation given by $x^a = x^a\left(\bar{x}^\alpha\right)$ in C^∞

$$\partial_b\, e^\alpha_a = \Gamma^c_{ab}\; e^\alpha_c - \bar{\Gamma}^\alpha_{\beta\lambda}\; e^\beta_a\; e^\lambda_b$$

where the tetrads of the *moving frames* are given by $e^\alpha_a = \partial_a\, \bar{x}^\alpha$ and $e^a_\alpha = \partial_\alpha\, x^a$. They satisfy $e^a_\alpha\; e^\alpha_b = \delta^a_b$ and $e^\alpha_a\; e^a_\beta = \delta^\alpha_\beta$. In addition, it can also be verified that

$$\begin{aligned}\partial_\beta\, e^a_\alpha &= \bar{\Gamma}^\lambda_{\alpha\beta}\; e^a_\lambda - \Gamma^a_{bc}\; e^b_\alpha\; e^c_\beta\\ \partial_b\, e^a_\alpha &= e^a_\lambda\; \bar{\Gamma}^\lambda_{\alpha\beta}\; e^\beta_b - \Gamma^a_{cb}\; e^c_\alpha\end{aligned}$$

From conventional tensor analysis, we know that Γ is a non-tensorial object, since its components transform as

$$\Gamma^c_{ab} = e^c_\alpha \, \partial_b \, e^\alpha_a + e^c_\alpha \, \overline{\Gamma}^\alpha_{\beta\lambda} \, e^\beta_a \, e^\lambda_b$$

However, it can be described as a kind of displacement field since it is what makes possible a comparison of vectors from point to point in C^∞. In fact the relation $\partial_b \, \omega_a = \Gamma^c_{ab} \, \omega_c$ defines the so-called metricity condition, i.e., the change (during a displacement) in the basis can be measured by the basis itself. This immediately translates into

$$\nabla_c \, g_{ab} = 0$$

where we have just applied the notion of a covariant derivative to an arbitrary tensor field T:

$$\begin{aligned}\nabla_k \, T^{ij\ldots s}_{lm\ldots r} = \partial_k \, T^{ij\ldots s}_{lm\ldots r} + \Gamma^i_{pk} \, T^{pj\ldots s}_{lm\ldots r} + \Gamma^j_{pk} \, T^{ip\ldots s}_{lm\ldots r} + \ldots + \Gamma^s_{pk} \, T^{ij\ldots p}_{lm\ldots r} \\ - \Gamma^p_{lk} \, T^{ij\ldots s}_{pm\ldots r} - \Gamma^p_{mk} \, T^{ij\ldots s}_{lp\ldots r} - \ldots - \Gamma^p_{rk} \, T^{ij\ldots s}_{lm\ldots p}\end{aligned}$$

such that $\left(\partial_k \, T\right)^{ij\ldots s}_{lm\ldots r} = \nabla_k \, T^{ij\ldots s}_{lm\ldots r}$.

The condition $\nabla_c \, g_{ab} = 0$ can be solved to give

$$\Gamma^c_{ab} = \frac{1}{2} \, g^{cd} \, \left(\partial_b \, g_{da} - \partial_d \, g_{ab} + \partial_a \, g_{bd}\right) + \Gamma^c_{[ab]} - g^{cd} \, \left(g_{ae} \, \Gamma^e_{[db]} + g_{be} \, \Gamma^e_{[da]}\right)$$

from which it is customary to define

$$\Delta^c_{ab} = \frac{1}{2} \, g^{cd} \, \left(\partial_b \, g_{da} - \partial_d \, g_{ab} + \partial_a \, g_{bd}\right)$$

as the Christoffel symbols (symmetric in their two lower indices) and

$$K^c_{ab} = \Gamma^c_{[ab]} - g^{cd} \, \left(g_{ae} \, \Gamma^e_{[db]} + g_{be} \, \Gamma^e_{[da]}\right)$$

as the components of the so-called contorsion tensor (anti-symmetric in the first two mixed indices).

Note that the components of the torsion tensor are given by

$$\Gamma^{a}_{[bc]} = \frac{1}{2}\, e^{a}_{\alpha} \left(\partial_{c} e^{\alpha}_{b} - \partial_{b} e^{\alpha}_{c} + e^{\beta}_{b}\, \overline{\Gamma}^{\alpha}_{\beta c} - e^{\beta}_{c}\, \overline{\Gamma}^{\alpha}_{\beta b} \right)$$

where we have set $\overline{\Gamma}^{\alpha}_{\beta c} = \overline{\Gamma}^{\alpha}_{\beta\lambda}\, e^{\lambda}_{c}$, such that for an arbitrary scalar field Φ we have

$$\left(\nabla_{a} \nabla_{b} - \nabla_{b} \nabla_{a} \right) \Phi = 2\, \Gamma^{c}_{[ab]}\, \nabla_{c} \Phi$$

The components of the curvature tensor R of C^{∞} are then given via the relation

$$\begin{aligned}
\left(\nabla_{q} \nabla_{p} - \nabla_{p} \nabla_{q} \right) T^{ab\ldots s}_{cd\ldots r} &= T^{ab\ldots s}_{wd\ldots r}\, R^{w}{}_{cpq} + T^{ab\ldots s}_{cw\ldots r}\, R^{w}{}_{dpq} + \ldots + T^{ab\ldots s}_{cd\ldots w}\, R^{w}{}_{rpq} \\
&\quad - T^{wb\ldots s}_{cd\ldots r}\, R^{a}{}_{wpq} - T^{aw\ldots s}_{cd\ldots r}\, R^{b}{}_{wpq} - \ldots - T^{ab\ldots w}_{cd\ldots r}\, R^{s}{}_{wpq} \\
&\quad - 2\, \Gamma^{w}_{[pq]}\, \nabla_{w} T^{ab\ldots s}_{cd\ldots r}
\end{aligned}$$

where

$$\begin{aligned}
R^{d}{}_{abc} &= \partial_{b} \Gamma^{d}_{ac} - \partial_{c} \Gamma^{d}_{ab} + \Gamma^{e}_{ac}\, \Gamma^{d}_{eb} - \Gamma^{e}_{ab}\, \Gamma^{d}_{ec} \\
&= B^{d}{}_{abc}(\Delta) + \hat{\nabla}_{b} K^{d}_{ac} - \hat{\nabla}_{c} K^{d}_{ab} + K^{e}_{ac}\, K^{d}_{eb} - K^{e}_{ab}\, K^{d}_{ec}
\end{aligned}$$

where $\hat{\nabla}$ denotes covariant differentiation with respect to the Christoffel symbols alone, and where

$$B^{d}{}_{abc}(\Delta) = \partial_{b} \Delta^{d}_{ac} - \partial_{c} \Delta^{d}_{ab} + \Delta^{e}_{ac}\, \Delta^{d}_{eb} - \Delta^{e}_{ab}\, \Delta^{d}_{ec}$$

are the components of the Riemann-Christoffel curvature tensor of C^{∞}.

From the components of the curvature tensor, namely, $R^{d}{}_{abc}$, we have (using the metric tensor to raise and lower indices)

$$\begin{aligned}
R_{ab} &\equiv R^{c}{}_{acb} = B_{ab}(\Delta) + \hat{\nabla}_{c} K^{c}_{ab} - K^{c}_{ad}\, K^{d}_{cb} - 2\, \hat{\nabla}_{b} \Gamma^{c}_{[ac]} + 2\, K^{c}_{ab}\, \Gamma^{d}_{[cd]} \\
R &\equiv R^{a}_{a} = B(\Delta) - 4\, g^{ab}\, \hat{\nabla}_{a} \Gamma^{c}_{[bc]} - 2\, g^{ac}\, \Gamma^{b}_{[ab]}\, \Gamma^{d}_{[cd]} - K_{abc}\, K^{acb}
\end{aligned}$$

where $B_{ab}(\Delta) \equiv B^{c}{}_{acb}(\Delta)$ are the components of the symmetric Ricci tensor and $B(\Delta) \equiv B^{a}{}_{a}(\Delta)$ is the Ricci scalar. Note that $K_{abc} \equiv g_{ad}\, K^{d}_{bc}$ and $K^{acb} \equiv g^{cd}\, g^{be}\, K^{a}_{de}$.

Now since

$$\begin{aligned}
\Gamma^{b}_{ba} &= \Delta^{b}_{ba} = \Delta^{b}_{ab} = \partial_{a} \left(\ln \sqrt{\det(g)} \right) \\
\Gamma^{b}_{ab} &= \partial_{a} \left(\ln \sqrt{\det(g)} \right) + 2\, \Gamma^{b}_{[ab]}
\end{aligned}$$

we see that for a continuous metric determinant, the so-called homothetic curvature vanishes:

$$H_{ab} \equiv R^{c}{}_{cab} = \partial_a \Gamma^c_{cb} - \partial_b \Gamma^c_{ca} = 0$$

Introducing the traceless Weyl tensor C, we have the following decomposition theorem:

$$\begin{aligned} R^{d}{}_{abc} = C^{d}{}_{abc} &+ \frac{1}{n-2}\left(\delta^d_b \; R_{ac} + g_{ac} \; R^d_{\ b} - \delta^d_c \; R_{ab} - g_{ab} \; R^d_{\ c}\right) \\ &+ \frac{1}{(n-1)(n-2)}\left(\delta^d_c \; g_{ab} - \delta^d_b \; g_{ac}\right) R \end{aligned}$$

which is valid for $n > 2$. For $n = 2$, we have

$$R^{d}{}_{abc} = K_G \left(\delta^d_b \; g_{ac} - \delta^d_c \; g_{ab}\right)$$

where

$$K_G = \frac{1}{2} R$$

is the Gaussian curvature of the surface. Note that (in this case) the Weyl tensor vanishes.

