correct answers because the same, tensorially incorrect, designation is applied to the flux density vectors **B**.

Interpreting the circuit for the machine with the field on the rotor, the field-coil axes fixed to the field structure oscillate with it. Incremental currents in this direct axis are coupled to the increments in armature free axes through the mutual inductance, shown on the circuit. The field increments Δi^{ds} are then interacting with absolute changes δi^{f} and δi^{b} on the armature.

The incremental quantities shown on the hunting circuit can be used with the values of flux and current obtained from the steady-state circuit derived by the same technique,(29) to give all of the components of positive and negative damping and synchronising torques at any hunting frequency.

CHAPTER VII

Circuit Models of Field Equations

7.1 Introduction

In Chapter III the concept of a tensor was introduced from a geometrical viewpoint. This was defined as the aggregate of a set of components, the whole making up the tensor, which transforms (singular) according to the very simple laws given in Section 3.2. It was seen that the components are all acting at a point in a geometrical or configuration space. In order to speak of the same tensor or vector at a different point in space one must resort to parallel displacement, which involves the metric tensor and covariant differentiation as shown in Appendix IV. In applying the laws and concepts of tensor analysis to electrical machines Kron adopted a radically different approach, based on Synge's 'Geometry of Dynamics'.(48) This involves a dynamical metric $L_{\alpha\beta}$ which gives invariant electrical or mechanical energy under transformation. There are therefore two different types of metric tensor, the purely geometrical, giving invariant distance, parallel displacement and so on, and the mechanical metric, derived from this, giving invariant energy and power.

In the application of tensors to field problems and to the equivalent circuits satisfying field equations it is necessary to revert to the geometrical metric. The equivalent network is then derived in general stationary curvilinear (usually orthogonal) co-ordinates, in which form the choice of co-ordinates does not affect the physical description of the field. The main advantage of Kron's circuit models is that quantities on the network represent line, surface and volume integrals of the field components, so that the whole space can be considered to be filled. Waves then propagate across the skeleton structure throughout the given region. The concept of integration is an essential part of the tensor model and Stokes' theorem, Ampere's law, Green's theorem, etc., are satisfied at every point.(49)

Before examining Kron's field circuit-models it is well to remember that similar circuit analogues have been in use for decades. Reference 50 lists about eleven hundred publications on applications of equivalent networks, electrolytic tanks, resistance paper, etc., to various types of fields. A typical node of an electrical network is shown in figure 7.1. This satisfies the difference equations(51)

$$i_1 + i_2 + i_3 + i_4 = 0$$

$$\frac{V_0 - V_1}{R} + \frac{V_0 - V_2}{R} + \frac{V_0 - V_3}{R} + \frac{V_0 - V_4}{R} = 0 \qquad (7.1)$$

or

$$V_0 = \frac{V_1 + V_2 + V_3 + V_4}{4}$$

which may be compared with the difference form of Laplace's equation in two dimensions

$$\nabla^2\phi = 0$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0}{k^2} = 0 \qquad (7.2)$$

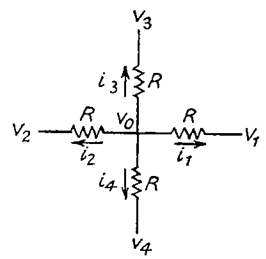

FIG. 7.1. Electrical network analogue (one node).

This analogue can therefore be used to give solutions of Laplace's equation, when the appropriate boundary conditions are applied. The equations and equivalent network can readily be extended to three dimensions and expressed in polar, cylindrical and other co-ordinates. In such analogues the dependent variables are read from the network branches and no concepts of line, surface or volume integration are explicitly used.

An equation of a more general type is that for heat conduction in an exothermic medium,[52]

$$D\nabla^2\theta - c\rho\frac{\partial\theta}{\partial t} = f(\theta) \qquad (7.3)$$

where D is a diffusion coefficient, θ is the temperature, c is the specific heat and ρ is the density of the material. In this case the transient nature of the equation leads to the addition of shunt capacitors on the network. The non-linear stored-energy function must be generated separately and injected at the proper place and time on the network. Figure 7.2 shows the type of analogue circuit required, with the necessary scanning and injection equipment.

Kron's models thus differ from conventional analogue networks in two respects, namely (*a*) the whole space is filled, by integration, and (*b*) the topology of the network does not change with transformation of the

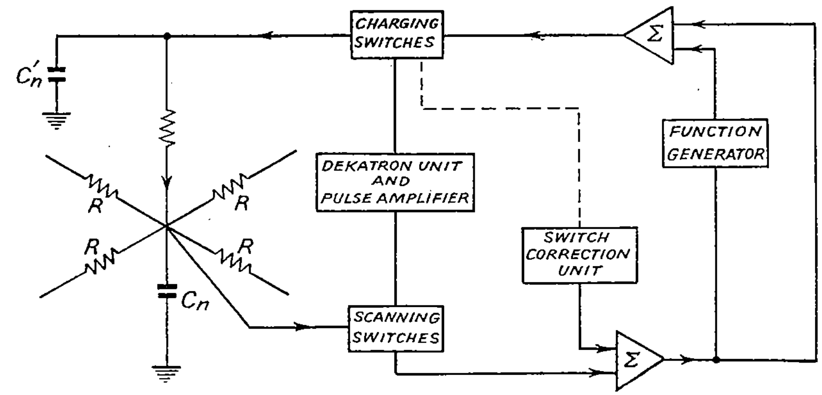

FIG. 7.2. Flow diagram of network, switches and function generation.

co-ordinate system, in other words, the network derived from the invariant form of the field equations is itself invariant.

7.2 Tensor densities

In setting up circuit models for field equations, two points of view must be combined. The field equations describe conditions at a point in space. In order to relate these point-conditions to circuit parameters, line, surface and volume integration of the field quantities must be used. Also, physical quantities at a point imply the use of densities, and tensor-densities must be introduced. Eddington[53] has described the situation thus . . . "Two different kinds of quantities are used in physics, intensity, or condition at a point expressed by tensors; and something, so much per unit, expressed by tensor density."

To develop the idea of tensor density, the first step is to define the elements of length, area and volume. All of these can be expressed in terms of the metric tensor

$$g_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \qquad (7.4)$$

The displacement of a point, or element of length, derived in Section 3.2 is given by

$$ds^2 = g_{\alpha\beta}\, du^\alpha du^\beta \qquad (7.5)$$

Using the system of unitary vectors $\mathbf{a}_\alpha$ as in Chapter III the elements of area and volume are derived as follows.

The element of area bounded by the co-ordinate curves u^2u^3

$$dS_1 = |ds_2 \times ds_3| \\ = |\mathbf{a}_2 \times \mathbf{a}_3| du^2 du^3 \\ = [\sqrt{(\mathbf{a}_2 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}] du^2 du^3 \tag{7.6}$$

and since

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

$$dS_1 = [\sqrt{(\mathbf{a}_2 \cdot \mathbf{a}_2)(\mathbf{a}_3 \cdot \mathbf{a}_3) - (\mathbf{a}_2 \cdot \mathbf{a}_3)(\mathbf{a}_3 \cdot \mathbf{a}_2)}] du^2 du^3 \\ = [\sqrt{g_{22} g_{33} - g_{23}^2}] du^2 du^3 = \sqrt{\bar{g}}\, du^2 du^3 \tag{7.7}$$

where

$$\bar{g} = \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix}$$

with similar expressions for dS_2 and dS_3

The volume element

$$d\tau = ds_1 \cdot ds_2 \times ds_3 = (\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3) du^1 du^2 du^3 \tag{7.8}$$

As in the previous case the right-hand side can be expanded and expressed in terms of scalar products giving

$$d\tau = \sqrt{g}\, du^1 du^2 du^3 \tag{7.9}$$

where

$$g = |g_{\alpha\beta}|$$

In this form the elements of length, area and volume are invariant. In general co-ordinates, volume is given by

$$V = \int\int\int \sqrt{g}\, du^1 du^2 du^3 \tag{7.10}$$

The mass of a medium of density ρ is therefore

$$m = \int\int\int \rho \sqrt{g}\, du^1 du^2 du^3 \tag{7.11}$$

This association with volume integration is the reason for the name 'tensor density'. It is usual to associate the term $\sqrt{g}$ with the vector or tensor function. This is then the quantity to be integrated with respect to the elements du^1, du^2 and du^3. Thus $\rho\sqrt{g}$ is called a scalar density. A tensor density is a quantity which transforms according to the tensor laws but which, in addition, is multiplied by $\sqrt{g}$. For example, if

$$A^{\alpha\beta\prime} = \sqrt{g} A^{\alpha\beta} = \sqrt{g} A^{kn} C_k^\alpha C_n^\beta \tag{7.12}$$

then $A^{\alpha\beta\prime}$ is a tensor density.

