1)

PROOF OF THE TETRAD POSTULATE

The tetrad postulate follows from the fact that a tensor is independent of the way it is written. The postulate follows from a consideration of the covariant derivative of a vector in two different bases. We denote these by 1 and 2. Thus:

$$(DX)_1 = (DX)_2 \quad \text{— (1)}$$

It follows that:

$$D_\mu q^a_{\ \nu} = 0. \quad \text{— (2)}$$

For those interested a detailed proof is given as follows, but eqn (1) is enough to know where the tetrad postulate comes from.

Detailed Proof

In the coordinate basis (see Carroll (3.129))

$$DX = (D_\mu X^\nu)\, dx^\mu \otimes \partial_\nu$$

$$= (\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda)\, dx^\mu \otimes \partial_\nu \quad \text{— (3)}$$

In the mixed basis:

$$DX = (D_\mu X^a)\, dx^\mu \otimes \hat{e}_{(a)} \quad \text{— (4)}$$

$$= (\partial_\mu X^a + \omega^a_{\ \mu b} X^b)\, dx^\mu \otimes \hat{e}_{(a)}$$

$$= \left(\partial_\mu (q^a_{\ \nu} X^\nu) + \omega^a_{\ \mu b}\, q^b_{\ \lambda} X^\lambda\right) dx^\mu \otimes (q^\sigma_{\ a}\, \partial_\sigma)$$

$$= q^\sigma_{\ a}\left(q^a_{\ \nu}\, \partial_\mu X^\nu + X^\nu \partial_\mu q^a_{\ \nu} + \omega^a_{\ \mu b}\, q^b_{\ \lambda} X^\lambda\right) dx^\mu \otimes \partial_\sigma$$

— (5)

where we have used the commutator rule. Now switch σ to ν and use

$$q^\nu_{\ a}\, q^a_{\ \nu} = 1 \quad \text{— (6)}$$

7) to obtain:

$$DX = \left(\partial_\mu X^\nu + q_a^\nu \partial_\mu q_\lambda^a X^\lambda + q_a^\nu q_\lambda^b \omega_{\mu b}^a X^\lambda \right) dx^\mu \otimes \partial_\nu \quad -(7)$$

Now compare eqns (3) and (7) to give:

$$\Gamma_{\mu\lambda}^\nu = q_a^\nu \partial_\mu q_\lambda^a + q_a^\nu q_\lambda^b \omega_{\mu b}^a \quad -(8)$$

multiply both sides of eqn (8) by q_ν^a:

$$q_\nu^a \Gamma_{\mu\lambda}^\nu = \partial_\mu q_\lambda^a + q_\lambda^b \omega_{\mu b}^a \quad -(9)$$

i.e. $$\boxed{D_\mu q_\lambda^a = \partial_\mu q_\lambda^a + \omega_{\mu b}^a q_\lambda^b - \Gamma_{\mu\lambda}^\nu q_\nu^a = 0} \quad -(10)$$

Quod erat demonstrandum.

Eqn (10) is known as the tetrad postulate, and is true for all connections.

Meaning of the Tetrad Postulate

The tetrad postulate means that the basis chosen for DX does not affect the result. The tetrad postulate originates in the definition of the tetrad itself:

$$V^a = q_\mu^a V^\mu \quad -(11)$$

where a refers to the tangent spacetime and μ to the base manifold.