Any $n-$dimensional manifold (for which $n > 1$) with constant sectional curvature R and vanishing torsion is called an Einstein space. It is described by the following simple relations:

$$\begin{aligned} R^{d}{}_{abc} &= \frac{1}{n(n-1)}\left(\delta^d_b \; g_{ac} - \delta^d_c \; g_{ab}\right) R \\ R_{ab} &= \frac{1}{n} g_{ab} \; R \end{aligned}$$

In the above, we note especially that

$$\begin{aligned} R^{d}{}_{abc} &= B^{d}{}_{abc}(\Delta) \\ R_{ab} &= B_{ab}(\Delta) \\ R &= B(\Delta) \end{aligned}$$

Furthermore, after some elaborate (if not tedious) algebra, we obtain, in general, the following *generalized* Bianchi identities:

$$
\begin{aligned}
&R^{a}{}_{bcd} + R^{a}{}_{cdb} + R^{a}{}_{dbc} = -2\left(\partial_d \Gamma^{a}_{[bc]} + \partial_b \Gamma^{a}_{[cd]} + \partial_c \Gamma^{a}_{[db]} + \Gamma^{a}_{eb}\,\Gamma^{e}_{[cd]} + \Gamma^{a}_{ec}\,\Gamma^{e}_{[db]} + \Gamma^{a}_{ed}\,\Gamma^{e}_{[bc]}\right) \\
&\nabla_e R^{a}_{bcd} + \nabla_c R^{a}_{bde} + \nabla_d R^{a}_{bec} = 2\left(\Gamma^{f}_{[cd]}\,R^{a}_{bfe} + \Gamma^{f}_{[de]}\,R^{a}_{bfc} + \Gamma^{f}_{[ec]}\,R^{a}_{bfd}\right) \\
&\nabla_a \left(R^{ab} - \frac{1}{2} g^{ab}\, R\right) = 2\, g^{ab}\,\Gamma^{c}_{[da]}\,R^{d}_{c} + \Gamma^{a}_{[cd]}\,R^{cdb}{}_{a}
\end{aligned}
$$

for any metric-compatible manifold endowed with both curvature and torsion.

In the last of the above set of equations, we have introduced the generalized Einstein tensor, i.e.,

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab}\, R$$

In particular, we also have the following specialized identities, i.e., the *regular* Bianchi identities:

$$
\begin{aligned}
&B^{a}{}_{bcd} + B^{a}{}_{cdb} + B^{a}{}_{dbc} = 0 \\
&\hat{\nabla}_e B^{a}_{bcd} + \hat{\nabla}_c B^{a}_{bde} + \hat{\nabla}_d B^{a}_{bec} = 0 \\
&\hat{\nabla}_a \left(B^{ab} - \frac{1}{2} g^{ab}\, B\right) = 0
\end{aligned}
$$

In general, these hold in the case of a symmetric, metric-compatible connection. Non-metric differential geometry is beyond the scope of our present consideration. We will need the identities presented in this section in the development of our semi-symmetric, metric-compatible unified field theory.

1.4 The structural equations

The results of the preceding section can be expressed in the language of exterior calculus in a somewhat more compact form.

In general, we can construct arbitrary $p-$forms $\omega^{ab\ldots e}_{cd\ldots f}$ through arbitrary $(p-1)$ forms $\alpha^{ab\ldots e}_{cd\ldots f}$, i.e.,

$$\omega^{ab\ldots e}_{cd\ldots f} = d\alpha^{ab\ldots e}_{cd\ldots f} = \frac{\partial \alpha^{ab\ldots e}_{cd\ldots f}}{\partial x^h} \wedge dx^h$$

The covariant exterior derivative is then given by

$$D\omega^{ab\ldots e}_{cd\ldots f} = \nabla_h\, \omega^{ab\ldots e}_{cd\ldots f} \wedge dx^h$$

i.e.,

$$D\omega^{ab\ldots e}_{cd\ldots f} = d\omega^{ab\ldots e}_{cd\ldots f} + (-1)^p \Big(\omega^{hb\ldots e}_{cd\ldots f} \wedge \Gamma^a_h + \omega^{ah\ldots e}_{cd\ldots f} \wedge \Gamma^b_h + \ldots + \omega^{ab\ldots h}_{cd\ldots f} \wedge \Gamma^e_h$$
$$- \omega^{ab\ldots e}_{hd\ldots f} \wedge \Gamma^h_c - \omega^{ab\ldots e}_{ch\ldots f} \wedge \Gamma^h_d - \ldots - \omega^{ab\ldots e}_{cd\ldots h} \wedge \Gamma^h_f \Big)$$

where we have defined the connection one-forms by

$$\Gamma^a_b \equiv \Gamma^a_{bc}\ \theta^c$$

with respect to the coframe basis θ^a.

Now we write the torsion two-forms τ^a as

$$\tau^a = D\theta^a = d\theta^a + \Gamma^a_b \wedge \theta^b$$

This gives the first structural equation. With respect to another local coordinate system (with coordinates $\bar{x}^\alpha$) in C^∞ spanned by the basis $\varepsilon^\alpha = e^\alpha_a\ \theta^a$, we see that

$$\tau^a = -\, e^a_\alpha\ \bar{\Gamma}^\alpha_{[\beta\lambda]}\ \varepsilon^\beta \wedge \varepsilon^\lambda$$

We shall again proceed to define the curvature tensor. For a triad of arbitrary vectors u, v, w, we may define the following relations with respect to the frame basis ω_a:

$$\nabla_u \nabla_v w \equiv u^c\ \nabla_c \left(v^b\ \nabla_b\, w^a\right) \omega_a$$
$$\nabla_{[u,v]}\, w \equiv \nabla_b\, w^a \left(u^c\ \nabla_c\, v^b - v^c\ \nabla_c\, u^b\right)$$

where ∇_u and ∇_v denote covariant differentiation in the direction of u and of v, respectively.

Then we have

$$\left(\nabla_u \nabla_v - \nabla_v \nabla_u\right) w = {}^*R^a{}_{bcd}\ w^b\ u^c\ v^d\ \omega_a$$

Note that

$${}^*R^a{}_{bcd} = \partial_c\, \Gamma^a_{bd} - \partial_d\, \Gamma^a_{bc} + \Gamma^e_{bd}\ \Gamma^a_{ec} - \Gamma^e_{bc}\ \Gamma^a_{ed} + 2\ \Gamma^e_{[cd]}\ \Gamma^a_{be}$$
$$= R^a{}_{bcd} + 2\ \Gamma^e_{[cd]}\ \Gamma^a_{be}$$

are the components of the extended curvature tensor ${}^*R$.

Define the curvature two-forms by

$$ {}^{*}R^{a}_{\ b} \equiv \frac{1}{2}\, {}^{*}R^{a}_{\ bcd}\ \theta^{c} \wedge \theta^{d} $$

The second structural equation is then

$$ {}^{*}R^{a}_{\ b} = d\Gamma^{a}_{b} + \Gamma^{a}_{c} \wedge \Gamma^{c}_{b} $$

The third structural equation is given by

$$ d^{2}\Gamma^{a}_{b} = d\,{}^{*}R^{a}_{b} - {}^{*}R^{a}_{\ c} \wedge \Gamma^{c}_{b} + \Gamma^{a}_{c} \wedge {}^{*}R^{c}_{\ b} = D\,{}^{*}R^{a}_{\ b} $$

which is equivalent to the generalized Bianchi identities given in the preceding section.

In fact the second and third structural equations above can be directly verified using the properties of exterior differentiation given in Section 1.2.

Now, as we have seen, the covariant exterior derivative of arbitrary one-forms ϕ^{a} is given by $D\phi^{a} = d\phi^{a} + \Gamma^{a}_{b} \wedge \phi^{b}$. Then

$$ \begin{aligned} DD\phi^{a} &= d\left(D\phi^{a}\right) + \Gamma^{a}_{b} \wedge D\phi^{b} \\ &= d\left(d\phi^{a} + \Gamma^{a}_{b} \wedge \phi^{b}\right) + \Gamma^{a}_{c} \wedge \left(d\phi^{c} + \Gamma^{c}_{d} \wedge \phi^{d}\right) \\ &= d\Gamma^{a}_{b} \wedge \phi^{b} - \Gamma^{a}_{b} \wedge \Gamma^{b}_{c} \wedge \phi^{c} \\ &= \left(d\Gamma^{a}_{b} + \Gamma^{a}_{c} \wedge \Gamma^{c}_{b}\right) \wedge \phi^{b} \end{aligned} $$

where we have used the fact that the $D\phi^{a}$ are two-forms. Therefore, from the second structural equation, we have

$$ DD\phi^{a} = {}^{*}R^{a}_{\ b} \wedge \phi^{b} $$

Finally, taking $\phi^{a} = \theta^{a}$, we give the fourth structural equation as

$$ DD\theta^{a} = D\tau^{a} = {}^{*}R^{a}_{\ b} \wedge \theta^{b} $$

or,

$$ d\tau^{a} = {}^{*}R^{a}_{\ b} \wedge \theta^{b} - \Gamma^{a}_{b} \wedge \tau^{b} $$

Remarkably, this is equivalent to the first generalized Bianchi identity given in the preceding section.