Now consider, following Eddington, the integration of a tensor over a given area. The integral

$$\int A^{\alpha\beta} \sqrt{g}\, dS$$

implies the summation of tensors at different points, and these have different coefficients of transformation. If the area becomes infinitesimally small the tensors approach a single point and the law of transformation of the whole integral approaches that of a tensor. Using tensor densities one can therefore integrate over infinitesimal areas. However, if the quantity under the integral sign is an absolute invariant (involving the given vector or tensor field), then each element of area or volume contributes to the integral. All elements will have the same coefficient of transformation whatever the vector location, and integration can be carried out over finite areas. For this reason, in tensor field theory, the integral form of Ampere's law and Faraday's law and the theorems of Gauss, Green and Stokes are written in invariant form.

The next consideration is therefore the invariant form of the vector quantities, gradient, divergence and curl.

7.3 Field equations

In Section 3.2 a vector $\mathbf{F}$ was expressed in physical components with respect to unit tangent direction vectors $\mathbf{i}$ and also with respect to the unitary tangent vectors $\mathbf{a}$.

In Cartesian co-ordinates $y^1 y^2 y^3 (\equiv x,y,z)$

Gradient,
$$\nabla\phi = \frac{\partial\phi}{\partial y^k} \mathbf{i}^k \qquad (k,1,2,3) \tag{7.13}$$

Divergence,
$$\nabla \cdot F = \frac{\partial F^k}{\partial y^k} \tag{7.14}$$

Curl,
$$\nabla \times F = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \dfrac{\partial}{\partial y^1} & \dfrac{\partial}{\partial y^2} & \dfrac{\partial}{\partial y^3} \\ F^1 & F^2 & F^3 \end{vmatrix} \tag{7.15}$$

Laplacian
$$\nabla^2\phi = \frac{\partial^2\phi}{(\partial y^1)^2} + \frac{\partial^2\phi}{(\partial y^2)^2} + \frac{\partial^2\phi}{(\partial y^3)^2} \tag{7.16}$$

In curvilinear co-ordinates the invariant form of these expressions can be obtained, using covariant or contravariant components, by writing the equivalent expressions in terms of covariant derivatives, where, for example,

$$X^{\alpha}_{,\gamma} = \frac{\delta X^{\alpha}}{\delta u^{\gamma}} = \frac{\partial X^{\alpha}}{\partial u^{\gamma}} + \Gamma^{\alpha}_{\beta\gamma} X^{\beta} \tag{7.17}$$

$$X_{\alpha,\gamma} = \frac{\delta X_{\alpha}}{\delta u^{\gamma}} = \frac{\partial X_{\alpha}}{\partial u^{\gamma}} - \Gamma^{\beta}_{\alpha\gamma} X_{\beta} \tag{7.18}$$

The gradient now becomes

$$\nabla\phi = \frac{\partial\phi}{\partial u^{i}}\,\mathbf{a}^{i} \tag{7.19}$$

The divergence is obtained by expanding

$$\frac{\delta f^{\alpha}}{\delta u^{\alpha}} = \frac{\partial f^{\alpha}}{\partial u^{\alpha}} + \Gamma^{\alpha}_{\beta\alpha} f^{\beta} \tag{7.20}$$

Expanding the second term on the right-hand side,

$$\Gamma^{\alpha}_{\beta\alpha} = \tfrac{1}{2} g^{\delta\pi} \frac{\partial g_{\alpha\pi}}{\partial u^{\beta}} \tag{7.21}$$

Now, if

$$g = |g_{\alpha\beta}|$$

$$\frac{\partial g}{\partial u^{\beta}} = G^{\delta\pi} \frac{\partial g_{\gamma\pi}}{\partial u^{\beta}} \tag{7.22}$$

where $G^{\delta\pi}$ is the cofactor of $g_{\delta\pi}$

Also,

$$g^{\gamma\lambda} = \frac{G^{\gamma\lambda}}{g}$$

or

$$G^{\delta\pi} = g^{\delta\pi} g \tag{7.23}$$

Thus

$$\frac{\partial g}{\partial u^{\beta}} = (g^{\delta\pi} g) \frac{\partial g_{\delta\pi}}{\partial u^{\beta}}$$

$$\Gamma^{\alpha}_{\beta\alpha} = \tfrac{1}{2} \frac{1}{g} \frac{\partial g}{\partial u^{\beta}} = \frac{\partial}{\partial u^{\beta}} (\log \sqrt{g}) \tag{7.24}$$

and

$$\nabla \cdot \mathbf{F} = \frac{\delta f^{\alpha}}{\delta u^{\alpha}} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} f^{\alpha})}{\partial u^{\alpha}} \tag{7.25}$$

The expression for the curl is obtained by a similar process using covariant components, namely,

$$\nabla \times \mathbf{F} = \frac{\delta f_{\alpha}}{\delta u^{\beta}} - \frac{\delta f_{\beta}}{\delta u^{\alpha}} = \frac{1}{\sqrt{g}} \begin{vmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \\ \frac{\partial}{\partial u^1} & \frac{\partial}{\partial u^2} & \frac{\partial}{\partial u^3} \\ f_1 & f_2 & f_3 \end{vmatrix} \tag{7.26}$$

The curl of a vector field can also be obtained by applying a generalised form of the skew-symmetric matrix whose components are ± 1. In three dimensional Cartesian co-ordinates these e-symbols are defined by[14]

$$e_{ijk} = +1, -1, 0 \tag{7.27}$$

depending on whether

(i) an even or odd permutation will restore the sequence 1, 2, 3, or,
(ii) two indices are equal.

From this definition

$$e^{ijk} e_{\alpha\beta\gamma} = \delta^{ijk}_{\alpha\beta\gamma} \tag{7.28}$$

The generalised δ-symbol has properties similar to those of the e-symbols with respect to permutations of upper and lower indices. If any two symbols are equal then that component is zero.

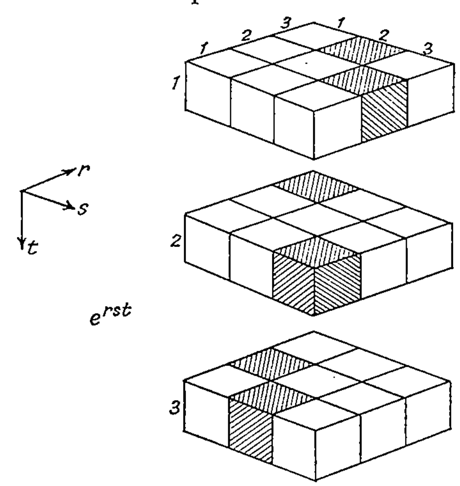

FIG. 7.3

The three-dimensional e-matrix will have its components located at the positions shown in figure 7.3. Any completely skew-symmetric matrix can have only two values for all its components, that is

$$A_{\alpha\beta\gamma} = A e_{\alpha\beta\gamma} \tag{7.29}$$

where $A_{123} = +A$, $A_{132} = -A$, $A_{113} = 0$, etc.

In this notation a determinant is expressed

$$|a^i_j| = a = e_{ijk}a^i_1 a^j_2 a^k_3 \tag{7.30}$$

and a vector product

$$\mathbf{D} = \mathbf{A} \times \mathbf{B} = e^{ijk}A_j B_k$$

for example

$$D^1 = e^{123}A_2B_3 + e^{132}A_3B_2 \tag{7.31}$$
$$= A_2B_3 - A_3B_2$$

(This leads later on to the required invariant form of the curl of a vector.)

