Proof of the Structure of the Homogeneous Field Equation in the Minkowski Limit.

(I)

To Prove:

If $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ — (1)

and $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ — (2)

then: $\partial_\mu \tilde{F}^{\mu\nu} = 0$ — (3)

Proof From eqn. (3):

$$\partial_\lambda \tilde{F}^{\lambda\rho} + \partial_\mu \tilde{F}^{\mu\rho} + \partial_\nu \tilde{F}^{\nu\rho} = 0 \quad — (4)$$

and using eqn. (3):

$$\frac{1}{2}\left(\partial_\lambda \left(\epsilon^{\lambda\rho\mu\nu} F_{\mu\nu} \right) + \partial_\mu \left(\epsilon^{\mu\rho\nu\lambda} F_{\nu\lambda} \right) + \partial_\nu \left(\epsilon^{\nu\rho\lambda\mu} F_{\lambda\mu} \right) \right) = 0 \quad — (5)$$

Using the Leibniz Theorem and the constancy of the Levi-Civita symbol we obtain:

$$\epsilon^{\lambda\rho\mu\nu} \partial_\lambda F_{\mu\nu} + \epsilon^{\mu\rho\nu\lambda} \partial_\mu F_{\nu\lambda} + \epsilon^{\nu\rho\lambda\mu} \partial_\nu F_{\lambda\mu} = 0, \quad — (6)$$

for example:

$$\epsilon^{1\rho 23} \partial_1 F_{23} + \ldots\ldots = 0 \quad — (7)$$

(II) More completely, eqn. (7) is:

$$\epsilon^{1\rho 23}\partial_1 F_{23} + \epsilon^{2\rho 31}\partial_2 F_{31} + \epsilon^{3\rho 12}\partial_3 F_{12} + \dots = 0 \quad -(8)$$

From the properties of the 4-D Levi Civita symbol in Minkowski spacetime, then in eqn (8):

$$\rho = 0 \quad -(9)$$

$$\Rightarrow \epsilon^{1023}\partial_1 F_{23} + \epsilon^{2031}\partial_2 F_{31} + \epsilon^{3012}\partial_3 F_{12} + \dots = 0 \quad -(10)$$

Now use: $\epsilon^{1023} = -\epsilon^{0123} = -1 \quad -(11)$

$$\epsilon^{2031} = -\epsilon^{0231} = \epsilon^{0213} = -\epsilon^{0123} = -1 \quad -(12)$$

$$\epsilon^{3012} = -\epsilon^{0312} = \epsilon^{0132} = -\epsilon^{0123} = -1 \quad -(13)$$

so eqn. (10) becomes:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} + \dots = 0 \quad -(14)$$

$\Rightarrow$ eqn. (1), Q.E.D.

In the general 4-D Evans manifold the proof is similar.