1.5 The geometry of distant parallelism

Let us now consider a special situation in which our $n-$dimensional manifold C^∞ is embedded *isometrically* in a flat $n-$dimensional (pseudo-)Euclidean space E^n (with coordinates $v^{\overline{m}}$) spanned by the constant basis $e_{\overline{m}}$ whose dual is denoted by $s^{\overline{n}}$. This embedding allows us to globally cover the manifold C^∞ in the sense that its geometric structure can be parametrized by the Euclidean basis $e_{\overline{m}}$ satisfying

$$\eta_{\overline{m}\overline{n}} = \langle e_{\overline{m}}, e_{\overline{n}} \rangle = diag\left(\pm 1, \pm 1, ..., \pm 1\right)$$

It is important to note that this situation is different from the one presented in Section 1.1, in which case we may refer the structural equations of C^∞ to the locally flat tangent space $T_x(M)$. The results of the latter situation (i.e., the *localized* structural equations) should not always be regarded as globally valid since the tangent space $T_x(M)$, though ubiquitous in the sense that it can be defined everywhere (at any point) in C^∞, cannot cover the whole structure of the curved manifold C^∞ without changing orientation from point to point.

One can construct geometries with special connections that will give rise to what we call *geometries with parallelism.* Among others, the geometry of *distant parallelism* is a famous case. Indeed, A. Einstein adopted this geometry in one of his attempts to geometrize physics, and especially to unify gravity and electromagnetism [5]. In its application to physical situations, the resulting field equations of a unified field theory based on distant parallelism, for instance, are quite remarkable in that the so-called energy-momentum tensor appears to be geometrized via the torsion tensor. We will therefore dedicate this section to a brief presentation of the geometry of distant parallelism in the language of Riemann-Cartan geometry.

In this geometry, it is possible to orient vectors such that their directions remain invariant after being displaced from a point to some distant point in the manifold. This situation is made possible by the vanishing of the curvature tensor, which is given by the integrability condition

$$R^d{}_{abc} = e^d_{\overline{m}} \left(\partial_b \partial_c - \partial_c \partial_b\right) e^{\overline{m}}_a = 0$$

where the connection is now given by

$$\Gamma^c_{ab} = e^c_{\overline{m}} \; \partial_b \, e^{\overline{m}}_a$$

where $e^{\overline{m}}_{a} = \partial_a \xi^{\overline{m}}$ and $e^{a}_{\overline{m}} = \partial_{\overline{m}} x^a$.

However, while the curvature tensor vanishes, one still has the torsion tensor given by

$$\Gamma^a_{[bc]} = \frac{1}{2} e^a_{\overline{m}} \left(\partial_c e^{\overline{m}}_b - \partial_b e^{\overline{m}}_c \right)$$

with the $e^{\overline{m}}_a$ acting as the components of a spin "potential". Thus the torsion can now be considered as the primary geometric object in the manifold C^{∞}_p endowed with distant parallelism.

Also, in general, the Riemann-Christoffel curvature tensor is non-vanishing as

$$B^d{}_{abc} = \hat{\nabla}_c K^d_{ab} - \hat{\nabla}_b K^d_{ac} + K^e_{ab} K^d_{ec} - K^e_{ac} K^d_{eb}$$

Let us now consider some facts. Taking the covariant derivative of the tetrad $e^{\overline{m}}_a$ with respect to the Christoffel symbols alone, we have

$$\hat{\nabla}_b e^{\overline{m}}_a = \partial_b e^{\overline{m}}_a - e^{\overline{m}}_d \Delta^d_{ab} = e^{\overline{m}}_c K^c_{ab}$$

i.e.,

$$K^c_{ab} = e^c_{\overline{m}} \hat{\nabla}_b e^{\overline{m}}_a = - e^{\overline{m}}_a \hat{\nabla}_b e^c_{\overline{m}}$$

In the above sense, the components of the contorsion tensor give the so-called *Ricci rotation coefficients*. Then from

$$\hat{\nabla}_c \hat{\nabla}_b e^{\overline{m}}_a = e^{\overline{m}}_d \left(\hat{\nabla}_c K^d_{ab} + K^e_{ab} K^d_{ec} \right)$$

it is elementary to show that

$$\left(\hat{\nabla}_c \hat{\nabla}_b - \hat{\nabla}_b \hat{\nabla}_c \right) e^{\overline{m}}_a = e^{\overline{m}}_d B^d{}_{abc}$$

Likewise, we have

$$\tilde{\nabla}_b e^{\overline{m}}_a = \partial_b e^{\overline{m}}_a - e^{\overline{m}}_d K^d_{ab} = e^{\overline{m}}_c \Delta^c_{ab}$$
$$\Delta^c_{ab} = e^c_{\overline{m}} \tilde{\nabla}_b e^{\overline{m}}_a = - e^{\overline{m}}_a \tilde{\nabla}_b e^c_{\overline{m}}$$

where now $\tilde{\nabla}$ denotes covariant differentiation with respect to the Ricci rotation coefficients alone. Then from

$$\widetilde{\nabla}_c \widetilde{\nabla}_b e_a^{\overline{m}} = e_d^{\overline{m}} \left(\widetilde{\nabla}_c \Delta^d_{ab} + \Delta^e_{ab} \Delta^d_{ec} \right)$$

we get

$$\left(\widetilde{\nabla}_c \widetilde{\nabla}_b - \widetilde{\nabla}_b \widetilde{\nabla}_c \right) e_a^{\overline{m}} = - e_d^{\overline{m}} \left(B^d{}_{abc} - 2\, \Delta^d_{ae}\, \Gamma^e_{[bc]} - \Delta^e_{ab}\, K^d_{ec} + \Delta^e_{ac}\, K^d_{eb} - K^e_{ab}\, \Delta^d_{ec} + K^e_{ac}\, \Delta^d_{eb} \right)$$

In this situation, one sees, with respect to the coframe basis $\theta^a = e^a_{\overline{m}}\, s^{\overline{m}}$, that

$$d\theta^a = - \Gamma^a_b \wedge \theta^b \equiv T^a$$

i.e.,

$$T^a = \Gamma^a_{[bc]}\, \theta^b \wedge \theta^c$$

Thus the torsion two-forms of this geometry are now given by T^a (instead of τ^a of the preceding section). We then realize that

$$D\theta^a = 0$$

Next, we see that

$$\begin{aligned} d^2\theta^a &= dT^a = - d\Gamma^a_b \wedge \theta^b + \Gamma^a_b \wedge d\theta^b \\ &= - \left(d\Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b \right) \wedge \theta^b \\ &= - {}^*R^a{}_b \wedge \theta^b \end{aligned}$$

But, as always, $d^2\theta^a = 0$, and therefore we have

$${}^*R^a{}_b \wedge \theta^b = 0$$

Note that in this case, ${}^*R^a{}_b \neq 0$ (in general) as

$${}^*R^a{}_{bcd} = 2\, \Gamma^e_{[cd]}\, \Gamma^a_{be}$$

will not vanish in general. We therefore see immediately that

$${}^*R^a{}_{bcd} + {}^*R^a{}_{cdb} + {}^*R^a{}_{dbc} = 0$$

giving the integrability condition

$$\Gamma^e_{[cd]}\, \Gamma^a_{be} + \Gamma^e_{[db]}\, \Gamma^a_{ce} + \Gamma^e_{[bc]}\, \Gamma^a_{de} = 0$$

Meanwhile, the condition

$$dT^{a} = 0$$

gives the integrability condition

$$\partial_d \, \Gamma^a_{[bc]} + \partial_b \, \Gamma^a_{[cd]} + \partial_c \, \Gamma^a_{[db]} = 0$$

Contracting, we find

$$\partial_c \, \Gamma^c_{[ab]} = 0$$

It is a curious fact that the last two relations somehow remind us of the algebraic structure of the components of the electromagnetic field tensor in physics.