Further properties of the e-symbols are as follows. Referring to equation (7.30) it follows that

$$e_{ijk}a^i_\alpha a^j_\beta a^k_\gamma = |a^k_\delta| e_{\alpha\beta\gamma} \tag{7.32}$$

and

$$e_{ijk}a^i_\alpha a^j_\beta a^k_\gamma = -e_{ijk}a^i_\gamma a^j_\beta a^k_\alpha \tag{7.33}$$

The e-symbols are not invariant under transformation of co-ordinates. Consider a holonomic, reversible transformation from, say, a Cartesian system y^m to a curvilinear system u^γ

$$C^m_\gamma = \frac{\partial y^m}{\partial u^\gamma} \tag{7.34}$$

$$dy^m = C^m_\gamma du^\gamma$$

The inverse of this transformation is

$$C^\gamma_m = \frac{\partial u^\alpha}{\partial y^m} \tag{7.35}$$

where

$$y^m = f(u^1u^2u^3)$$
$$u^\gamma = f(y^1y^2y^3)$$

The Jacobian is defined by

$$J = \left| \frac{\partial u^\gamma}{\partial y^m} \right| \tag{7.36}$$

and

$$\frac{1}{J} = \left| \frac{\partial y^m}{\partial u^\gamma} \right|$$

Now from equation (7.32), putting $a^i_\alpha \equiv C^i_\alpha$

$$\left| \frac{\partial u^\gamma}{\partial y^m} \right| e_{\alpha\beta\gamma} = e_{knm}C^k_\alpha C^n_\beta C^m_\gamma \tag{7.37}$$

$$e_{\alpha\beta\gamma} = \left| \frac{\partial u^\gamma}{\partial y^m} \right|^{-1} e_{knm}C^k_\alpha C^n_\beta C^m_\gamma \tag{7.38}$$

and

$$e^{\alpha\beta\gamma} = \left| \frac{\partial u^\gamma}{\partial y^m} \right| e^{knm}C^\alpha_k C^\beta_n C^\gamma_m \tag{7.39}$$

An object transforming in this way is called a relative tensor of weight W (the power to which the determinant is raised). Thus $e^{\alpha\beta\gamma}$ and $e_{\alpha\beta\gamma}$ are relative tensors of weight $+1$ and -1 respectively. In terms of the metric tensor,

$$g_{mn} = \frac{\partial u^\gamma}{\partial y^m} \cdot \frac{\partial u^\gamma}{\partial y^n}$$

and

$$|g_{mn}| = g' = \left| \frac{\partial u^\gamma}{\partial y^m} \right|^2$$

It therefore follows that the quantities

$$\varepsilon_{knm} = \sqrt{g'}\, e_{knm} \tag{7.40}$$

$$\varepsilon^{knm} = \frac{1}{\sqrt{g'}} e^{knm} \tag{7.41}$$

transform as absolute tensors. The covariant quantity e_{knm} is a tensor density. The contravariant form is called a tensor capacity. In three-dimensional orthogonal co-ordinates, with the usual notation,

$$g_{\alpha\beta} = \begin{array}{c|c|c|c} \alpha\diagdown\beta & 1 & 2 & 3 \\ \hline 1 & h_1^2 & & \\ 2 & & h_2^2 & \\ 3 & & & h_3^2 \end{array} \tag{7.42}$$

$$\sqrt{g_{11}} = h_1, \quad \sqrt{g_{22}} = h_2, \quad \sqrt{g_{33}} = h_3$$
$$g_{12} = g_{23} = g_{31} = 0$$
$$\sqrt{g} = h_1h_2h_3 \tag{7.43}$$

In the general three-dimensional completely skew-symmetric matrix $A_{\alpha\beta\gamma}$, the single value A is not an invariant and

$$A = JA'$$

also

$$\sqrt{g} = J\sqrt{g'}$$

The curl of a vector field can now be expressed with respect to the system $\mathbf{a}^{\pi}$

$$(\text{curl } \mathbf{F})^{\gamma} \equiv G^{\gamma} = -\varepsilon^{\gamma\beta\alpha} f_{\beta,\alpha} = -\frac{1}{\sqrt{g}} e^{\gamma\beta\alpha} f_{\beta,\alpha} \tag{7.44}$$

and Stokes' theorem becomes

$$\iint G^{\gamma} n_{\gamma}\, dS = \int f_{\gamma} \frac{du^{\gamma}}{ds}\, ds \tag{7.45}$$

where n_{γ} is the normal unitary vector.

Each side of equation (7.45) is now an absolute invariant and the integration difficulties mentioned in Section 7.2 do not arise.

The Laplacian, in absolute form, is obtained by equating the contravariant vector in equation (7.25) to the gradient of a scalar field,

$$f^{\alpha} = g^{\alpha\beta} \frac{\partial \phi}{\partial u^{\beta}} \tag{7.46}$$

Summarising, the invariant forms of the expressions used in field equations are

Gradient, $$\nabla \phi = \frac{\partial \phi}{\partial u^{\alpha}} \mathbf{a}^{\alpha} \tag{7.47}$$

Divergence, $$\nabla \cdot \mathbf{F} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^{\alpha}} (f^{\alpha} \sqrt{g}) \tag{7.48}$$

Curl, $$\nabla \times \mathbf{F} = \frac{1}{\sqrt{g}} \begin{vmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \\ \dfrac{\partial}{\partial u^1} & \dfrac{\partial}{\partial u^2} & \dfrac{\partial}{\partial u^3} \\ f_1 & f_2 & f_3 \end{vmatrix} \tag{7.49}$$

Laplacian, $$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^{\beta}} \left(g^{\beta\gamma} \sqrt{g} \frac{\partial \phi}{\partial u^{\gamma}} \right) \tag{7.50}$$

Stokes' theorem $$\iint G^{\alpha} n_{\alpha}\, dS = \int f_{\alpha} \frac{du^{\alpha}}{ds}\, ds \tag{7.51}$$

Green's theorem $$\iiint f^{\gamma}_{,\gamma}\, d\tau = \iint f^{\gamma} n_{\gamma}\, dS \tag{7.52}$$

Gauss' theorem $$\iint D^{\gamma} n_{\gamma}\, dS = \iiint \rho\, d\tau \tag{7.53}$$

Ampere's law, $$\int H_{\gamma} \frac{du^{\gamma}}{ds}\, ds - \frac{\partial}{\partial t} \iint D^{\gamma} n_{\gamma}\, dS - \iint I^{\gamma} n_{\gamma}\, dS = 0 \tag{7.54}$$

Faraday's law, $$\int E_{\gamma} \frac{du^{\gamma}}{ds}\, ds + \frac{\partial}{\partial t} \iint B^{\gamma} n_{\gamma}\, dS = 0 \tag{7.55}$$

In these equations the elements of area and volume are

$$dS_1 = \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} du^2 du^3, \text{ etc.}$$

$$d\tau = \sqrt{g}\, du^1 du^2 du^3$$

7.4 Maxwell's equations[54,55,56]

In setting up the equivalent circuit model for the electromagnetic field, the components of the vector fields are first expressed in covariant or contravariant form as shown in Section 7.3. Once the covariant or contravariant nature of one of the vectors has been decided then the field equations determine the nature of all other components. The contravariant or covariant components are then related to the electrical circuit equations for open and closed meshes, namely,

$$\begin{aligned} &\text{closed meshes} \quad v_{\gamma} = Z_{\gamma\alpha} i^{\alpha} \\ &\text{open meshes} \quad I^{\alpha} = Y^{\alpha\beta} V_{\beta} \end{aligned} \tag{7.56}$$

It will be found, however, that some covariant quantities, having contravariant curl components, are represented on the network by contravariant quantities. Consider the equations of the electromagnetic field,

$$\begin{aligned} \text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \text{curl } \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \text{div } \mathbf{B} &= 0 \\ \text{div } \mathbf{D} &= \rho \\ \mathbf{B} &= \mu \mathbf{H} \\ \mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{J} &= \sigma \mathbf{E} \end{aligned} \tag{7.57}$$

If covariant components of the electric field vector, E_α be taken, the index notation gives the following components of the remaining terms in the equations,

E covariant
Curl **E** contravariant (equation 7.44)
B contravariant (equation 7.57)
H covariant (equation 7.57)
Curl **H** contravariant (equation 7.44)
J and **D** contravariant (equation 7.57)
$\boldsymbol{\mu}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ doubly contravariant (equation 7.57)

The physical components of the vectors are taken as a starting point. These are, with respect to the unit direction vectors,

$$E_{(k)} = E_{(1)}\mathbf{i}^1 + E_{(2)}\mathbf{i}^2 + E_{(3)}\mathbf{i}^3 \tag{7.58}$$

In orthogonal co-ordinate systems the transformation matrix from these to covariant or contravariant components is

$C_\alpha^{(k)} =$ (7.59)

$(k)\diagdown\alpha$	1	2	3
1	h_1		
2		h_2	
3			h_3

Quantities associated with line integrals are then transformed by

$$E_\alpha = C_\alpha^{(k)} E_{(k)} \tag{7.60}$$

for example $E_1 = h_1 E_{(1)}$, etc. (7.61a)

And similarly $H_1 = h_1 H_{(1)}$, etc. (7.61b)

Quantities associated with surface or volume integrals are transformed by two steps,

(i) a transformation as above to covariant or contravariant components,

(ii) these components are multiplied by $\sqrt{g}$ to express them as vector or tensor densities.