Finally, from the contraction of the components $B^{d}{}_{abc}$ of the Riemann-Christoffel curvature tensor (the Ricci tensor), one defines the regular Einstein tensor by

$$\hat{G}_{ab} \equiv B_{ab} - \frac{1}{2} \, g_{ab} \, B \equiv k \, E_{ab}$$

where k is a physical coupling constant and E_{ab} are the components of the so-called energy-momentum tensor. We therefore see that

$$\begin{aligned} E_{ab} = & \frac{1}{k} \left(K^c_{ad} \, K^d_{cb} - \hat{\nabla}_c \, K^c_{ab} + 2 \, \hat{\nabla}_b \, \Gamma^c_{[ac]} - 2 \, K^c_{ab} \, \Gamma^d_{[cd]} \right) \\ & - \frac{1}{2k} \, g_{ab} \left(4 \, g^{cd} \, \hat{\nabla}_c \, \Gamma^e_{[de]} + 2 \, g^{ce} \, \Gamma^d_{[cd]} \, \Gamma^f_{[ef]} + K_{cde} \, K^{ced} \right) \end{aligned}$$

In addition, the following two conditions are satisfied:

$$\begin{aligned} & E_{[ab]} = 0 \\ & \hat{\nabla}_a \, E^{ab} = 0 \end{aligned}$$

We have now seen that, in this approach, the energy-momentum tensor (matter field) is fully geometrized. This way, gravity arises from torsional (spin) interaction (possibly on the microscopic scales) and becomes an *emergent* phenomenon rather than a fundamental one. This seems rather speculative. However, it may have profound consequences.

1.6 Spin frames

A spin frame is described by the anti-symmetric tensor product

$$\Omega^{ik} = \frac{1}{2}\left(\theta^i \otimes \theta^k - \theta^k \otimes \theta^i\right) = \theta^i \wedge \theta^k \equiv \frac{1}{2}\left[\theta^i, \theta^k\right]$$

In general, then, for arbitrary vector field fields A and B, we can form the commutator

$$\left[A, B\right] = A \otimes B - B \otimes A$$

Introducing another vector field C, we have the so-called Jacobi identity

$$\left[A, \left[B, C\right]\right] + \left[B, \left[C, A\right]\right] + \left[C, \left[A, B\right]\right] = 0$$

With respect to the local coordinate basis elements $E_i = \partial_i$ of the tangent space $T_x(M)$, we see that, astonishingly enough, the anti-symmetric product $\left[A, B\right]$ is what defines the Lie (exterior) derivative of B with respect to A:

$$L_A B \equiv \left[A, B\right] = \left(A^i\ \partial_i B^k - B^i\ \partial_i A^k\right) \frac{\partial}{\partial X^k}$$

(Note that $L_A A = \left[A, A\right] = 0$.) The terms in the round brackets are just the components of our Lie derivative which can be used to define a diffeomorphism invariant (i.e., by taking $A^i = \xi^i$ where ξ represents the displacement field in a neighborhood of coordinate points).

Furthermore, for a vector field U and a tensor field T, both arbitrary, we have (in component notation) the following:

$$\begin{aligned} L_U\, T^{ij\ldots s}_{kl\ldots r} &= \partial_m T^{ij\ldots s}_{kl\ldots r}\ U^m + T^{ij\ldots s}_{ml\ldots r}\ \partial_k U^m + T^{ij\ldots s}_{km\ldots r}\ \partial_l U^m + \ldots + T^{ij\ldots s}_{kl\ldots m}\ \partial_r U^m \\ &\quad - T^{mj\ldots s}_{kl\ldots r}\ \partial_m U^i - T^{im\ldots s}_{kl\ldots r}\ \partial_m U^j - \ldots - T^{ij\ldots m}_{kl\ldots r}\ \partial_m U^s \end{aligned}$$

It is not immediately apparent whether these transform as components of a tensor field or not. However, with the help of the torsion tensor and the relation

$$\partial_k U^i = \nabla_k U^i - \Gamma^i_{mk}\ U^m = \nabla_k U^i - \left(\Gamma^i_{km} - 2\ \Gamma^i_{[km]}\right) U^m$$

we can write

$$\begin{aligned} L_U\, T^{ij\ldots s}_{kl\ldots r} &= \nabla_m T^{ij\ldots s}_{kl\ldots r}\ U^m + T^{ij\ldots s}_{ml\ldots r}\ \nabla_k U^m + T^{ij\ldots s}_{km\ldots r}\ \nabla_l U^m + \ldots + T^{ij\ldots s}_{kl\ldots m}\ \nabla_r U^m \\ &\quad - T^{mj\ldots s}_{kl\ldots r}\ \nabla_m U^i - T^{im\ldots s}_{kl\ldots r}\ \nabla_m U^j - \ldots - T^{ij\ldots m}_{kl\ldots r}\ \nabla_m U^s \\ &\quad + 2\ \Gamma^i_{[mp]}\ T^{mj\ldots s}_{kl\ldots r}\ U^p + 2\ \Gamma^j_{[mp]}\ T^{im\ldots s}_{kl\ldots r}\ U^p + \ldots + 2\ \Gamma^s_{[mp]}\ T^{ij\ldots m}_{kl\ldots r}\ U^p \\ &\quad - 2\ \Gamma^m_{[kp]}\ T^{ij\ldots s}_{ml\ldots r}\ U^p - 2\ \Gamma^m_{[lp]}\ T^{ij\ldots s}_{km\ldots r}\ U^p - \ldots - 2\ \Gamma^m_{[rp]}\ T^{ij\ldots s}_{kl\ldots m}\ U^p \end{aligned}$$

Hence, noting that the components of the torsion tensor, namely, $\Gamma^{i}_{[kl]}$, indeed transform as components of a tensor field, it is seen that the $L_U\, T^{ij\ldots s}_{kl\ldots r}$ do transform as components of a tensor field. Apparently, the beautiful property of the Lie derivative (applied to an arbitrary tensor field) is that it is connection-independent even in a curved manifold.

If we now apply the commutator to the frame basis of the base manifold C^{∞} itself, we see that (for simplicity, we again refer to the coordinate basis of the tangent space $T_x(M)$)

$$[\omega_a, \omega_b] = \left(\partial_a X^i\ \partial_i \partial_b X^k - \partial_b X^i\ \partial_i \partial_a X^k\right) \frac{\partial}{\partial X^k}$$

Again, writing the tetrads simply as $e^i_a = \partial_a X^i$, $e^a_i = \partial_i x^a$, we have

$$[\omega_a, \omega_b] = \left(\partial_a e^k_b - \partial_b e^k_a\right) \frac{\partial}{\partial X^k}$$

i.e.,

$$[\omega_a, \omega_b] = -\, 2\, \Gamma^{c}_{[ab]}\ \omega_c$$

Therefore, in the present formalism, the components of the torsion tensor are by themselves proportional to the so-called *structure constants* Ψ^{c}_{ab} of our rotation group:

$$\Psi^{c}_{ab} = -\, 2\, \Gamma^{c}_{[ab]} = -\, e^c_i \left(\partial_a e^i_b - \partial_b e^i_a\right)$$

As before, here the tetrad represents a spin potential.

Also note that

$$\Psi^{d}_{ab}\ \Psi^{e}_{dc} + \Psi^{d}_{bc}\ \Psi^{e}_{da} + \Psi^{d}_{ca}\ \Psi^{e}_{db} = 0$$

We therefore observe that, as a consequence of the present formalism of differential geometry, spin fields (*objects of anholonomicity*) in the manifold C^{∞} are generated *directly* by the torsion tensor.

2. The New Semi-Symmetric Unified Field Theory of the Classical Fields of Gravity and Electromagnetism

In this part, we develop our semi-symmetric unified field theory on the foundation of Riemann-Cartan geometry presented in Sections 1.1-1.6. We shall concentrate on physical events in the four-dimensional space-time manifold S^4 with the usual Lorentzian signature. As we will see, the choice of a semi-symmetric Cartan torsion will lead to a set of physically meaningful field equations from which we will obtain not only the generally covariant Lorentz equation of motion of a charged particle, but also its generalizations.

We are mainly concerned with the dynamical equations governing a cluster of individual particles and their multiple field interactions and also the possibility of defining geometrically and phenomenologically conserved currents in the theory. We will therefore not assume dimensional (i.e., structural) homogeneity with regard to the particles. Classically, a point-like (i.e., structureless) particle which characterizes a particular physical field is only a mere idealization which is not subject, e.g., to any possible dilation when interacting with other particles or fields. Still within the classical context, we relax this condition by assigning a structural configuration to each individual particle. Therefore, the characteristic properties of the individual particles allow us to describe a particle as a field in a physically meaningful sense. In this sense, the particle-field duality is abolished on the phenomenological level as well. In particular, this condition automatically takes into account both the rotational and reflectional symmetries of individual particles which have been developed separately. As such, without having to necessarily resort to particle isotropy, the symmetry group in our theory is a general one, i.e., it includes all rotations about all possible axes and reflections in any plane in the space-time manifold S^4.

The presence of the semi-symmetric torsion causes any local (hyper)surface in the space-time manifold S^4 to be *non-orientable* in general. As a result, the trajectories of individual particles generally depend on the twisted path they trace in S^4. It is important to note that this torsion is the generator of the so-called microspin, e.g., in the simplest case, a spinning particle is simply a point-rotation in the sense of the so-called Cosserat continuum theory [10]. As usual, the semi-symmetric torsion tensor enters the curvature tensor as an integral part via the general (semi-symmetric) connection. This way, all classical physical fields, not just the gravitational field, are intrinsic to the space-time geometry.