When they are so expressed, the $\sqrt{g}$ coefficients in the elements of area or volume have been transferred to the new density terms and the elements of area or volume in the invariant equations are then $(\Delta u^2 \Delta u^3)$ and $(\Delta u^1 \Delta u^2 \Delta u^3)$, etc. The laws of transformation are as follows,

Contravariant vector

$$B^{\alpha'} = \sqrt{g}B^\alpha = \sqrt{g}g^{\alpha\beta}B_\beta = \sqrt{g}g^{\alpha\beta}C_\beta^{(k)}B_{(k)} \tag{7.62}$$

For example,

$$B^{1'} = h_1h_2h_3 \frac{1}{(h_1)^2} h_1 B_{(1)} = h_2h_3B_{(1)}, \text{ etc.} \tag{7.63}$$

Contravariant tensors, ($\boldsymbol{\mu\epsilon}$ and $\boldsymbol{\sigma}$)

$$\mu^{\alpha\beta'} = \sqrt{g}\mu^{\alpha\beta} = \sqrt{g}g^{\delta\beta}\delta_\delta^\alpha\mu = \sqrt{g}g^{\delta\beta}\mu \tag{7.64}$$

(δ_δ^α is the square, unit matrix)

For example,

$$\mu^{11'} = h_1h_2h_3 \frac{1}{(h_1)^2}\mu = \frac{h_2h_3}{h_1}\mu, \text{ etc.} \tag{7.65}$$

In curvilinear co-ordinates, an element of length

$$ds_1 = \sqrt{g_{11}}\, du^1, \text{ etc.,} \tag{7.66}$$

therefore the voltage interval along a branch of the network will be

$$E_{(1)}\, ds_1 = E_{(1)}h_1\,\Delta u^1 = E_1\,\Delta u^1 \tag{7.67}$$

The flux threading a mesh containing a field of flux density $B_{(1)}$ will be given by

$$\begin{aligned} B_{(1)}\, ds_2\, ds_3 &= B_{(1)}h_2\,\Delta u^2 h_3\,\Delta u^3 \\ &= B_{(1)}h_2h_3\,\Delta u^2\,\Delta u^3 \\ &= B^{1'}\,\Delta u^2\,\Delta u^3 \end{aligned} \tag{7.68}$$

Thus the vector or tensor density components can be integrated with respect to the elements Δu^1, Δu^2, Δu^3 of the co-ordinate system.

The equations in orthogonal curvilinear co-ordinates can now be written with vector or tensor density components.

$$\frac{\partial E_3}{\partial u^2} - \frac{\partial E_2}{\partial u^3} = -\frac{\partial B^{1'}}{\partial t}$$

and two similar equations.

$$\frac{\partial H_3}{\partial u^2} - \frac{\partial H_2}{\partial u^3} = J^{1'} + \frac{\partial D^{1'}}{\partial t}$$

and two similar equations.

$$\frac{\partial B^{1'}}{\partial u^1} + \frac{\partial B^{2'}}{\partial u^2} + \frac{\partial B^{3'}}{\partial u^3} = 0$$

$$\frac{\partial D^{1'}}{\partial u^1} + \frac{\partial D^{2'}}{\partial u^2} + \frac{\partial D^{3'}}{\partial u^3} = \rho' \qquad (\rho' = \sqrt{g}\,\rho)$$

$$J^{1'} = \sigma^{11'}E_1, \text{ etc.}$$
$$D^{1'} = \epsilon^{11'}E_1, \text{ etc.} \qquad (7.69)$$
$$B^{1'} = \mu^{11'}H_1, \text{ etc.}$$

Kron's equivalent circuit is shown in figure 7.4. Each circuit element of this network is a lumped equivalent of an element of space with distributed

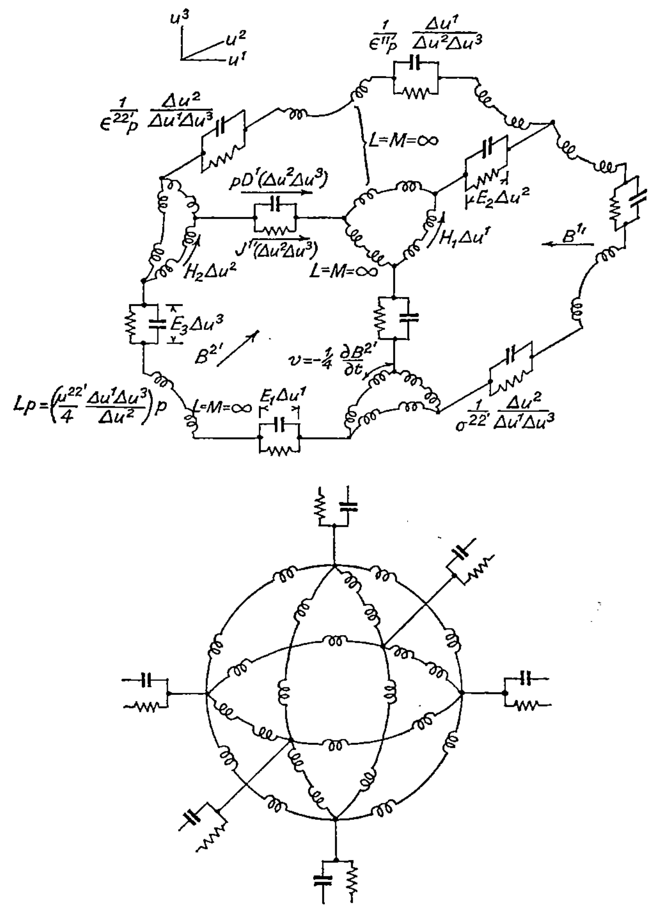

FIG. 7.4. Equivalent circuit for Maxwell's equations (one block).

parameters. This 'concentration' of currents, voltages and fluxes is shown in figure 7.5. The voltages across the capacitors round a mesh in the u^1u^3 plane add to give the line integral

$$\Delta E_1\,\Delta u^1 - \Delta E_3\,\Delta u^3 \qquad (7.70)$$

and

$$\left(\frac{\partial E_1}{\partial u^3}\,\Delta u^3\right)\Delta u^1 - \left(\frac{\partial E_3}{\partial u^1}\,\Delta u^1\right)\Delta u^3 = -\frac{\partial}{\partial t}(B^{2'}\Delta u^1\,\Delta u^3)$$

or

$$\left(\frac{\partial E_1}{\partial u_3} - \frac{\partial E_3}{\partial u^1}\right)\Delta u^1\,\Delta u^3 = -\frac{\partial}{\partial t}(B^{2'}\Delta u^1\,\Delta u^3) \qquad (7.71)$$

The covariant magnetic vector H_2 is represented by current, which is contravariant. However, the curl of **H** is contravariant and this is given by

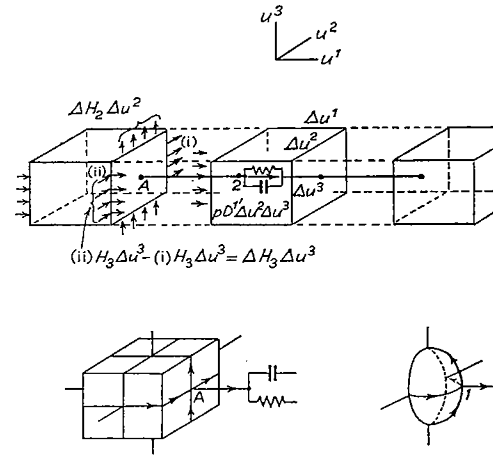

FIG. 7.5. Simulation of blocks by "concentrated" currents.

the residual current flowing out from the magnetic meshes into the capacitor-resistor circuits. In figure 7.5 the current in the lumped RC connecting circuit is the integral of $J^{1'}$ and $pD^{1'}$ over the area $\Delta u^2\,\Delta u^3$, namely

$$(J^{1'} + pD^{1'})\Delta u^2\,\Delta u^3 \qquad (7.72)$$

On this circuit the sheets of current flowing into the junction A give residuals

$$\Delta H_3\,\Delta u^3 - \Delta H_2\,\Delta u^2 = (J^{1'} + pD^{1'})\Delta u^2\,\Delta u^3$$

or

$$\left(\frac{\partial H_3}{\partial u^2} - \frac{\partial H_2}{\partial u^3}\right)\Delta u^2\,\Delta u^3 = (J^{1'} + pD^{1'})\Delta u^2\,\Delta u^3 \qquad (7.73)$$

corresponding to Ampere's law.

Similar analysis shows that the divergence equations are also satisfied.

The operation of the circuit model depends on the ideal transformers coupling the diagonal magnetic circuits as shown. On synthesising a network from a set of operational equations it is often found that the currents in the meshes must be constrained by ideal transformers. The transformers included by Kron[55] in the circuit of figure 7.4 perform two main functions.

(i) They eliminate currents that would circulate in the smaller magnetic coupling-meshes at the corners of the elementary cubes and constrain the network currents to flow in the larger meshes only. The currents in these are always associated with curl **H**. Similarly the voltage circulation round these meshes is always equated to the terms $p(\mu\mathbf{H})$. The network therefore propagates electromagnetic waves and not simply currents and voltages.

(ii) Maxwell's equations are satisfied even in the presence of an arbitrary scalar potential field ϕ. In terms of a magnetic vector potential **A**, where

$$\mathbf{B} = \text{curl}\,\mathbf{A} \text{ webers/sq. metre} \tag{7.74}$$

and

$$\mathbf{A} = \mu\int_{\text{vol.}} \frac{\mathbf{J}}{r}\,d\tau \text{ webers/metre} \tag{7.75}$$

$$\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\text{curl}\,\mathbf{A}) \tag{7.76}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{7.77}$$

In the presence of a scalar potential field

$$\mathbf{E} = -\,\text{grad}\,\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{7.78}$$

$$\text{curl}\,\mathbf{E} = -\,\text{curl grad}\,\phi - \frac{\partial}{\partial t}(\text{curl}\,\mathbf{A}) \tag{7.79}$$

and since curl grad $\phi = 0$

the equation

$$\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

is again satisfied.

With the transformers coupled diagonally across the larger meshes, voltage corresponding to a source-free potential field may be impressed diametrically across the corner magnetic meshes. These will cause no currents to circulate in these smaller meshes and the ideal coupling means that the voltage gradients will adjust themselves across the whole network so that they cause no additional currents in the larger meshes either. The network will therefore propagate waves as if the scalar potential field did not exist.

The whole model is seen to be a three-dimensional orthogonal network energised in both closed and open meshes. It can be divided in this way as shown in figure 7.6 to give the transverse electric (TE) and transverse magnetic (TM) modes of oscillation. These two-dimensional networks are in fact those usually derived separately by difference techniques. In three dimensions this division corresponds to the two four-dimensional tensor equations usually associated with relativistic electrodynamics,[11,53]

$$\frac{\partial F_{\alpha\beta}}{\partial u^{\gamma}} + \frac{\partial F_{\beta\gamma}}{\partial u^{\alpha}} + \frac{\partial F_{\gamma\alpha}}{\partial u^{\beta}} = 0 \tag{7.80a}$$

$$\frac{1}{\sqrt{-a}}\left[\frac{\partial}{\partial x^{\beta}}(\sqrt{-a}\,H^{\alpha\beta})\right] = S^{\alpha} \tag{7.80b}$$

where $u^{\alpha} = u^1, u^2, u^3, u^4$

and $u^4 = ict \quad (i = \sqrt{-1}$ and c is the velocity of light)

$$F_{\alpha\beta} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & \frac{B_3}{(h_3)^2} & -\frac{B_2}{(h_2)^2} & \frac{i}{c}\frac{E_1}{\sqrt{g}} \\ 2 & -\frac{B_3}{(h_3)^2} & & \frac{B_1}{(h_1)^2} & \frac{i}{c}\frac{E_2}{\sqrt{g}} \\ 3 & \frac{B_2}{(h_2)^2} & -\frac{B_1}{(h_1)^2} & & \frac{i}{c}\frac{E_3}{\sqrt{g}} \\ 4 & -\frac{i}{c}\frac{E_1}{\sqrt{g}} & -\frac{i}{c}\frac{E_2}{\sqrt{g}} & -\frac{i}{c}\frac{E_3}{\sqrt{g}} & \end{array} \tag{7.81}$$

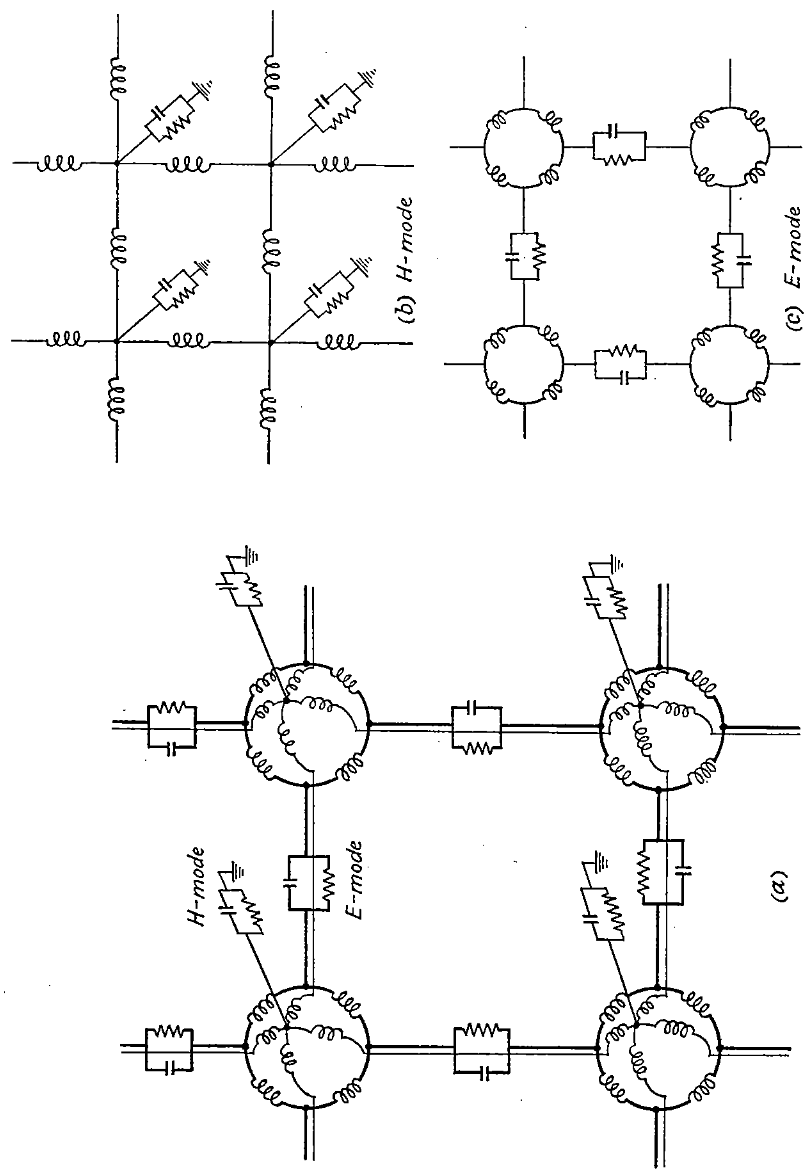

FIG. 7.6

$$H^{\alpha\beta} = \begin{array}{c|c|c|c|c|} & 1 & 2 & 3 & 4 \\ \hline 1 & & \dfrac{H_3}{\sqrt{g}} & -\dfrac{H_2}{\sqrt{g}} & -ic\,\dfrac{D_1}{(h_1)^2} \\ \hline 2 & -\dfrac{H_3}{\sqrt{g}} & & \dfrac{H_1}{\sqrt{g}} & -ic\,\dfrac{D_2}{(h_2)^2} \\ \hline 3 & \dfrac{H_2}{\sqrt{g}} & -\dfrac{H_1}{\sqrt{g}} & & -ic\,\dfrac{D_3}{(h_3)^2} \\ \hline 4 & \dfrac{D_1}{(h_1)^2} & \dfrac{D_2}{(h_2)^2} & \dfrac{D_3}{(h_3)^2} & \\ \hline \end{array} \qquad (7.82)$$

$$a_{\alpha\beta} = \begin{array}{c|c|c|c|c|} & 1 & 2 & 3 & 4 \\ \hline 1 & -(h_1)^2 & & & \\ \hline 2 & & -(h_2)^2 & & \\ \hline 3 & & & -(h_3)^2 & \\ \hline 4 & & & & c^2 \\ \hline \end{array} \qquad (7.83)$$

$$S^{\alpha} = \begin{array}{|c|c|c|c|} 1 & 2 & 3 & 4 \\ \hline \dfrac{J_1}{(h_1)^2} & \dfrac{J_2}{(h_2)^2} & \dfrac{J_3}{(h_3)^2} & \rho \\ \hline \end{array} \qquad (7.84)$$