2.1 A semi-symmetric connection based on a semi-simple (transitive) rotation group

Let us now work in *four* space-time dimensions (since this number of dimensions is most relevant to physics). For a semi-simple (transitive) rotation group, we can show that

$$[\omega_a, \omega_b] = -\gamma \in_{abcd} \varphi^c \theta^d$$

where $\in_{abcd} = \sqrt{\det(g)}\, \varepsilon_{abcd}$ are the components of the completely anti-symmetric four-dimensional Levi-Civita permutation tensor and φ is a vector field normal to a three-dimensional space (hypersurface) $\sum(t)$ defined as the time section $ct = x^0 = const.$

(where c denotes the speed of light in vacuum) of S^4 with local coordinates z^A. It satisfies $\varphi_a \, \varphi^a = \gamma = \pm 1$ and is given by

$$\varphi_a = \frac{1}{6} \gamma \in_{abcd} \in^{ABC} \lambda_A^b \, \lambda_B^c \, \lambda_C^d$$

where

$$\begin{aligned} &\lambda_A^a \equiv \partial_A x^a \, , \, \lambda_a^A \equiv \partial_a z^A \\ &\lambda_A^b \, \lambda_a^A = \delta_a^b - \gamma \, \varphi_a \, \varphi^b \\ &\lambda_A^a \, \lambda_a^B = \delta_A^B \end{aligned}$$

More specifically,

$$\in_{ABC} \varphi_d = \in_{abcd} \lambda_A^a \, \lambda_B^b \, \lambda_C^c$$

from which we find

$$\in_{abcd} = \in_{ABC} \lambda_a^A \, \lambda_b^B \, \lambda_c^C \, \varphi_d + \Lambda_{abcd}$$

where

$$\Lambda_{abcd} = \gamma \left(\in_{ebcd} \varphi_a + \in_{aecd} \varphi_b + \in_{abed} \varphi_c \right) \varphi^e$$

Noting that $\Lambda_{abcd} \, \varphi^d = 0$, we can define a completely anti-symmetric, three-index, four-dimensional "permutation" tensor by

$$\Phi_{abc} \equiv \in_{abcd} \varphi^d = \gamma \in_{ABC} \lambda_a^A \, \lambda_b^B \, \lambda_c^C$$

Obviously, the hypersurface $\Sigma(t)$ can be thought of as representing the position of a material body at any time t. As such, it acts as a boundary of the so-called world-tube of a family of world-lines covering an arbitrary four-dimensional region in S^4.

Meanwhile, in the most general four-dimensional case, the torsion tensor can be decomposed according to

$$\begin{aligned} &\Gamma^c_{[ab]} = \frac{1}{3} \left(\delta_b^c \, \Gamma^d_{[ad]} - \delta_a^c \, \Gamma^d_{[bd]} \right) + \frac{1}{6} \in^c{}_{abd} \in^d{}_{pqr} g^{qs} \, g^{rt} \, \Gamma^p_{[st]} + g^{cd} \, Q_{dab} \\ &Q_{abc} + Q_{bca} + Q_{cab} = 0 \\ &Q^a{}_{ab} = Q^a{}_{ba} = 0 \end{aligned}$$

In our special case, the torsion tensor becomes completely anti-symmetric (in its three indices) as

$$\Gamma^{c}_{[ab]} = -\frac{1}{2}\,\gamma\, g^{ce} \in_{abed} \varphi^{d}$$

from which we can write

$$\varphi^{a} = -\frac{1}{3} \in^{abcd} \Gamma_{b[cd]}$$

where, as usual, $\Gamma_{b[cd]} = g_{be}\,\Gamma^{e}_{[cd]}$. Therefore, at this point, the full connection is given by (with the Christoffel symbols written explicitly)

$$\Gamma^{c}_{ab} = \frac{1}{2}\, g^{cd}\left(\partial_{b}\, g_{da} - \partial_{d}\, g_{ab} + \partial_{a}\, g_{bd}\right) - \frac{1}{2}\,\gamma \in^{c}{}_{abd}\, \varphi^{d}$$

We shall call this special connection "*semi-symmetric*". This gives the following simple conditions:

$$\begin{aligned}
&\Gamma^{c}_{(ab)} = \Delta^{c}_{ab} = \frac{1}{2}\, g^{cd}\left(\partial_{b}\, g_{da} - \partial_{d}\, g_{ab} + \partial_{a}\, g_{bd}\right)\\
&K^{c}_{ab} = \Gamma^{c}_{[ab]} = -\frac{1}{2}\,\gamma \in^{c}{}_{abd}\, \varphi^{d}\\
&\Gamma^{b}_{[ab]} = 0\\
&\Gamma^{b}_{ab} = \Gamma^{b}_{ba} = \partial_{a}\left(\ln\sqrt{\det(g)}\right)
\end{aligned}$$

Furthermore, we can extract a projective metric tensor ϖ from the torsion (via the structure constants) as follows:

$$\varpi_{ab} = g_{ab} - \gamma\, \varphi_{a}\, \varphi_{b} = 2\, \Gamma^{c}_{[ad]}\, \Gamma^{d}_{[cb]}$$

In three dimensions, the above relation gives the so-called Cartan metric.

Finally, we are especially interested in how the existence of torsion affects a coordinate frame spanned by the basis ω_{a} and its dual θ^{b} in a geometry endowed with distant parallelism. Taking the *four-dimensional curl* of the coframe basis θ^{b}, we see that

$$\begin{aligned}
\left[\nabla, \theta^{a}\right] &= 2\, d\theta^{a} = 2\, T^{a}\\
&= -\gamma \in^{\overline{mnpq}} \left(\partial_{\overline{m}}\, e^{a}_{\overline{n}}\right) \varphi_{\overline{p}}\, e_{\overline{q}}
\end{aligned}$$

where $\nabla = \theta^b \nabla_b = s^{\overline{m}} \partial_{\overline{m}}$ and $\in^{abcd} = \dfrac{1}{\sqrt{\det(g)}} \varepsilon^{abcd}$. From the metricity condition of the tetrad (with respect to the basis of E^n), namely, $\nabla_b e_a^{\overline{m}} = 0$, we have

$$
\begin{aligned}
\partial_b e_a^{\overline{m}} &= \Gamma^c_{ab} e_c^{\overline{m}} \\
\partial^{\overline{n}} e_a^{\overline{m}} &= \eta^{\overline{n}\overline{p}} e^b_{\overline{p}} \partial_b e_a^{\overline{m}} = e_c^{\overline{m}} \Gamma^c_{ab} e^{\overline{n}b}
\end{aligned}
$$

It is also worthwhile to note that from an equivalent metricity condition, namely, $\nabla_a e^b_{\overline{m}} = 0$, one finds

$$\partial_{\overline{n}} e^a_{\overline{m}} = - \Gamma^a_{bc} e^b_{\overline{m}} e^c_{\overline{n}}$$

Thus we find

$$\left[\nabla, \theta^a\right] = - \gamma \in^{bcde} \Gamma^a_{[bc]} \varphi_d \omega_e$$

In other words,

$$T^a = d\theta^a = - \frac{1}{2} \gamma \in^{bcde} \Gamma^a_{[bc]} \varphi_d \omega_e$$

For the frame basis, we have

$$\left[\nabla, \omega_a\right] = - \gamma \in^{bcde} \Gamma_{a[bc]} \varphi_d \omega_e$$

At this point it becomes clear that the presence of torsion in S^4 rotates the frame and coframe bases themselves. The basics presented here constitute the reality of the so-called *spinning frames.*

2.2 Construction of the semi-symmetric field equations

In the preceding section, we have introduced the semi-symmetric connection

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(\partial_b g_{da} - \partial_d g_{ab} + \partial_a g_{bd}\right) - \frac{1}{2} \gamma \in^c{}_{abd} \varphi^d$$

based on the semi-simple rotation group

$$\left[\omega_a, \omega_b\right] = - \gamma \in_{abcd} \varphi^c \theta^d$$

Now we are in a position to construct a classical unified field theory of gravity and electromagnetism based on this connection. We shall then call the resulting field equations semi-symmetric, hence the name semi-symmetric unified field theory. (Often the terms "symmetric" and "asymmetric" refer to the metric rather than the connection.)

Using the results we have given in Section 1.3, we see that the curvature tensor built from our semi-symmetric connection is given by

$$R^{d}_{\ abc} = B^{d}_{\ abc} - \frac{1}{2}\,\gamma\left(\in^{d}_{\ ace}\hat{\nabla}_b\,\varphi^e - \in^{d}_{\ abe}\hat{\nabla}_c\,\varphi^e\right) + \frac{3}{2}\,\gamma\left(g_{eb}\,\delta^{def}_{acg} - g_{ec}\,\delta^{def}_{abg}\right)\varphi_f\,\varphi^g$$

As before, the generalized Ricci tensor is then given by the contraction $R_{ab} = R^{c}_{\ acb}$, i.e.,

$$R_{ab} = B_{ab} - \frac{1}{2}\left(g_{ab} - \gamma\,\varphi_a\,\varphi_b\right) - \frac{1}{2}\,\gamma \in_{ab}^{\ \ cd}\hat{\nabla}_c\,\varphi_d$$

Then we see that its symmetric and anti-symmetric parts are given by

$$R_{(ab)} = B_{ab} - \frac{1}{2}\left(g_{ab} - \gamma\,\varphi_a\,\varphi_b\right)$$

$$R_{[ab]} = -\frac{1}{2}\,\gamma \in_{ab}^{\ \ cd} F_{cd}$$

where

$$F_{ab} = \frac{1}{2}\left(\partial_a\,\varphi_b - \partial_b\,\varphi_a\right)$$

are the components of the intrinsic spin tensor of the first kind in our unified field theory. Note that we have used the fact that $\hat{\nabla}_a\,\varphi_b - \hat{\nabla}_b\,\varphi_a = \partial_a\,\varphi_b - \partial_b\,\varphi_a$.