The elements of these matrices are obtained from those defined by equations (7.69), to conform with the invariant equations. They give the same results. For example,

$$J_1 = \frac{1}{\sqrt{g}} g_{11} J^{1'}$$

$$\frac{J_1}{(h_1)^2} = \frac{1}{\sqrt{g}} J^{1'} \qquad (7.85)$$

$$\frac{1}{\sqrt{-a}}\frac{\partial}{\partial u^2}(\sqrt{-a}H^{12}) = \frac{1}{ch_1h_2h_3}\frac{\partial}{\partial u^2}\left(ch_1h_2h_3\frac{H_3}{\sqrt{g}}\right)$$

$$= \frac{1}{\sqrt{g}}\frac{\partial H_3}{\partial u^2} \tag{7.86}$$

and with the first row of matrix (7.82) equation (7.80b) gives

$$\frac{1}{\sqrt{-a}}\left[\frac{\partial}{\partial u^2}(\sqrt{-a}H^{12}) + \frac{\partial}{\partial u^3}(\sqrt{-a}H^{13}) + \frac{\partial}{\partial u^4}(\sqrt{-a}H^{14})\right] = S^1 \tag{7.87}$$

which becomes

$$\frac{\partial H_3}{\partial u^2} - \frac{\partial H_2}{\partial u^3} - \frac{\partial D^{1'}}{\partial t} = J^{1'} \tag{7.88}$$

On the model the closed meshes satisfy the covariant equation (7.80a) and the open meshes satisfy the contravariant equation (7.80b). The network as a whole combines the two sets into one orthogonal network satisfying the complete four-dimensional space-time formulation of the electromagnetic equations.

7.5 Network Analogues

When Kron had established the tensor-density technique which enabled him to draw equivalent circuits for the electromagnetic field, the same

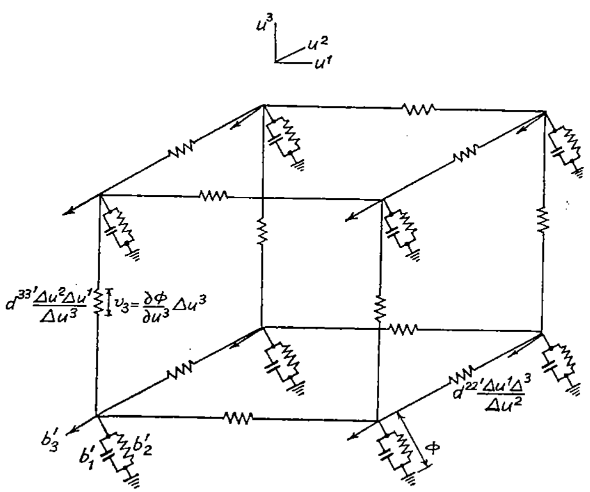

FIG. 7.7. Equivalent circuit for the diffusion equation.

reasoning was then applied to circuit models for other types of partial differential equations.[57,49] In these circuits, transient and wave phenomena are represented by electromagnetic waves. They are wholly analogue circuits. Some examples will now be considered.

The equivalent circuit for the diffusion equation[58]

$$\text{div } \mathbf{d} \text{ grad } \phi = b \tag{7.89}$$

is shown in figure 7.7. In many cases

$$b = b_1\frac{\partial \phi}{\partial t} + b_2\phi + b_3 \tag{7.90}$$

The terms of equation (7.89) in tensor density form are given by equations (7.60) to (7.64). In this equation,

d is doubly contravariant

b is a scalar

ϕ is a scalar

grad ϕ is a covariant vector.

On the network the voltage difference along Δu^1 is

$$\frac{\partial \phi}{\partial u^1}\Delta u^1$$

Now,

$$\text{current} = \text{conductance} \times \text{voltage}$$

$$J^{1'}\Delta u^2 \Delta u^3 = d^{11'}\frac{\Delta u^2 \Delta u^3}{\Delta u^1}\frac{\partial \phi}{\partial u^1}\Delta u^1 \tag{7.91}$$

$$J_{(1)}h_2h_3\,\Delta u^2\,\Delta u^3 = \left(d^{11'}\frac{\partial \phi}{\partial u^1}\right)\Delta u^2\,\Delta u^3$$

$$= \left[d_{(1)}\frac{h_2h_3}{h_1}\frac{\partial \phi}{\partial u^1}h_1\right]\Delta u^2\,\Delta u^3$$

$$= \left[d_{(1)}\frac{\partial \phi}{\partial u^1}\right]h_2h_3\,\Delta u^2\,\Delta u^3 \tag{7.92}$$

where $$d^{11'} = d_{(1)}\frac{h^2h^3}{h_1}$$

The components of the divergence, over the intervals Δu^1, Δu^2, Δu^3 are summed, to give

$$\left[\frac{\partial}{\partial u^1}\left(d^{11'}\frac{\partial\phi}{\partial u^1}\right)+\frac{\partial}{\partial u^2}\left(d^{22'}\frac{\partial\phi}{\partial u^2}\right)+\frac{\partial}{\partial u^3}\left(d^{33'}\frac{\partial\phi}{\partial u^3}\right)\right]\Delta u^1\,\Delta u^2\,\Delta u^3$$
$$= bh_1h_2h_3\,\Delta u^1\,\Delta u^2\,\Delta u^3$$
$$= b'\,\Delta u^1\,\Delta u^2\,\Delta u^3 \qquad (7.93)$$

which satisfies simultaneously, Kirchhoff's laws and the diffusion equation.

An example involving the curl of a vector field is that of solenoidal fluid flow. In this form of stream flow, examined here without sources, there exists a stream function ψ which has the form of a vector potential. The flow is described by the following equations,

$$\begin{aligned} \operatorname{div}\rho\mathbf{v} &= 0\\ \operatorname{curl}\mathbf{v} &= \boldsymbol{\Gamma}\\ \rho\mathbf{v} &= \operatorname{curl}\psi \qquad (7.94)\\ \operatorname{curl}(\rho^{-1}\operatorname{curl}\psi) &= \boldsymbol{\Gamma}\\ \operatorname{div}\operatorname{curl}\psi &= 0 \end{aligned}$$

In these equations, ρ is the fluid density, $\mathbf{v}$ is a velocity vector field. $\boldsymbol{\Gamma}$ is called the vorticity and it may have the form

$$\boldsymbol{\Gamma} = d_1\frac{\partial\psi}{\partial t} + d_2\psi + d_3 \qquad (7.95)$$

Again the equations decide the covariant or contravariant nature of the terms, once one of them has been fixed, thus

$\mathbf{v}$ covariant (voltage)
ψ covariant (current)
Curl $\mathbf{v}$ contravariant (mesh voltage)
ρ doubly contravariant (conductivity)
ρ^{-1} doubly covariant (resistivity)
$\boldsymbol{\Gamma}$ contravariant (mesh voltage)

On the network shown in figure 7.8, v is the branch voltage drop, curl ψ is the residual current flowing from the magnetic meshes into the branches and

$$\operatorname{curl}\psi' = \rho'\mathbf{v}' \qquad (7.96)$$

The contravariant components of conductivity are, from equation 7.64

$$\rho^{11'} = \rho\frac{h_2h_3}{h_1} \qquad (7.97)$$

The branch conductance is

$$\rho^{11'}\frac{\Delta u^2\,\Delta u^3}{\Delta u^1} \qquad (7.98)$$

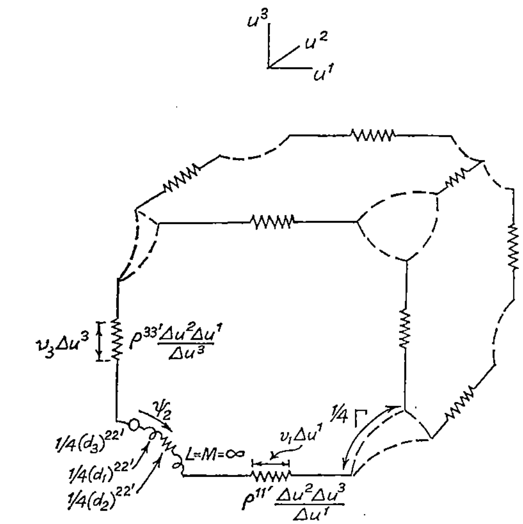

FIG. 7.8. Equivalent circuit for solenoidal fluid flow (one block).