Note that if

$$\varphi^a = \gamma\,\delta^a_0$$

then the torsion tensor becomes covariantly constant throughout the space-time manifold, i.e.,

$$\nabla_d\,\Gamma^c_{[ab]} = \hat{\nabla}_d\,\Gamma^c_{[ab]} = 0$$

This special case may indeed be anticipated as in the present theory, the two fundamental geometric objects are the metric and torsion tensors.

Otherwise, in general let us define a vector-valued gravoelectromagnetic potential A via

$$\varphi^a = \lambda \; A^a$$

where

$$\lambda = \left(\frac{\gamma}{A_a \; A^a} \right)^{1/2}$$

Letting $\in = \lambda^2 \; \gamma$, we then have

$$R_{ab} = B_{ab} - \frac{1}{2} \left(g_{ab} - \in A_a \; A_b \right) - \frac{1}{2} \, \gamma \in_{ab}^{\;\;cd} \left(\lambda \; \overline{F}_{cd} + H_{cd} \right)$$

where

$$\overline{F}_{ab} = \frac{1}{2} \left(\partial_a \, A_b - \partial_b \, A_a \right)$$
$$H_{ab} = - \frac{1}{2} \left(A_a \, \partial_b \, \lambda - A_b \; \partial_a \, \lambda \right)$$

We may call $\overline{F}_{ab}$ the components of the intrinsic spin tensor of the second kind. The components of the anti-symmetric field equation then take the form

$$R_{[ab]} = - \frac{1}{2} \, \gamma \in_{ab}^{\;\;cd} \left(\lambda \; \overline{F}_{cd} + H_{cd} \right)$$

Using the fact that

$$\partial_a \, F_{bc} + \partial_b \, F_{ca} + \partial_c \, F_{ab} = 0$$

we obtain

$$\hat{\nabla}_a \, R^{[ab]} = 0$$

The dual of the anti-symmetric part of the generalized Ricci tensor is then given by

$$\widetilde{R}_{[ab]} = \frac{1}{2} \in_{abcd} \; R^{[cd]} = - \frac{1}{2} \left(\partial_a \, \varphi_b - \partial_b \, \varphi_a \right)$$

i.e.,

$$\widetilde{R}_{[ab]} = -\left(\lambda\ \overline{F}_{ab} + H_{ab}\right)$$

We therefore see that

$$\partial_a\ \widetilde{R}_{[bc]} + \partial_b\ \widetilde{R}_{[ca]} + \partial_c\ \widetilde{R}_{[ab]} = 0$$

At this point, the components of the intrinsic spin tensor take the following form:

$$\overline{F}_{ab} = -\frac{1}{2\lambda}\left(\in_{abcd}\ R^{[cd]} + 2\ H_{ab}\right)$$

The generalized Einstein field equation is then given by

$$G_{ab} = R_{ab} - \frac{1}{2}\ g_{ab}\ R = k\ T_{ab}$$

where k is a coupling constant, $R = R^a_{\ a} = B - \frac{3}{2}$ (in our geometrized units) is the generalized Ricci scalar, and T_{ab} are the components of the energy-momentum tensor of the coupled matter and spin fields. Taking the covariant divergence of the generalized Einstein tensor with the help of the relations

$$\nabla_a R^{ab} = \hat{\nabla}_a R^{ab} - \Gamma^b_{[ac]}\ R^{[ac]}$$
$$\nabla_a R = \partial_a R = \partial_a B$$
$$F_{ab}\ \varphi^b = -\frac{1}{2}\ \varphi^b\ \hat{\nabla}_b \varphi_a$$

we obtain

$$\nabla_a G^{ab} = \hat{\nabla}_a G^{ab} - \gamma\ F^{\ b}_{a}\ \varphi^a$$

On the other hand, using the integrability condition

$$\in^{abcd}\ \hat{\nabla}_b\ \hat{\nabla}_c\ \varphi_d = \in^{abcd}\ \partial_b\ \partial_c\ \varphi_d = 0$$

we have

$$\hat{\nabla}_a R^{ab} = \hat{\nabla}_a B^{ab} - \frac{1}{2}\ \gamma\ \hat{\nabla}_a\left(\varphi^a\ \varphi^b\right)$$

Therefore

$$\nabla_a G^{ab} = \hat{\nabla}_a \hat{G}^{ab} + \frac{1}{2}\gamma\left(\varphi^b \hat{\nabla}_a \varphi^a + \varphi^a \hat{\nabla}_a \varphi^b\right) - \gamma F^b{}_a \varphi^a$$

where, as before, $\hat{G}_{ab} = B_{ab} - \frac{1}{2} g_{ab} B$. But as $\hat{\nabla}_a \hat{G}^{ab} = 0$, we are left with

$$\nabla_a G^{ab} = \frac{1}{2}\gamma\left(\varphi^b \hat{\nabla}_a \varphi^a + \varphi^a \hat{\nabla}_a \varphi^b\right) - \gamma F^b{}_a \varphi^a$$

We may notice that in general the above divergence does not vanish.

We shall now seek a possible formal correspondence between our present theory and both general relativistic gravitomagnetism and Maxwellian electrodynamics. We shall first assume that particles do not necessarily have point-like structure. Now let the rest (inertial) mass of a particle and the speed of light in vacuum (again) be denoted by m and c, respectively. Also, let ϕ represent the scalar gravoelectromagnetic potential and let g_a and B_a denote the components of the gravitational spin potential and the electromagnetic four-potential, respectively. We now make the following ansatz:

$$\lambda = const. = -\frac{\bar{g}}{2mc^2}$$
$$A_a = \partial_a \phi + v\, g_{0a} = \partial_a \phi + g_a + B_a$$

where v is a constant and

$$\bar{g} = (1 + m)\, n + 2\,(1 + s_\pi)\, e$$

is the *generalized gravoelectromagnetic charge*. Here n is the structure constant (i.e., a volumetric number) which is different from zero for structured particles, s_π is the spin constant, and e is the electric charge (or, more generally, the electromagnetic charge).

Now let the gravitational vorticity tensor be given by

$$\omega_{ab} = \frac{1}{2}\left(\partial_a g_b - \partial_b g_a\right)$$

which vanishes in spherically symmetric (i.e., centrally symmetric) situations. Next, the electromagnetic field tensor is given as usual by

$$f_{ab} = \partial_b B_a - \partial_a B_b$$

The components of the intrinsic spin tensor can now be written as

$$\overline{F}_{ab} = \omega_{ab} - \frac{1}{2} f_{ab}$$

As a further consequence, we have $H_{ab} = 0$ and therefore

$$\overline{F}_{ab} = -\frac{1}{2\lambda} \in_{abcd} R^{[cd]} = \frac{mc^2}{\overline{g}} \in_{ab}^{\ \ cd} R_{[cd]}$$

The electromagnetic field tensor in our unified field theory is therefore given by

$$f_{ab} = -2\left(\frac{mc^2}{\overline{g}} \in_{ab}^{\ \ cd} R_{[cd]} - \omega_{ab}\right)$$

Here we see that when the gravitational spin is present, the electromagnetic field does interact with the gravitational field. Otherwise, in the presence of a centrally symmetric gravitational field we have

$$f_{ab} = -\frac{2mc^2}{\overline{g}} \in_{ab}^{\ \ cd} R_{[cd]}$$

and there is no physical interaction between gravity and electromagnetism.

2.3 Equations of motion

Now let us take the unit vector field φ to represent the unit velocity vector field, i.e.,

$$\varphi^a = u^a = \frac{dx^a}{ds}$$

where ds is the (infinitesimal) world-line satisfying

$$1 = g_{ab} \frac{dx^a}{ds} \frac{dx^b}{ds}$$

This selection defines a general material object in our unified field theory as a hypersurface $\sum(t)$ whose world-velocity u is normal to it. Indeed, we will soon see some profound physical consequences.