The branch resistivity is

$$\rho'_{11} = (\rho)^{-1}\frac{h_1}{h_2h_3} \qquad (7.99)$$

and the branch resistance is

$$\rho'_{11}\frac{\Delta u^1}{\Delta u^2\,\Delta u_3} \qquad (7.100)$$

In the integrated form, since one component of curl ψ' is the contravariant vector G^1, the integrated branch current is $G^{1'}\Delta u^2\,\Delta u^3$. Thus

$$G^{1'}\Delta u^2\,\Delta u^3 = \rho^{11'}\frac{\Delta u^2\,\Delta u^3}{\Delta u^1}\,v'_1\,\Delta u^1$$
$$= J^{1'}\Delta u^2\,\Delta u^3 \qquad \text{the branch current.} \qquad (7.101)$$

Also,

$$\rho^{-1}\operatorname{curl}\psi = \mathbf{v} \qquad (7.102)$$

for example

$$\rho'_{11} \frac{\Delta u^1}{\Delta u^2 \Delta u^3} G^{1'} \Delta u^2 \Delta u^3 = v'_1 \Delta u^1 \tag{7.103}$$

the branch voltage.

Around a mesh, the voltage v is equal to

$$\text{curl}\,(\rho^{-1}\,\text{curl}\,\psi) = \mathbf{\Gamma} \tag{7.104}$$

The divergence equations are also satisfied in accordance with Kirchhoff's laws.

In studies of theoretical elasticity, tensor analysis has always been widely used.(59) The equations can be conveniently handled in index notation and the concepts can be expressed in an elegant mathematical form.

There are three sets of equations to be simultaneously satisfied,

(i) the stress equilibrium equations,

(ii) the stress-strain relationship,

(iii) the compatibility relationship.

The last of these ensures that the displacements due to any given stress distribution will be single valued—a condition automatically satisfied by the electrical analogue network. In generalised co-ordinates, covariant differential equations are used. To set up the equivalent circuit it is first decided that (say) the force vector f^α is to be represented by current (contravariant), and displacements, by voltage (covariant). The index notation then gives the nature of other quantities, for example, the stress becomes a doubly contravariant tensor-density(60) and

$$\sigma'^{11} = \frac{h_2 h_3}{h_1} \sigma_{(1)(1)}, \quad \text{etc.} \tag{7.105}$$

(i) The equilibrium equations can now be expressed in terms of displacement s_β

$$\frac{\delta \sigma^{\alpha\beta'}}{\delta u^\beta} + \rho' f^\alpha = \rho' g^{\alpha\beta} \frac{\partial^2 s_\beta}{\partial t^2} \tag{7.106}$$

where, again ρ' is the material density $(= \sqrt{g}\rho)$. This equation involves the covariant derivative of a tensor density and as shown in Appendix IV

$$\frac{\delta \sigma'^{\alpha\beta}}{\delta u^\beta} = \frac{\partial \sigma'^{\alpha\beta}}{\partial u^\beta} + \Gamma^\alpha_{\beta\gamma} \sigma'^{\beta\gamma} \tag{7.107}$$

where

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\pi} \left(\frac{\partial g_{\pi\gamma}}{\partial u^\beta} + \frac{\partial g_{\pi\beta}}{\partial u^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial u^\pi} \right)$$

Equation (7.107) expands to give a set of simultaneous equations. For example, in two dimensions,

$$\frac{\delta \sigma^{1\beta'}}{\delta u^\beta} = \frac{\partial \sigma'^{11}}{\partial u^1} + \Gamma^1_{11}\sigma'^{11} + \Gamma^1_{12}\sigma'^{12} + \frac{\partial \sigma'^{12}}{\partial u^2} + \Gamma^1_{21}\sigma'^{21} + \Gamma^1_{22}\sigma'^{22} \tag{7.108}$$

where

$$\Gamma^1_{11} = g^{11}\Gamma_{11,1} = \frac{1}{h_1}\frac{\partial h_1}{\partial u^1}, \quad \text{etc.}$$

$$\frac{\delta \sigma^{2\beta'}}{\delta u^\beta} = \text{similar expression.}$$

The tensor-density equilibrium-equation in three dimensions thus becomes

$$\frac{\partial \sigma'^{11}}{\partial u^1} + \frac{\partial \sigma'^{12}}{\partial u^2} + \frac{\partial \sigma'^{13}}{\partial u^3} + \frac{1}{h_1}\frac{\partial h_1}{\partial u^1}\sigma'^{11} + \frac{2}{h_1}\frac{\partial h_1}{\partial u^2}\sigma'^{12} + \frac{2}{h_1}\frac{\partial h_1}{\partial u^3}\sigma'^{13}$$
$$- \frac{h^2}{(h_1)^2}\frac{\partial h_2}{\partial u^1}\sigma'^{22} - \frac{h_3}{(h_1)^2}\frac{\partial h_3}{\partial u^1}\sigma'^{33} + \rho' f^{1'} = \rho' \frac{1}{h_1}\frac{\partial^2 s_1}{\partial t^2} \tag{7.109}$$

and another two similar equations.

(ii) In the second set the stresses (σ) are given in terms of strains (e) by

$$\sigma_{ii} = \lambda\xi + 2\mu e_{ii}$$
$$\sigma_{ij} = 2\mu e_{ij} \tag{7.110}$$

where λ and μ are Lame's constants and the dilatation

$$\xi = e_{11} + e_{22} + e_{33} \tag{7.111}$$

From these equations

$$\sigma_{11} = (2\mu + \lambda)e_{11} + \lambda e_{22} + \lambda e_{33}$$
$$\sigma_{12} = 2\mu e_{12}, \tag{7.112}$$
$$\text{etc.}$$

The relation between strain and displacement is given by

$$e_{ij} = \tfrac{1}{2}(s_{i,j} + s_{j,i}) \tag{7.113}$$

where the comma denotes covariant differentiation. In cylindrical co-ordinates $u^1u^2(r,\theta)$ for example,

$$e_{11} = \tfrac{1}{2}\left(\frac{\partial s_1}{\partial u^1} - \Gamma^1_{11}s_1 - \Gamma^2_{11}s_2 + \frac{\partial s_1}{\partial u^1} - \Gamma^1_{11}s_1 - \Gamma^1_{11}s_2\right) = \frac{\partial s_1}{\partial u^1}$$
$$e_{12} = \tfrac{1}{2}\left(\frac{\partial s_1}{\partial u^2} + \frac{\partial s_2}{\partial u^1} - \frac{2s_2}{u^1}\right) \tag{7.114}$$
$$e_{22} = \frac{\partial s_2}{\partial u^2} + u^1 s_1$$

In this co-ordinate system the stress tensor-densities, in terms of displacements can therefore be written,

$$\begin{aligned}
\sigma'^{rr} &= (2\mu - \lambda) r \frac{\partial s_r}{\partial r} + \frac{\lambda}{r}\frac{\partial s_\theta}{\partial \theta} + \lambda s_r \\
\sigma'^{r\theta} &= \frac{\mu}{r}\frac{\partial s_r}{\partial \theta} + \frac{\mu}{r}\frac{\partial s_\theta}{\partial r} - \frac{2\mu}{r^2} s_\theta \qquad (7.115)\\
\sigma'^{\theta\theta} &= \frac{\lambda}{r}\frac{\partial s_r}{\partial r} + \frac{2\mu + \lambda}{r^3}\frac{\partial s_\theta}{\partial \theta} + \frac{2\mu + \lambda}{r^2} s_r
\end{aligned}$$

The equations in Cartesian and curvilinear co-ordinates, together with the corresponding circuits have been given by Kron[60] and Soroka.[61]

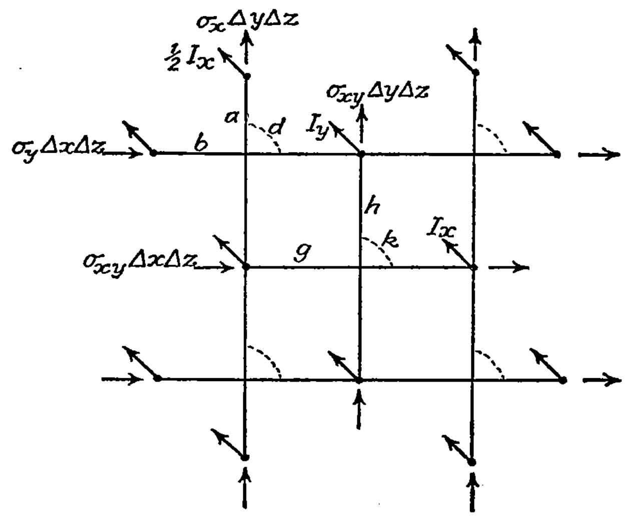

FIG. 7.9.

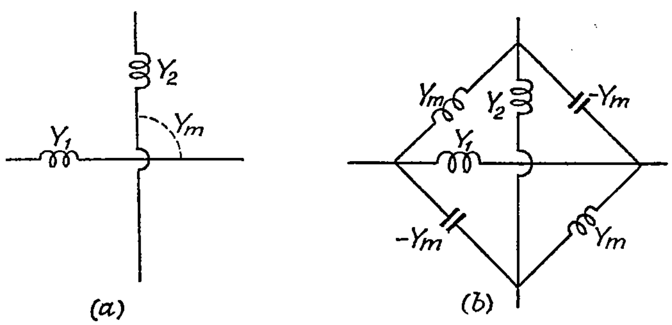

FIG. 7.10

Investigation of the practical application has been carried out by Carter[62] and Gross.[63]

In Cartesian co-ordinates (x,y,z), with displacements u, v, w, the equations have the form

$$\begin{aligned}
\sigma^{xx} &= (2\mu + \lambda)\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\\
\partial^{yy} &= (2\mu + \lambda)\frac{\partial v}{\partial y} + \lambda\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}\right)\\
\sigma^{zz} &= (2\mu + \lambda)\frac{\partial w}{\partial y} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\\
\sigma^{yz} &= \sigma^{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad (7.116)\\
\sigma^{zx} &= \sigma^{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\\
\sigma^{xy} &= \sigma^{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}$$

The equivalent circuit for the case of two-dimensional plane stress is shown in figure 7.9. The mutual coupling indicated can be represented for any given frequency of oscillation by the bridge circuit shown in figures 7.10a and 7.10b. The small oscillation circuit therefore becomes as in figure 7.11 and similarly for curvilinear co-ordinates. The parameters for figure 7.9 are shown in table VI.

TABLE VI

Symbol	Admittance
a	$(2\mu + \lambda)\dfrac{\Delta y \Delta z}{\Delta x}$
b	$(2\mu + \lambda)\dfrac{\Delta x \Delta z}{\Delta y}$
d	$\lambda \Delta z$
g	$\mu\dfrac{\Delta z \Delta x}{\Delta y}$
h	$\mu\dfrac{\Delta y \Delta z}{\Delta x}$
k	$\mu \Delta z$

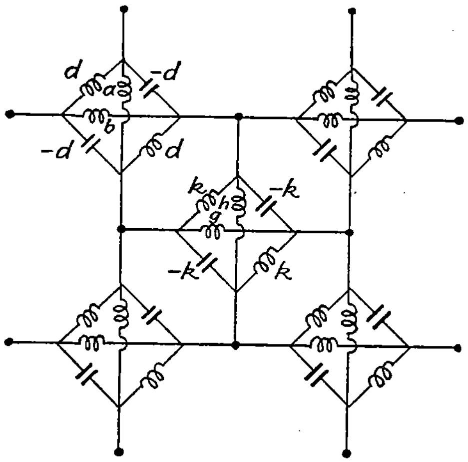

FIG. 7.11

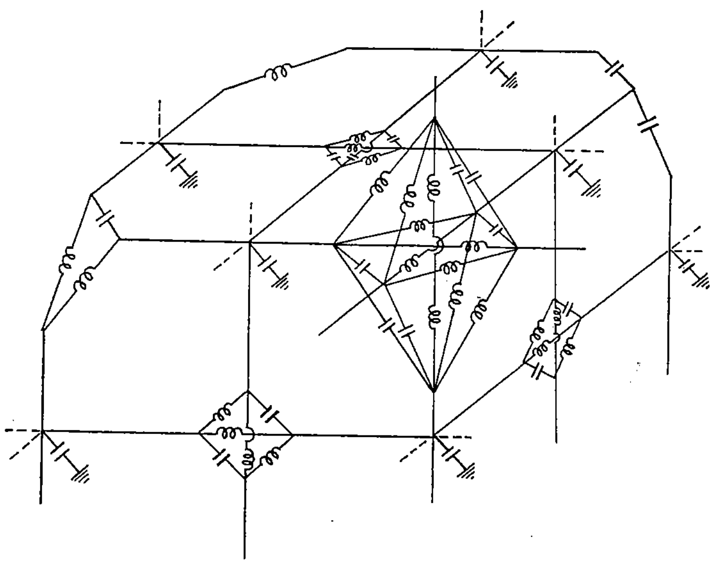

FIG. 7.12

In three dimensions the network has the form shown in figure 7.12.

The general stress equilibrium equations can be written in terms of the displacements. In a homogeneous isotropic medium these have the form[(64)]

$$(\lambda + \mu)\frac{\partial \xi}{\partial u^i} + \mu \nabla^2 \mathbf{s} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{s}}{\partial t^2} \tag{7.117}$$

A more general form of this equation is[(57)]

$$\text{grad}\,(\lambda + 2\mu)\,\text{div}\,\mathbf{s} - \text{curl}\,\mu\,\text{curl}\,\mathbf{s} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{s}}{\partial t^2} \tag{7.118}$$

It is seen that in equation (7.118), in equivalent circuit terminology,

curl **s** is a mesh voltage
μ is an admittance
μ curl **s** is a current
curl μ curl **s** is a residual current, etc.

The equations (7.117) and (7.118) satisfy Kirchhoff's second law. The networks for curvilinear co-ordinates are given in detail by Kron, Soroka, Carter and Gross.

The circuit shown in figure 7.7 has been applied by Kron to the nuclear reactor diffusion equation.[(65)] The equation describing the rate of change of neutron density in a reactor is derived from the fundamental neutron balance equation[(66)]

$$\text{production} - \text{leakage} - \text{absorption} = \frac{\partial n}{\partial t} \tag{7.119}$$

When the neutron diffusion coefficient D is independent of position this becomes

$$S + D\nabla^2\phi - \Sigma\phi = \frac{\partial n}{\partial t} \tag{7.120}$$

where S is a source term, Σ is the total macroscopic neutron absorption cross section, and ϕ is the neutron flux (nv). More generally,

$$\text{div}\,\mathbf{D}\,\text{grad}\,\phi - \Sigma\phi + S = \frac{\partial n}{\partial t} \tag{7.121}$$

and in the steady state $\partial n/\partial t = 0$

The steady-state equation is clearly satisfied by a circuit of resistances of the form of figure 7.7 with the appropriate parameters and impressed values, for example, the diffusion coefficient is a doubly contravariant tensor density

$$D^{11'} = \frac{h_2 h_3}{h_1} D_{(1)(1)}$$

$$\Sigma' = \sqrt{g}\, \Sigma \qquad (7.122)$$

and

$$S' = \sqrt{g}\, S$$

The circuit has been extended by Kron, to five dimensions, to include multiple energy groups and time variations.

APPENDIX I

Conditions of Integrability

The expression

$$Adx + Bdy + Cdz = 0$$

can be integrated only if certain relationships hold among the coefficients. To show this, suppose the expression to have an integral

$$f(xyz) = k$$

On integration this will give

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$$

and

$$\frac{\partial f}{dx} = aA, \quad \frac{\partial f}{\partial y} = aB, \quad \frac{\partial f}{\partial z} = aC$$

Hence

$$\frac{\partial}{\partial y}(aC) = \frac{\partial^2 f}{\partial y\, \partial z} = \frac{\partial^2 f}{\partial z\, \partial y} = \frac{\partial}{\partial z}(aB)$$

$$\frac{\partial}{\partial z}(aB) - \frac{\partial}{\partial y}(aC) = 0$$

$$a\frac{\partial B}{\partial z} + B\frac{\partial a}{\partial z} - a\frac{\partial C}{\partial y} - C\frac{\partial a}{\partial y} = 0$$

or

$$a\left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) + B\frac{\partial a}{\partial z} - C\frac{\partial a}{\partial y} = 0$$

Similarly,

$$a\left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right) + C\frac{\partial a}{\partial x} - A\frac{\partial a}{\partial z} = 0$$

$$a\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) + A\frac{\partial a}{\partial y} - B\frac{\partial a}{\partial x} = 0$$