Invoking this condition, we immediately obtain the following equation of motion:

$$\nabla_a G^{ab} = \frac{1}{2}\gamma\left(u^b\, \nabla_a u^a + \frac{Du^b}{Ds}\right) - \gamma\, F^b{}_a\, \varphi^a$$

where we have used the following relations:

$$\Gamma^c_{(ab)} = \Delta^c_{ab}$$

$$\Gamma^c_{[ab]} = -\frac{1}{2}\gamma \in^c{}_{abd}\, u^d$$

$$\frac{Du^a}{Ds} = u^b\, \nabla_b u^a = \frac{du^a}{ds} + \Gamma^a_{(bc)}\, u^b\, u^c = \frac{du^a}{ds} + \Delta^a_{bc}\, u^b\, u^c = u^b\, \hat{\nabla}_b u^a$$

What happens now if we insist on guaranteeing the conservation of matter and spin? Letting

$$\nabla_a G^{ab} = 0$$

and inserting the value of λ, we obtain the equation of motion

$$\frac{Du^a}{Ds} = -\frac{\overline{g}}{mc^2}\,\overline{F}^a_b\, u^b - u^a\, \nabla_b u^b$$

i.e., the *generalized* Lorentz equation of motion

$$\frac{Du^a}{Ds} = \frac{\overline{g}}{2mc^2}\left(f^a_b - 2\,\omega^a{}_b\right)u^b - u^a\, \nabla_b u^b$$

From the above equation of motion we may derive special equations of motion such as those in the following cases:

1. For an electrically charged, non-spinning, incompressible, structureless (point-like) particle moving in a static, centrally symmetric gravitational field, we have $m \neq 0,\ e \neq 0,\ s_\pi = 0,\ n = 0,\ \nabla_a u^a = 0,\ f_{ab} \neq 0,\ \omega_{ab} = 0$. Therefore its equation of motion is given by

$$\frac{Du^a}{Ds} = \frac{e}{mc^2}\, f^a_b\, u^b$$

   which is just the standard, relativistically covariant Lorentz equation of motion.

2. For an electrically charged, spinning, incompressible, structureless particle moving in a non-static, spinning gravitational field, we have

$m \neq 0, e \neq 0, s_\pi \neq 0, n = 0, \nabla_a u^a = 0, f_{ab} \neq 0, \omega_{ab} \neq 0$. Therefore its equation of motion is given by

$$\frac{Du^a}{Ds} = \frac{(1 + s_\pi)}{mc^2} e \left(f^a_{\ b} - 2\, \omega^a{}_b \right) u^b$$

3. For a neutral, non-spinning, incompressible, structureless particle moving in a static, centrally symmetric gravitational field, we have $m \neq 0, e = 0, s_\pi = 0, n = 0, \nabla_a u^a = 0, f_{ab} = 0, \omega_{ab} = 0$. Therefore its equation of motion is given by the usual geodesic equation of motion

$$\frac{Du^a}{Ds} = 0$$

   In general, this result does not hold for arbitrary incompressible bodies with structure.

4. For a neutral, static, non-spinning, compressible body moving in a static, non-spinning, centrally symmetric gravitational field, we have $m \neq 0, e = 0, s_\pi = 0, n \neq 0, \nabla_a u^a \neq 0, f_{ab} = 0, \omega_{ab} = 0$. Therefore its equation of motion is given by

$$\frac{Du^a}{Ds} = -\, u^a\, \nabla_b u^b$$

   which holds for non-Newtonian fluids in classical hydrodynamics.

5. For an electrically charged, non-spinning, compressible body moving in a static, non-spinning, centrally symmetric gravitational field, we have $m \neq 0, e \neq 0, s_\pi = 0, n \neq 0, \nabla_a u^a \neq 0, f_{ab} \neq 0, \omega_{ab} = 0$. Therefore its equation of motion is given by

$$\frac{Du^a}{Ds} = \frac{n(1 + m)}{mc^2} e\, f^a_b\, u^b - u^a\, \nabla_b u^b$$

   which holds for a variety of classical Maxwellian fluids.

6. For a neutral, spinning, compressible body moving in a non-static, spinning gravitational field, the parametric (structural) condition is given by $m \neq 0, e = 0, s_\pi \neq 0, n \neq 0, \nabla_a u^a \neq 0, f_{ab} = 0, \omega_{ab} \neq 0$. Therefore its equation of motion is given by

$$\frac{Du^a}{Ds} = -\frac{n(1+m)}{mc^2}\,\omega^a_{\ b}\, u^b - u^a\, \nabla_b u^b$$

Note that the exact equation of motion for massless, neutral particles cannot be directly extracted from the general form of our equation of motion.

We now proceed to give the most general form of the equation of motion in our unified field theory. Using the general identity (see Section 1.3)

$$\nabla_a G^{ab} = 2\, g^{ab}\, \Gamma^c_{[da]}\, R^d_{\ c} + \Gamma^a_{[cd]}\, R^{cdb}_{\ \ \ \ a}$$

we see that

$$\nabla_a G^{ab} = \gamma \left(\in^b_{\ cda} R^{[cd]} + \frac{1}{2} \in_{cdea} R^{cdeb} \right) u^a$$

After some algebra, we can show that the above relation can also be written in the form

$$\frac{Du^a}{Ds} = - \in^a_{\ bcd} R^{[bc]}\, u^d$$

Note that the above general equation of motion is true whether the covariant divergence of the generalized Einstein tensor vanishes or not. Otherwise, let $\Phi^a = \nabla_b G^{ba}$ represent the components of the non-conservative vector of the coupled matter and spin fields. Our equation of motion can then be written alternatively as

$$\frac{Du^a}{Ds} = \frac{1}{2} \in_{bcde} R^{bcda}\, u^e - \gamma\, \Phi^a$$

Let us once again consider the conservative case, in which $\Phi^a = 0$. We now have the relation

$$\frac{1}{2} \in_{bcde} R^{bcda}\, u^e = -\frac{2\bar{g}}{mc^2}\, \bar{F}^a_{\ b}\, u^b - u^a\, \nabla_b u^b$$

i.e.,

$$\frac{1}{2}\left(\in_{cdhb} R^{cdha} + \frac{4\bar{g}}{mc^2}\, \bar{F}^a_{\ b} \right) u^b = - u^a\, \nabla_b u^b$$

For a structureless spinning particle, we are left with

$$\left(\in_{cdhb} R^{cdha} + \frac{4(1+s_\pi)}{mc^2} e \, \overline{F}^{a}_{\ b} \right) u^b = 0$$

for which the general solution may read

$$\overline{F}_{ab} = e \frac{mc^2}{4(1+s_\pi)} \left(\in_{acde} R_b^{\ cde} - \in_{bcde} R_a^{\ cde} \right) + S_{ab}$$

where $S_{ab} \neq 0$ are the components of a generally asymmetric tensor satisfying

$$S_{ab} \, u^b = -e \frac{mc^2}{4(1+s_\pi)} \in_{acde} R_b^{\ cde} \, u^b$$

In the case of a centrally symmetric gravitational field, this condition should again allow us to determine the electromagnetic field tensor from the curvature tensor alone.

Now, with the help of the decomposition

$$\begin{aligned} R^d_{\ abc} = C^d_{\ abc} &+ \frac{1}{2} \left(\delta^d_b \, R_{ac} + g_{ac} \, R^d_{\ b} - \delta^d_c \, R_{ab} - g_{ab} \, R^d_{\ c} \right) \\ &+ \frac{1}{6} \left(\delta^d_c \, g_{ab} - \delta^d_b \, g_{ac} \right) R \end{aligned}$$

we obtain the relation

$$\in_{bcde} R^{bcda} = \in_{bcde} \left(C^{bcda} + \frac{1}{2} \left(g^{ac} \, R^{[bd]} - g^{ab} \, R^{[cd]} \right) \right)$$

However, it can be shown that the last two terms in the above relation cancel each other, since

$$\in_{bcde} g^{ac} \, R^{[bd]} = \in_{bcde} g^{ab} \, R^{[cd]} = -\gamma \, g^{ac} \left(\partial_e u_c - \partial_c u_e \right)$$

therefore we are left with the simple relation

$$\in_{bcde} R^{bcda} = \in_{bcde} C^{bcda}$$

If the space-time under consideration is conformally flat (i.e., $C^d_{\ abc} = 0$), we obtain the following integrability condition for the curvature tensor:

$$\in_{bcde} R^{bcda} = 0$$

It is easy to show that this is generally true if the components of the curvature tensor are of the form

$$R_{abcd} = \frac{1}{12}\left(g_{ac}\; g_{bd} - g_{ad}\; g_{bc}\right) B + P_{abcd}$$

where

$$P_{abcd} = \varepsilon\left(g_{ac}\; g_{bd} - g_{ad}\; g_{bc}\right) \overline{F}_{rs}\; \overline{F}^{rs}$$

with ε being a constant of proportionality. In this case, the generalized Ricci tensor is completely symmetric, i.e.,

$$R_{(ab)} = \frac{1}{4}\, g_{ab}\left(B + 12\varepsilon\, \overline{F}_{rs}\; \overline{F}^{rs}\right)$$
$$R_{[ab]} = 0$$

We also have

$$R = B + 12\varepsilon\, \overline{F}_{ab}\; \overline{F}^{ab}$$

such that the variation $\delta S = 0$ of the action integral

$$S = \int\!\!\iiint \sqrt{\det(g)}\; R\, d^4x = \int\!\!\iiint \sqrt{\det(g)}\left(B + 12\varepsilon\, \overline{F}_{ab}\; \overline{F}^{ab}\right) d^4x$$

where $dV = \sqrt{\det(g)}\; dx^0\, dx^1\, dx^2\, dx^3 = \sqrt{\det(g)}\; d^4x$ defines the elementary four-dimensional volume, gives us a set of generalized Einstein-Maxwell equations. Note that in this special situation, the expression for the curvature scalar is true irrespective of whether the Ricci scalar B is constant or not. Furthermore, this gives a generalized Einstein space endowed with a generally non-vanishing spin density. Electromagnetism, in this case, appears to be inseparable from the gravitational vorticity and therefore becomes an emergent phenomenon. Also, the motion then becomes purely geodesic:

$$\frac{du^a}{ds} + \Delta^a_{bc}\; u^b\; u^c = 0$$
$$\overline{F}_{ab}\; u^b = 0$$

2.4 The conserved gravoelectromagnetic currents of the theory

Interestingly, we can obtain more than one type of conserved gravoelectromagnetic current from the intrinsic spin tensor of the present theory.

We have seen in Section 2.2 that the intrinsic spin tensor in the present theory is given by

$$\overline{F}_{ab} = \frac{mc^2}{\overline{g}} \in_{ab}{}^{cd} R_{[cd]}$$

We may note that

$$\hat{j}^a \equiv \hat{\nabla}_b \overline{F}^{ba} = 0$$

which is a covariant "source-free" condition in its own right.

Now, we shall be particularly interested in obtaining the conservation law for the gravoelectromagnetic current in the most general sense. Define the absolute (i.e., global) gravoelectromagnetic current via the total covariant derivative as follows:

$$j^a \equiv \nabla_b \overline{F}^{ba} = \frac{mc^2}{\overline{g}} \in^{abcd} \nabla_d R_{bc}$$

Now, with the help of the relation

$$\nabla_c \overline{F}_{ab} + \nabla_a \overline{F}_{bc} + \nabla_b \overline{F}_{ca} = -2\left(\Gamma^d_{[ab]} \overline{F}_{cd} + \Gamma^d_{[bc]} \overline{F}_{ad} + \Gamma^d_{[ca]} \overline{F}_{bd}\right)$$

we see that

$$j^a = -\frac{6mc^2}{\overline{g}} g^{ce} \delta^{ab}_{cd} R_{[be]} u^d$$

Simplifying, we have

$$j^a = \frac{6mc^2}{\overline{g}} R^{[ab]} u_b$$

At this moment, we have nothing definitive to say about gravoelectromagnetic charge confinement. We cannot therefore speak of a globally admissible gravoelectromagnetic current density yet. However, we can show that our current is indeed conserved. As a start, it is straightforward to see that we have the relative conservation law

$$\hat{\nabla}_a j^a = 0$$

However, this is not the most desired conservation law as we are looking for the most generally covariant one.

Now, with the help of the relations

$$\in^{abcd} \nabla_c F_{ab} = - \in^{abcd} \left(\Gamma^e_{[ac]} F_{eb} + \Gamma^e_{[bc]} F_{ae} \right)$$

$$\Gamma^a_{[bc]} = - \frac{1}{2} \gamma \in^a{}_{bcd} u^d$$

we obtain

$$\nabla_a R^{[ab]} = - 2 F^{ab} u_a$$

Therefore

$$u_b \nabla_a R^{[ab]} = 0$$

Using this result together with the fact that

$$R^{[ab]} \nabla_a u_b = - \frac{1}{2} \gamma \in^{abcd} F_{ab} F_{cd} = 0$$

we see that

$$\nabla_a j^a = \frac{6mc^2}{\bar{g}} \left(u_b \nabla_a R^{[ab]} + R^{[ab]} \nabla_a u_b \right) = 0$$

i.e., our gravoelectromagnetic current is conserved in a fully covariant manner.

Let us now consider a region in our space-time manifold in which the gravoelectromagnetic current vanishes. We have, from the boundary condition $j^a = 0$, the governing equation

$$R_{[ab]} u^b = 0$$

which is equivalent to the following integrability condition:

$$\in^{abcd} u_a \left(\partial_c u_d - \partial_d u_c \right) = 0$$

In three dimensions, if in general $curl\, u \neq 0$, this gives the familiar integrability condition

$$u \cdot curl\, u = 0$$

where the dot represents three-dimensional scalar product.

We are now in a position to define the phenomenological gravoelectromagnetic current density which shall finally allow us to define gravoelectromagnetic charge confinement.

However, in order to avoid having extraneous sources, we do not in general expect such confinement to hold globally. From our present perspective, what we need is a relative (i.e., local) charge confinement which can be expressed solely in geometric terms.

Therefore we first define the spin tensor density (of weight $+2$) as

$$\bar{f}^{ab} \equiv \det(g)\,\bar{F}^{ab} = \frac{mc^2}{\bar{g}}\,\sqrt{\det(g)}\;\varepsilon^{abcd}\;R_{[cd]}$$

The phenomenological (i.e., relative) gravoelectromagnetic current density is given here by

$$\bar{j}^a = \partial_b\,\bar{f}^{ab} = \frac{mc^2}{\bar{g}}\left(\partial_b\sqrt{\det(g)}\right)\varepsilon^{abcd}\;R_{[cd]}$$

i.e.,

$$\bar{j}^a = \frac{mc^2}{2\bar{g}}\,\varepsilon^{abcd}\;g^{rs}\,\left(\partial_b\,g_{rs}\right)R_{[cd]}$$

Meanwhile, using the identity

$$\partial_a\,g^{bc} = -\,g^{br}\;g^{cs}\;\partial_a\,g_{rs}$$

we see that

$$\left(\partial_a\,g^{rs}\right)\left(\partial_b\,g_{rs}\right) = \left(\partial_a\,g_{rs}\right)\left(\partial_b\,g^{rs}\right)$$

Using this result and imposing continuity on the metric tensor, we finally see that

$$\partial_a\,\bar{j}^a = \frac{mc^2}{2\bar{g}}\,\varepsilon^{abcd}\left(\frac{1}{2}\,g^{rs}\;g^{pq}\;\left(\partial_a\,g_{rs}\right)\left(\partial_b\,g_{pq}\right) - \left(\partial_a\,g^{rs}\right)\left(\partial_b\,g_{rs}\right)\right)R_{[cd]} = 0$$

which is the desired local conservation law. In addition, it is easy to show that

$$\hat{\nabla}_a\,\bar{j}^a = 0$$

Unlike the geometric current represented by j^a, the phenomenological current density given by $\bar{j}^a$ corresponds directly to the hydrodynamical analogue of a gravoelectromagnetic current density if we set

$$\bar{j}^a = \det(g)\;\rho\;u^a$$

which defines charge confinement in our gravoelectrodynamics. Combining this relation with the previously given equivalent expression for j^a, we obtain

$$\rho = \frac{mc^2}{2\bar{g}} \in^{abcd} u_a \, g^{rs} \left(\partial_b \, g_{rs}\right) R_{[cd]}$$

i.e.,

$$\rho = \frac{mc^2}{\bar{g}} \in^{abcd} u_a \, \Gamma^h_{hb} \, R_{[cd]}$$

for the gravoelectromagnetic charge density. Note that this is a pseudo-scalar.

At this point, it becomes clear that the gravoelectromagnetic charge density is generated by the properties of the curved space-time itself, i.e., the non-unimodular character of the space-time geometry, for which $\sqrt{\det(g)} \neq 1$ and $\Gamma^h_{hb} \neq 0$, and the torsion (intrinsic spin) of space-time which in general causes material points (whose characteristics are given by $\bar{g}$) to rotate on their own axes such that in a finite region in the space-time manifold, an "individual" energy density emerges. Therefore, in general, a material body is simply a collection of individual material points confined to interact gravoelectrodynamically with each other in a finite region in our curved space-time. More particularly, this can happen in the absence of either the electromagnetic field or the gravitational vorticity, but not in the absence of both fields. To put it more simply, it requires both local curvature and torsion to generate a material body out of an energy field.

3. Final Remarks

At this point, we may note that we have not considered the conditions for the balance of spin (intrinsic angular momentum) in detail. This may be done, in a straightforward manner, by simply expressing the anti-symmetric part of the generalized Ricci tensor in terms of the so-called spin density tensor as well as the couple stress tensor. This can then be used to develop a system of equations governing the balance of energy-momentum in our theory. Therefore, we also need to obtain a formal representation for the energy-momentum tensor in terms of the four-momentum vector. This way, we obtain a set of constitutive equations which characterize the theory.

This work has simply been founded on the feeling that it *could* be physically correct as a unified description of physical phenomena due to its manifest simplicity. Perhaps there remains nothing more beyond the simple appreciation of that possibility. It is valid for a large class of particles and (space-time) continua in which the coordinate points themselves are allowed to rotate and translate. Since the particles are directly related to the coordinate points, they are but intrinsic objects in the space-time manifold, just as the fields are.

It remains, therefore, to consider a few physically meaningful circumstances in greater detail for the purpose of finding particular solutions to the semi-symmetric field equations of our theory.

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